A CHARACTERIZATION OF FINITE ÉTALE MORPHISMS IN TENSOR TRIANGULAR GEOMETRY

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Abstract. We provide a characterization of finite étale morphisms in tensor triangular geometry. They are precisely those functors which have a conservative right adjoint, satisfy Grothendieck–Neeman duality, and for which the relative dualizing object is trivial (via a canonically-defined map).

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1. Introduction

The purpose of this note is to give a characterization of “finite étale morphisms” in tensor triangular geometry. We follow the notation, terminology, and perspective of [BDS16]. In particular, we will work in the context of rigidly-compactly generated tensor-triangulated categories [BDS16, Def. 2.7]. The kind of characterization we have in mind is analogous to the following well-known characterization of smashing localizations:

1.1. Theorem. Smashing localizations of a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ are precisely those geometric functors $f^* : \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tensor-triangulated categories whose right adjoint $f_*$ is fully faithful.

We will recall a proof in Remark 3.9 below. Smashing localizations include, for example, restriction to a quasi-compact open subset of the Balmer spectrum. More generally, the tensor-triangular analogue of an étale morphism is extension-of-scalars with respect to a commutative separable algebra (with smashing localizations being the special case of idempotent algebras). Finite étale morphisms are, by definition, extension-of-scalars with respect to a compact commutative separable algebra (see Definition 4.1). Most smashing localizations are not finite étale morphisms, just as most open immersions are not proper. We will prove:

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1.2. Theorem. Finite étale extensions of a rigidly-compactly generated tensor-triangulated category $\mathcal{T}$ are precisely those geometric functors $f^* : \mathcal{T} \to \mathcal{S}$ between rigidly-compactly generated tensor-triangulated categories which satisfy the following three properties:

(a) $f^*$ satisfies Grothendieck–Neeman duality;
(b) the right adjoint $f_*$ is conservative;
(c) the canonical map $1_\mathcal{S} \to \omega_f$ is an isomorphism.

The terminology and notation will be explained in Section 4. We just remark that under hypothesis (a), the algebra $f_!(1_\mathcal{S})$ is rigid (a.k.a. dualizable) and hence has an associated trace map. This corresponds by adjunction to a canonical map $1_\mathcal{S} \to \omega_f$ from the unit to the relative dualizing object, which hypothesis (c) asserts is an isomorphism.

The keys to the theorem are the robust monadicity theorems which hold for triangulated categories and a deeper understanding of strongly separable algebras. Indeed, we begin the paper in Section 2 with a treatment of strongly separable algebras in arbitrary symmetric monoidal categories which may be of independent interest. We prove, in particular, that a rigid commutative algebra is separable if and only if it is strongly separable if and only if its canonically-defined trace form is nondegenerate (Corollary 2.38). We then turn in Section 3 to tensor-triangulated categories and the role separable algebras play in that setting. A key tool is a strengthened version of separable monadicity (Proposition 3.8). We define finite étale morphisms and prove the main theorem (Theorem 4.7) in Section 4. In Section 5, we illustrate the theorem by giving some examples and non-examples of finite étale morphisms in equivariant homotopy theory, algebraic geometry, and derived algebra.

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2. Strongly separable algebras

We begin with a discussion of separable algebras in an arbitrary symmetric monoidal category. Although separable algebras are well-understood at this level of generality, we would like to clarify the notion of strongly separable algebra. Our main goal is to show that the equivalent characterizations of classical strongly separable algebras over fields established by [Agu00] have suitable generalizations to arbitrary symmetric monoidal categories. The main punch-line is that a rigid commutative algebra is separable if and only if it is strongly separable if and only if its trace form is nondegenerate (see Corollary 2.38). Moreover, this is the case if and only if it has the (necessarily unique) structure of a special symmetric Frobenius algebra.

2.1. Terminology. Throughout this section we work in a fixed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$. The symmetry isomorphism will be denoted $\tau : A \otimes B \sim B \otimes A$. An object $A$ in $\mathcal{C}$ is rigid (a.k.a. dualizable) if there exists an object $DA$ such that $DA \otimes -$ is right adjoint to $A \otimes -$. An algebra $A$ is an associative unital monoid in $\mathcal{C}$. The multiplication and unit maps will be denoted $\mu : A \otimes A \to A$ and $u : 1 \to A$.

2.2. Definition. An algebra $(A, \mu, u)$ is separable if there exists a map $\sigma : A \to A \otimes A$ such that
(σ1) $\mu \circ \sigma = \text{id}_A$, and
(σ2) $(1 \otimes \mu) \circ (\sigma \otimes 1) = \sigma \circ \mu = (\mu \otimes 1) \circ (1 \otimes \sigma)$ as maps $A \otimes A \to A \otimes A$.

In other words, $A$ is separable if the multiplication map $\mu : A \otimes A \to A$ admits an $(A,A)$-bilinear section.

2.3. Remark. If we precompose such a section $\sigma$ with the unit $u : 1 \to A$, we obtain a map $\kappa := \sigma \circ u : 1 \to A \otimes A$ which satisfies
(κ1) $\mu \circ \kappa = u$, and
(κ2) $(1 \otimes \mu) \circ (\kappa \otimes 1) = (\mu \otimes 1) \circ (1 \otimes \kappa)$ as maps $A \to A \otimes A$.

Conversely, given such a $\kappa$, the map $A \to A \otimes A$ displayed in (κ2) satisfies axioms (σ1) and (σ2). Thus, an algebra $A$ is separable if and only if it admits a map $\kappa : 1 \to A \otimes A$ satisfying (κ1) and (κ2). Such a map $\kappa$ is called a separability idempotent.

2.4. Remark. We refer the reader to [AG60], [CHR65], [KO74], [Pie82, Chapter 10], and [For17] for further information about separable algebras and their role in classical representation theory and algebraic geometry. The following notion of a strongly separable algebra was originally studied by Kanzaki and Hattori [Hat65, Kan62]:

2.5. Definition. An algebra $A$ is strongly separable if there exists a map $\kappa : 1 \to A \otimes A$ satisfying (κ1), (κ2) and
(κ3) $\kappa = \tau \circ \kappa$.

In other words, $A$ is strongly separable if it admits a symmetric separability idempotent.

2.6. Remark. For classical algebras over a field, [Agu00] provides several equivalent characterizations of strongly separable algebras. Our present goal is to clarify the extent to which these characterizations hold in an arbitrary symmetric monoidal category. For this purpose, the graphical calculus of string diagrams will be very convenient. We refer the reader to [Sel11, §§3–4] and [PS13, §2] for more information concerning these diagrams and suffice ourselves to remark that it is a routine exercise to convert a proof involving string diagrams into a detailed proof using commutative diagrams.

2.7. Notation. We'll read our string diagrams from bottom to top. The multiplication map $\mu : A \otimes A \to A$ and the map $\kappa : 1 \to A \otimes A$ will be represented by

while the unit $u : 1 \to A$ and the identity $\text{id}_A : A \to A$ will be represented by

Thus, for example, axiom (κ2) reads

(2.8)
2.9. **Proposition.** An algebra \((A, \mu, u)\) is strongly separable if and only if there exists a morphism \(\kappa : 1 \to A \otimes A\) satisfying \((\kappa 2)\) and
\[
\mu \circ \tau \circ \kappa = u.
\]

**Proof.** By definition an algebra is strongly separable if it admits a morphism \(\kappa\) satisfying \((\kappa 1)\), \((\kappa 2)\), and \((\kappa 3)\). It is immediate that \((\kappa 1)\) and \((\kappa 3)\) together imply \((\kappa 4)\). It is also immediate that \((\kappa 3)\) and \((\kappa 4)\) together imply \((\kappa 1)\). Thus, the claim will be established if we can prove that \((\kappa 2)\) and \((\kappa 4)\) together imply \((\kappa 3)\).

Using Notation 2.7, observe:

\[
\begin{align*}
(\kappa 3) & = (\kappa 4) & = (\mu) \\
(\kappa 2) & = (\kappa 4) & = (\tau) \\
(\kappa 1) & = (\eta) & = (\mu) \\
(\kappa 3) & = (\tau) & = (\mu)
\end{align*}
\]

This establishes \(\kappa = \tau \circ \kappa\) which is axiom \((\kappa 3)\). \(\square\)

2.12. **Corollary.** Any commutative separable algebra is strongly separable.

**Proof.** This follows from Proposition 2.9 since axiom \((\kappa 4)\) coincides with axiom \((\kappa 1)\) when the algebra is commutative. \(\square\)

2.13. **Remark.** For string diagrams involving a rigid object \(A\), we’ll use the direction of a string to indicate whether it represents \(A\) or its dual \(DA\). For example, the unit \(1 \to DA \otimes A\) and counit \(A \otimes DA \to 1\) are represented by

respectively, and the unit-counit relations are given by

\[
\begin{align*}
\text{and} & \quad \text{and}
\end{align*}
\]

2.14. **Definition.** Let \(A\) be a rigid algebra in the symmetric monoidal category \(\mathcal{C}\). Its **trace map** \(\text{tr} : A \to 1\) is given by

\[
A \simeq A \otimes 1 \xrightarrow{\id \otimes \eta} A \otimes DA \otimes A \xrightarrow{1 \otimes \tau} A \otimes A \otimes DA \xrightarrow{\mu \otimes 1} A \otimes DA \xrightarrow{\varepsilon} 1.
\]
2.16. Remark. To explain this definition, recall that every endomorphism $f : A \to A$ of the rigid object $A$ has an associated “trace” $\text{Tr}(f) : 1 \to 1$ given as

$$1 \xrightarrow{2} DA \otimes A \xrightarrow{1 \otimes f} DA \otimes A \xrightarrow{\tau} A \otimes DA \xrightarrow{\epsilon} 1.$$ 

Moreover, post-composition by the map

$$DA \otimes A \xrightarrow{\tau} A \otimes DA \xrightarrow{\epsilon} 1 \xrightarrow{1}$$

induces a function $C(A,A) \simeq C(1,DA \otimes A) \to C(1,1)$ which sends $f$ to $\text{Tr}(f)$. On the other hand, the multiplication map $\mu : A \otimes A \to A$ corresponds by adjunction to a morphism $A \to DA \otimes A$ given by

$$A \simeq 1 \otimes A \xrightarrow{n \otimes 1} DA \otimes A \otimes A \xrightarrow{1 \otimes \tau} DA \otimes A \otimes A \xrightarrow{1 \otimes \mu} DA \otimes A.$$ 

Post-composition by this map provides a function $C(1,A) \to C(1,DA \otimes A) \simeq C(A,A)$ which sends a morphism $a : 1 \to A$ to “left multiplication by $a$”:

$$L_a : A \simeq A \otimes 1 \xrightarrow{1 \otimes a} A \otimes A \xrightarrow{\tau} A \otimes A \xrightarrow{\mu} A.$$ 

The map (2.15) defining the trace map $\text{tr} : A \to 1$ is readily checked to equal the composite of (2.18) and (2.17). Post-composition by the trace map thus provides the function

$$C(1,A) \xrightarrow{\text{tr}} C(1,1) \xrightarrow{a \mapsto \text{Tr}(L_a)}$$

Thus $\text{tr} : A \to 1$ is morally the map which sends an “element” of $A$ to the trace of left multiplication by that element.

2.19. Definition. The trace form of a rigid algebra $A$ is the map $t : A \otimes A \to 1$ defined as the composite

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\mu} 1.$$ 

2.20. Remark. The trace map and trace form of a rigid algebra are given by the following string diagrams:

$$\text{tr} = \quad \text{and} \quad t =$$

2.21. Remark. A map $f : A \otimes A \to 1$ is said to be an “invariant” form (also called an “associative” form) if $f \circ (\mu \otimes 1) = f \circ (1 \otimes \mu)$. Note that any form which factors through $\mu$ (such as the trace form of a rigid algebra) is necessarily invariant by the associativity of the multiplication. The converse is also true: A form $A \otimes A \to 1$ is invariant if and only if it factors through $\mu$. In fact, we obtain a bijection

$$\{\text{maps } A \to 1\} \simeq \{\text{invariant forms } A \otimes A \to 1\}$$

given by $\theta \mapsto \theta \circ \mu$ with inverse $f \mapsto f \circ (u \otimes 1) = f \circ (1 \otimes u)$. 
2.22. Remark. A map \( f : A \otimes A \to 1 \) is said to be a “symmetric” form if \( f = f \circ \tau \). Note that if \( A \) is a commutative algebra then every invariant form is automatically symmetric.

2.23. Proposition. The trace form of a rigid algebra is symmetric.

Proof. First note that we can rewrite the trace form as follows

\[
(2.24)
\]

as has already been mentioned in Remark 2.16. Next we establish

\[
(2.25)
\]

which is an equality of morphisms \( A \otimes A \to DA \otimes A \). By adjunction this can be checked after applying \( A \otimes - \) and post-composing with \( A \otimes DA \otimes A \overset{\epsilon \otimes 1}{\to} A \):

Next note that:

\[
(2.26)
\]

Then

\[
(2.24) \quad (2.25) \quad (2.26) \quad (2.25) \quad (2.24)
\]

establishes that the trace form is symmetric. \( \square \)

2.27. Remark. Intuition for why the trace form is symmetric comes from the fact that for any two endomorphisms \( f, g : A \to A \), we have \( \text{Tr}(f \circ g) = \text{Tr}(g \circ f) \). Hence, at least morally, \( t(a, b) = \text{Tr}(L_{ab}) = \text{Tr}(L_a \circ L_b) = \text{Tr}(L_b \circ L_a) = \text{Tr}(L_{ba}) = t(b, a) \).
2.28. **Definition.** If $A$ is a rigid object in a symmetric monoidal category, then every map $f : A \otimes A \to 1$ gives rise to two morphisms $A \to DA$ by adjunction (moving each copy of $A$ to the right-hand side). These two maps $A \to DA$ coincide when $f$ is symmetric, and are given by

\[
(f^* : A \simeq 1 \otimes A \xrightarrow{\eta \otimes 1} DA \otimes A \otimes A \xrightarrow{1 \otimes f} DA \otimes 1 \simeq DA).
\]

We say that a symmetric form $f : A \otimes A \to 1$ is **nondegenerate** if $f^* : A \to DA$ is an isomorphism.

2.30. **Proposition.** The trace form of a strongly separable rigid algebra is nondegenerate. Moreover, a strongly separable rigid algebra has a unique symmetric separability idempotent, which is given by

\[
(1 \xrightarrow{\eta} DA \otimes A \xrightarrow{(t^*)^{-1} \otimes 1} A \otimes A).
\]

**Proof.** Let $\kappa$ be a symmetric separability idempotent. We’ll start by showing that the composite

\[
A \simeq 1 \otimes A \xrightarrow{\kappa \otimes 1} A \otimes A \xrightarrow{1 \otimes t} A \otimes 1 \simeq A
\]

is the identity, where $t$ denotes the trace form (Def. 2.19). First note:

\[
(2.33)
\]

Then observe that

\[
(2.32)
\]

and

\[
(2.33)
\]

which shows that (2.32) is the identity map. Now $\kappa$ is symmetric by assumption and the trace form $t$ is symmetric by Proposition 2.23. Hence:
In other words, the composite (2.32) coincides with the other composite
\[
A \simeq A \otimes 1 \xrightarrow{1 \otimes \kappa} A \otimes A \otimes A \xrightarrow{t \otimes 1} 1 \otimes A \simeq A.
\]
It follows that the map \( \kappa^* : DA \to A \) given by
\[
DA \simeq 1 \otimes DA \xrightarrow{\kappa \otimes 1} A \otimes A \otimes DA \xrightarrow{1 \otimes \kappa} A \otimes 1 \simeq A
\]
is an inverse to \( t^* : A \to DA \). Indeed:

In particular, the trace form is nondegenerate. Moreover, one can readily check that \((t^* \otimes 1) \circ \kappa = \eta\) from which it follows that \( \kappa \) is given by (2.31).

2.34. Theorem. A rigid algebra is strongly separable if and only if its trace form is nondegenerate.

Proof. The only if part is provided by Proposition 2.30. Conversely, suppose \( A \) is a rigid algebra whose trace form \( t : A \otimes A \to 1 \) is nondegenerate. Write \( \theta : A \cong DA \) for the associated isomorphism (that is, \( \theta = t^* \) in the notation of Def. 2.28) and define \( \kappa : 1 \to A \otimes A \) by

(2.35)
\[
1 \xrightarrow{\eta} DA \otimes A \xrightarrow{\theta^{-1} \otimes 1} A \otimes A.
\]

First we check that \( t \) and \( \kappa \) form a self-duality in the sense that

(2.36)
\[
(2.35) = (2.35) \quad \text{and} \quad (2.29) = (2.29)
\]

and

(2.37)
\[
(2.35) = (2.35) \quad \text{and} \quad (2.29) = (2.29)
\]
Armed with this relationship between $t$ and $\kappa$, the fact that $t$ is symmetric (by Proposition 2.23) implies that $\kappa$ is also symmetric:

This establishes axiom $(\kappa 3)$. Next we establish $(\kappa 2)$, visualized in string diagrams in (2.8). It suffices to check equality after post-composition by the isomorphism $\theta \otimes \text{id}_A$. Then by adjunction it suffices to check equality after applying $A \otimes -$ and post-composing by $A \otimes DA \otimes A \xrightarrow{\epsilon \otimes 1} A$. Indeed

where the equality $(\dag)$ is the fact that the trace form is an invariant form (Remark 2.21). Finally, we establish $(\kappa 1)$. Observe that

and note that we showed $(\ddagger)$ was a consequence of $(\kappa 2)$ in the proof of Proposition 2.30. Precomposing with the unit we obtain

which is $(\kappa 1)$.

2.38. Corollary. A rigid commutative algebra is separable if and only if it is strongly separable if and only if its trace form is nondegenerate.

Proof. Every commutative separable algebra is strongly separable (Corollary 2.12) hence the claim follows from Theorem 2.34.
2.39. **Remark.** A rigid strongly separable algebra $A$ is automatically self-dual, since the nondegeneracy of the trace form provides an isomorphism $A \cong DA$.

2.40. **Example.** Consider the case where $\mathcal{C} = R\text{-Mod}$ is the category of $R$-modules for $R$ a commutative ring. An $R$-algebra $A$ is rigid precisely when it is finitely generated and projective (equivalently, finitely presented and flat) as an $R$-module. The trace map $A \to R$ is $a \mapsto \text{Tr}(L_a)$ where $L_a : A \to A$ denotes left multiplication by $a$, and the trace form $t : A \otimes A \to R$ is given by $t(a \otimes b) = \text{Tr}(L_{ab})$. In this example, the argument in Remark 2.27 shows immediately that the trace form is symmetric. It turns out that over a field $R = k$, a separable algebra is automatically rigid (that is, finite-dimensional), as shown by [VZ66, Prop. 1.1]. It was partly to clarify such finiteness assumptions that led the author to write this section on strongly separable algebras in arbitrary symmetric monoidal categories.

2.41. **Example.** An idempotent algebra in a symmetric monoidal category is an algebra $(A, \mu, u)$ whose multiplication map $\mu : A \otimes A \to A$ is an isomorphism. This is equivalent to the equality $u \otimes A = A \otimes u$ of morphisms $A \to A \otimes A$ (which then serve as an inverse to $\mu$). It is also equivalent to the switch map $\tau : A \otimes A \to A \otimes A$ being equal to the identity map $A \otimes A \to A \otimes A$. Idempotent algebras are thus examples of commutative (strongly) separable algebras. They have a (unique) separability idempotent given by $\mu^{-1} \circ u : 1 \to A \otimes A$. However, they are usually not rigid. For example, take $\mathcal{C} = R\text{-Mod}$ for $R$ a commutative ring. The idempotent $R$-algebra $R[1/s]$ is rarely finitely generated as an $R$-module. Indeed, this would imply that the principal open $D(s) \subset \text{Spec}(R)$ is both an open and closed subset of $\text{Spec}(R)$; see the argument in [San19, Example 7.4], for example.

2.42. **Remark.** A discussion of separable algebras would not be complete without saying something about their relationship with Frobenius algebras:

2.43. **Definition.** A Frobenius algebra in a symmetric monoidal category is an object $A$ equipped with both an algebra structure $(A, \mu, u)$ and a coalgebra structure $(A, \Delta, c)$ such that the Frobenius law holds:

$$(1 \otimes \mu) \circ (\Delta \otimes 1) = \Delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \Delta).$$

See, for example, [Koc04, 3.6.8]. We say that $A$ is a symmetric Frobenius algebra if the invariant form

$$A \otimes A \xrightarrow{\Delta} A \xleftarrow{1} 1$$

is symmetric. Thus, every commutative Frobenius algebra is symmetric. A special Frobenius algebra is a Frobenius algebra such that $\mu \circ \Delta = \text{id}_A$.

2.44. **Remark.** If $(A, \mu, u, \Delta, c)$ is a Frobenius algebra then the underlying object $A$ is necessarily self-dual (cf. Rem. 2.39). Indeed, the two maps $c \circ \mu : A \otimes A \to 1$ and $\Delta \circ u : 1 \to A \otimes A$ provide a self-duality.

2.45. **Remark.** The following relationship between strongly separable algebras and special symmetric Frobenius algebras is well-known classically; we include a proof for precision and completeness. The interested reader will find more concerning these ideas in [LP07, Section 2.5], [FRS02, Section 3.3] and [Fau13], among other sources.

2.46. **Proposition.** An algebra admits the structure of a special symmetric Frobenius algebra if and only if it is rigid and strongly separable. In this case, the
special symmetric Frobenius structure is unique: The counit $A \to \mathbb{1}$ is the trace map (Def. 2.14) and the comultiplication $A \to A \otimes A$ is the map corresponding (Rem. 2.3) to the unique symmetric separability idempotent (Prop. 2.30).

Proof. If $A$ is a strongly separable rigid algebra with (unique) symmetric separability idempotent $\kappa : 1 \to A \otimes A$ then the corresponding map $A \to A \otimes A$ is coassociative. Indeed, using both descriptions provided by $(\kappa 2)$ we have:

This provides $A$ with the structure of a coalgebra with counit $A \to 1$ given by the trace map. For the counital axiom just observe that

where the last equalities (†) were established in the proof of Proposition 2.30. Alternatively, one can use the description (2.31) of the unique separability idempotent and check the counital diagrams after post-composition by the isomorphism $t^* : A \cong DA$. This establishes that a strongly separable rigid algebra admits the structure of a special symmetric Frobenius algebra.

Now suppose that $(A, \mu, u, \Delta, c)$ is a special symmetric Frobenius algebra. Every Frobenius algebra is self-dual (Remark 2.44) and the comultiplication $\Delta : A \to A \otimes A$ satisfies $(\sigma 2)$. In our case, it also satisfies $(\sigma 1)$ since $A$ is assumed to be special. Symmetry of the associated separability idempotent $\Delta \circ u : 1 \to A \otimes A$ then follows from the assumed symmetry of $c \circ \mu : A \otimes A \to A$ via the self-duality (as in the beginning of the proof of Theorem 2.34). Thus $A$ is strongly separable with symmetric separability idempotent $\Delta \circ u$.

To establish uniqueness, first observe that the following commutative diagram

shows that the comultiplication $\Delta$ of a Frobenius algebra is determined by $\Delta \circ u$ and $\mu$. If $(A, \mu, u, \Delta_1, c_1)$ and $(A, \mu, u, \Delta_2, c_2)$ are two special symmetric Frobenius structures on the (rigid strongly separable) algebra $(A, \mu, u)$ then $\Delta_1 \circ u = \Delta_2 \circ u$ by the uniqueness of symmetric separability idempotents (Prop. 2.30) and hence $\Delta_1 = \Delta_2$. Moreover, since $c_i \circ \mu : A \otimes A \to \mathbb{1}$ and $\Delta_i \circ u : 1 \to A \otimes A$ form a self-duality for each $i = 1, 2$, we have:
That is, \( c_1 \circ \mu = c_2 \circ \mu \). Precomposing by the unit we conclude that \( c_1 = c_2 \).

Finally, we have already proved that if \( A \) admits a special symmetric Frobenius structure then it is rigid and strongly separable and consequently the trace map and the symmetric separability idempotent provide it with a special symmetric Frobenius structure. These thus provide the unique such structure. □

3. SEPARABLE ALGEBRAS AND TRIANGULATED CATEGORIES

In this section, we recall the relationship between separable algebras and tensor-triangulated categories established in [Bal11].

3.1. Remark. Recall from [Bal11, Section 5] that for each \( 2 \leq N \leq \infty \), there is the notion of an \( N \)-triangulated category (or triangulated category of order \( N \)) which includes as part of the structure a distinguished class of \( n \)-triangles for each \( n \leq N \) which are required to satisfy suitable higher octahedral axioms. A 2-triangulated category is precisely the same thing as a pre-triangulated category, while the usual notion of triangulated category (in the sense of Verdier) lies between the notion of 2-triangulated and 3-triangulated. An \( N \)-triangulated functor is a functor which commutes with the suspension and preserves distinguished \( N \)-triangles (equivalently, preserves distinguished \( n \)-triangles for all \( n \leq N \)).

3.2. Example. The homotopy category \( \text{Ho}(\mathcal{C}) \) of a stable \( \infty \)-category has the structure of an \( \infty \)-triangulated category.

3.3. Remark. A tensor-triangulated category is a triangulated category equipped with a closed symmetric monoidal structure which is compatible with the triangulation in the sense of [HPS97, Definition A.2.1]. For \( 2 \leq N \leq \infty \), we similarly have the notion of an \( N \)-tensor-triangulated category by replacing all instances of “triangulated” in the definition with “\( N \)-triangulated”. By an \((N-)\)tensor-triangulated functor we mean an \((N-)\)triangulated functor which is also a strong symmetric monoidal functor.

3.4. Example. The homotopy category \( \text{Ho}(\mathcal{C}) \) of a presentably symmetric monoidal [NS17, Def. 2.1] stable \( \infty \)-category is an \( \infty \)-tensor-triangulated category.

3.5. Example. If \( A \) is a commutative separable algebra in an \( N \)-tensor-triangulated category \( \mathcal{T} \) \( (2 \leq N \leq \infty) \) then the Eilenberg–Moore category \( A \text{-Mod}_\mathcal{T} \) inherits the structure of an \( N \)-tensor-triangulated category such that the extension-of-scalars functor \( F_A : \mathcal{T} \to A \text{-Mod}_\mathcal{T} \) is an \( N \)-tensor-triangulated functor. The distinguished \( n \)-triangles in \( A \text{-Mod}_\mathcal{T} \) \( (n \leq N) \) are precisely those which are created by the forgetful functor \( U_A : A \text{-Mod}_\mathcal{T} \to \mathcal{T} \). This is established by [Bal11, Theorem 5.17] and [Bal14, Section 1].

3.6. Remark. The main theorem of [DS18] states that if \( \mathcal{T} \) is an idempotent-complete triangulated category, then any triangulated adjunction \( F : \mathcal{T} \rightleftarrows \mathcal{S} : G \) is essentially monadic (that is, monadic up to idempotent completion and killing the kernel of \( G \)) whenever the Eilenberg–Moore category inherits a triangulation from \( \mathcal{T} \):

\[
(S/\ker G)^\sharp \cong GF \text{-Mod}_\mathcal{T}.
\]

This theorem also holds (with the same proof) in the 2-category of \( N \)-triangulated categories for any \( 2 \leq N \leq \infty \). In this case, the equivalence (3.7) is an equivalence of \( N \)-triangulated categories. To be clear, this is under the hypothesis
that the Eilenberg–Moore category $GF : \text{Mod}_\mathcal{T}$ inherits an $N$-triangulation from the $N$-triangulation of $\mathcal{T}$ (see [DS18, Remark 1.8]). This is a strong hypothesis on the adjunction but, as established by Balmer (Example 3.5), does hold in the separable case. The following proposition clarifies the situation with the tensor:

3.8. **Proposition.** Let $F : \mathcal{T} \rightarrow \mathcal{S}$ be an $(N)$-tensor-triangulated functor with $\mathcal{T}$ idempotent complete. Suppose $F$ admits a right adjoint $G$ and that the $F \dashv G$ adjunction satisfies the right projection formula [BDS15, Definition 2.7]. If the commutative algebra $G(1) \in \mathcal{T}$ is separable then we have an induced equivalence

$$\left(\mathcal{S}/\ker G\right)^\sharp \cong G(1) \cdot \text{Mod}_\mathcal{T}$$

of $(N)$-tensor-triangulated categories.

**Proof.** By [BDS15, Lemma 2.8], the projection formula implies that the monad of the adjunction is the monad associated to the algebra $G(1)$. The separability of this monad implies that every $s \in \mathcal{S}$ is a direct summand of $FG(s)$. It then follows from the projection formula that the thick subcategory $\ker G$ is a tensor-ideal. Thus $\mathcal{S}/\ker G$ and its idempotent completion $(\mathcal{S}/\ker G)^\sharp$ inherit tensor-structures from $\mathcal{S}$. On the other hand, the Kleisli category $G(1) \cdot \text{Free}_\mathcal{T}$ inherits a tensor-structure from $\mathcal{T}$ such that the canonical functor $\mathcal{T} \rightarrow G(1) \cdot \text{Free}_\mathcal{T}$ is a strict symmetric monoidal functor. The canonical functor $K : G(1) \cdot \text{Free}_\mathcal{T} \rightarrow \mathcal{S}$ then inherits the structure of a strong symmetric monoidal functor from the corresponding structure on $F$. Since $G(1)$ is separable, the Eilenberg–Moore category inherits a triangulation from $\mathcal{T}$ (Example 3.5) hence by [DS18] and Remark 3.6, we have equivalences

$$\left(G(1) \cdot \text{Free}_\mathcal{T}\right)^\sharp \cong \left(\mathcal{S}/\ker G\right)^\sharp \cong G(1) \cdot \text{Mod}_\mathcal{T}.$$ 

The first functor is a strong symmetric monoidal equivalence. It follows that the second functor is also a symmetric monoidal equivalence since the tensor structure on $G(1) \cdot \text{Mod}_\mathcal{T} \cong \left(G(1) \cdot \text{Free}_\mathcal{T}\right)^\sharp$ is the idempotent completion of the tensor structure on the Kleisli category (see [Pau15, Section 1.1] and [Bal14, Section 1]).

3.9. **Remark.** As an application of the proposition, we can provide a proof of Theorem 1.1 stated in the Introduction which characterizes smashing localizations of rigidly-compactly generated categories.

**Proof of Theorem 1.1.** The $(\Rightarrow)$ direction is well-known: Any smashing localization of a rigidly-compactly generated category is a geometric functor to a rigidly-compactly generated category (whose right adjoint is fully faithful); see [HPS97, Section 3.3]. For the $(\Leftarrow)$ direction, recall that smashing localizations are nothing but extension-of-scalars with respect to idempotent algebras. Suppose $f^* : \mathcal{D} \rightarrow \mathcal{E}$ is a geometric functor whose right adjoint $f_*$ is fully faithful. The multiplication map $f_*(1_e) \otimes f_*(1_e) \rightarrow f_*(1_e)$ becomes, under the projection formula, the counit $f_*(\epsilon) : f_*f^*(f_*1_e) \approx f_*(1_e) \otimes f_*(1_e) \rightarrow f_*(1_e)$. This is an isomorphism since $f_*$ is fully faithful. Thus $f_*(1_e)$ is an idempotent algebra. Idempotent algebras are separable, so Proposition 3.8 gives the result. 

4. **Finite étale morphisms**

The idea that extension-of-scalars with respect to a commutative separable algebra provides tensor triangular geometry with an analogue of an étale extension goes back to the work of Balmer [Bal15, Bal16a, Bal16b]. Here we focus on finite étale extensions of rigidly-compactly generated categories.
4.1. **Definition.** A geometric functor \( f^* : \mathcal{D} \to \mathcal{C} \) between rigidly-compactly generated \((N-)\) tensor-triangulated categories is **finite étale** if there exists a compact commutative separable algebra \( A \) in \( \mathcal{D} \) and an \((N-)\) tensor-triangulated equivalence\(^1\) \( \mathcal{C} \cong A - \operatorname{Mod}_\mathcal{D} \) such that the functor \( f^* \) becomes isomorphic to the extension-of-scalars functor \( F_A : \mathcal{D} \to A - \operatorname{Mod}_\mathcal{D} \).

4.2. **Remark.** It follows from [Bal16a, Theorem 4.2] that if \( \mathcal{D} \) is rigidly-compactly generated then \( A - \operatorname{Mod}_\mathcal{D} \) is also rigidly-compactly generated (for \( A \) a commutative separable algebra in \( \mathcal{D} \)). Thus, there is no loss of generality in considering only geometric functors between rigidly-compactly generated categories.

4.3. **Remark.** Recall from [BDS16] that a geometric functor \( f^* : \mathcal{D} \to \mathcal{C} \) between rigidly-compactly generated tensor-triangulated categories has a right adjoint \( f_* : \mathcal{C} \to \mathcal{D} \) which itself has a right adjoint \( f^! : \mathcal{D} \to \mathcal{C} \). The relative dualizing object of \( f^* \) is the object \( \omega_f := f^!(1_\mathcal{D}) \in \mathcal{C} \). Recall that \( f^* \) is said to satisfy Grothendieck–Neeman duality if the right adjoint \( f_* \) preserves compact objects. (A number of equivalent definitions are provided by [BDS16, Theorem 3.3].) In this case, the commutative algebra \( f_*(\mathbb{1}_\mathcal{C}) \) is compact-rigid. Hence it has a trace map \( f_*(\mathbb{1}_\mathcal{C}) \to 1_\mathcal{D} \) (Def. 2.14), which corresponds to a map \( \mathbb{1}_\mathcal{C} \to \omega_f \).

4.4. **Remark.** In general, morphisms \( \mathbb{1}_\mathcal{C} \to \omega_f \) can be identified with morphisms \( f_*(\mathbb{1}_\mathcal{C}) \to 1_\mathcal{D} \) by adjunction and these can be identified as in Remark 2.21 with the invariant forms on the algebra \( f_*(\mathbb{1}_\mathcal{C}) \):

\[
\{ \mathbb{1}_\mathcal{C} \to \omega_f \} \xrightarrow{\sim} \{ f_*(\mathbb{1}_\mathcal{C}) \to 1_\mathcal{D} \} \xrightarrow{\sim} \{ \text{invariant forms } f_*(\mathbb{1}_\mathcal{C}) \otimes f_*(\mathbb{1}_\mathcal{C}) \to 1_\mathcal{D} \}.
\]

Also recall (Def. 2.28) that an invariant form \( f_*(\mathbb{1}_\mathcal{C}) \otimes f_*(\mathbb{1}_\mathcal{C}) \to 1_\mathcal{D} \) is nondegenerate if the adjoint morphism \( f_*(\mathbb{1}_\mathcal{C}) \to Df_*(\mathbb{1}_\mathcal{C}) \) is an isomorphism. (Note that these invariant forms are automatically symmetric since the algebra \( f_*(\mathbb{1}_\mathcal{C}) \) is commutative.) On the other hand, recall from [BDS16, (2.18)] that we have an isomorphism \( f_*(\omega_f) \cong Df_*(\mathbb{1}_\mathcal{C}) \).

4.5. **Lemma.** Let \( \theta : \mathbb{1}_\mathcal{C} \to \omega_f \) be any morphism. The map \( f_*(\mathbb{1}_\mathcal{C}) \to Df_*(\mathbb{1}_\mathcal{C}) \) which is adjoint to the invariant form on \( f_*(\mathbb{1}_\mathcal{C}) \) corresponding to \( \theta \) coincides with the map

\[
(4.6) \quad f_*(\mathbb{1}_\mathcal{C}) \xrightarrow{f_*^{(\theta)}} f_*(\omega_f) \cong Df_*(\mathbb{1}_\mathcal{C}).
\]

Consequently, the invariant form associated to \( \theta \) is nondegenerate if and only if \( f_*(\theta) \) is an isomorphism.

**Proof.** This is a straightforward verification. From the definition of the isomorphism \( f_*(\omega_f) \cong Df_*(\mathbb{1}_\mathcal{C}) = [f_*(\mathbb{1}_\mathcal{C}), 1_\mathcal{D}] \) in [BDS16, (2.18)], one sees that the morphism (4.6) is obtained by going along the top of the following commutative diagram

\[
\begin{array}{c}
\xymatrix{ f_*(\omega_f) \ar[r]^{f_*^{(\theta)}} & f_*(\omega_f) \otimes f_*(\mathbb{1}_\mathcal{C}) \ar[r]^{[1, \operatorname{hax}]} & f_*(\mathbb{1}_\mathcal{C}), f_*(\mathbb{1}_\mathcal{C}) \ar[r]^{[1, \operatorname{hax}]} & f_*(\mathbb{1}_\mathcal{C}), 1_\mathcal{D} \\
f_*(\mathbb{1}_\mathcal{C}) \ar[r]^{\text{coev}} & f_*(\mathbb{1}_\mathcal{C}) \otimes f_*(\mathbb{1}_\mathcal{C}) \ar[r]^{[1, \operatorname{hax}]} & [f_*(\mathbb{1}_\mathcal{C}), f_*(\omega_f)] \ar[r]^{[1, f_*(\theta)]} & [f_*(\mathbb{1}_\mathcal{C}), f_*(\mathbb{1}_\mathcal{C})] }\end{array}
\]

\(^1\)For tensor-triangulated categories in the usual sense of Verdier, the category of modules \( A - \operatorname{Mod}_\mathcal{D} \) is a priori only a pre-tensor-triangulated category, but this does not cause any trouble for the definition. Since \( \mathcal{C} \) is tensor-triangulated by assumption, the equivalence \( \mathcal{C} \cong A - \operatorname{Mod}_\mathcal{D} \) just forces \( A - \operatorname{Mod}_\mathcal{D} \) to be tensor-triangulated as well. This technicality doesn’t arise when working in the 2-category of \( N \)-tensor-triangulated categories for any \( 2 \leq N \leq \infty \).
while the adjoint of the associated invariant form is obtained by going along the bottom.

4.7. **Theorem.** Let $f^* : D \to C$ be a geometric functor between rigidly-compactly generated tensor-triangulated categories. Then $f^*$ is a finite étale morphism (Definition 4.1) if and only if the following three conditions hold:

(a) $f^*$ satisfies Grothendieck–Neeman duality;
(b) the right adjoint $f_*$ is conservative;
(c) the map $\mathbb{1}_C \to \omega_f$ adjoint to the trace map is an isomorphism.

**Proof.** ($\Rightarrow$) If $f^*$ is finite étale then it is extension-of-scalars with respect to the compact separable commutative algebra $f_*(\mathbb{1}_C)$. By the separable Neeman–Thomason Localization Theorem established by Balmer [Bal16a, Theorem 4.2], the compact objects in $C$ are precisely the thick subcategory generated by the image $f^*(D^c)$ of the compact objects in $D$. Thus, by the projection formula, the fact that $f_*(\mathbb{1}_C)$ is compact ensures that $f_*(c)$ is compact for all $c \in C$. Thus, $f^*$ satisfies Grothendieck–Neeman duality. The right adjoint $f_*$ is certainly conservative (in fact faithful). Moreover, by Corollary 2.38, the commutative rigid separable algebra $f_*(1)$ is strongly separable, so that its trace form is nondegenerate. By Lemma 4.5, this means that the canonical map $\mathbb{1}_C \to \omega_f$ becomes an isomorphism after applying $f_*$. But this means the canonical map is an isomorphism since $f_*$ is conservative.

($\Leftarrow$) If $f^*$ satisfies Grothendieck–Neeman duality then the commutative algebra $f_*(\mathbb{1}_C)$ is rigid hence has a trace map (so that part (c) makes sense). Moreover, by Lemma 4.5, if the map $\mathbb{1}_C \to \omega_f$ adjoint to the trace map is an isomorphism then the trace form is nondegenerate; hence by Corollary 2.38, $f_*(1)$ is a (strongly) separable algebra. By Proposition 3.8, we have a tensor-triangulated equivalence $C \to f_*(1)$-Mod$_D$ compatible with the two adjunctions. Here we use the assumption that $f_*$ is conservative and the fact that $C$ is idempotent complete (since it has small coproducts). Therefore $f^*$ is finite étale. □

4.8. **Remark.** Although in part (b) of Theorem 4.7 we just assume $f_*$ is conservative, it follows from the other hypotheses that it is actually faithful. It also follows from (a) and (c) that $f^*$ has the full Wirthmüller isomorphism of [BDS16, Theorem 1.9].

5. **Examples**

We will now discuss some examples of finite étale morphisms with an eye to future applications.

5.1. **Example.** Let $G$ be a compact Lie group. It was proved in [BDS15, Theorem 1.1] that for any finite index subgroup $H \leq G$, the restriction functor $\text{SH}(G) \to \text{SH}(H)$ between equivariant stable homotopy categories is finite étale. We can use Theorem 4.7 to improve this to an if and only if statement:

5.2. **Theorem.** Let $G$ be a compact Lie group and let $H \leq G$ be a closed subgroup. The restriction functor $\text{res}^G_H : \text{SH}(G) \to \text{SH}(H)$ is finite étale if and only if $H$ has finite index in $G$.

**Proof.** As already mentioned, the “if” part is [BDS15, Theorem 1.1]. For the “only if” part recall that the relative dualizing object for $\text{res}^G_H$ is the representation sphere $S^{L(H;G)}$ for the tangent $H$-representation at the coset $eH \in G/H$ (see [May03] and
By Theorem 4.7, if $\text{res}_H^G$ is finite étale, the canonical morphism $1_{SH(H)} \to S^{L(H,G)}$ is an isomorphism. Restricting to the trivial subgroup, we obtain an isomorphism $S^0 \to S^{\dim(G/H)}$ in the nonequivariant stable homotopy category $SH$. The dimension of (the suspension spectrum of) a sphere is recovered by rational cohomology. Hence $\dim(G/H) = 0$. The compact 0-dimensional manifold $G/H$ is just a finite collection of points. That is, $H$ has finite index in $G$. □

5.3. Example. Let $p_n : S^1 \to S^1$ denote the degree $n$ map $z \mapsto z^n$ on the unit circle. The induced functor $p_n^* : SH(S^1) \to SH(S^1)$ is not finite étale (for $n \geq 2$). Indeed, this amounts to the question of whether the quotient $S^1 \to S^1/C_n$ by the subgroup of $n$th roots of unity induces a finite étale morphism $SH(S^1/C_n) \to SH(S^1)$. But [San19, Proposition 3.2] establishes that inflation $\text{infl}_G^G : SH(G/N) \to SH(G)$ never satisfies Grothendieck–Neeman duality except when $N = 1$ is the trivial subgroup.

5.4. Remark. Another way of appreciating why Example 5.3 is not finite étale is to look at its behaviour on the Balmer spectrum, which we know due to [BGH20, BS17]. The points of $\text{Spec}(SH(S^1)^c)$ are of the form $P(H, C)$ for $H$ a closed subgroup of $S^1$ and $C \in \text{Spec}(SH^c)$. The closed subgroups of $S^1$ are, in addition to $S^1$ itself, the finite cyclic groups $C_m$ ($m \geq 1$) realized as the roots of unity in $S^1$. Consider the map on the Balmer spectrum

$$\varphi := \text{Spec}(p_n^*) : \text{Spec}(SH(S^1)^c) \to \text{Spec}(SH(S^1)^c)$$

induced by the degree $n$ map $p_n : S^1 \to S^1$. One can show that $\varphi(P(C_m, C)) = P(C_{\text{lcm}(m,n)}/n, C)$. For example, taking $n = 2$ and fixing the nonequivariant prime $C$, it maps

$$m \mapsto \begin{cases} m/2 & \text{if } 2 \mid m \\ m & \text{if } 2 \nmid m. \end{cases}$$

In particular, we find that the fibers have cardinality

$$|\varphi^{-1}(\{P(C_N, C)\})| = \begin{cases} 1 & \text{if } 2 \mid N \\ 2 & \text{if } 2 \nmid N. \end{cases}$$

For example, the fiber over $P(C_1, C)$ consists of two points $\{P(C_1, C), P(C_2, C)\}$. Moreover, if the nonequivariant prime $C$ is 2-local then $P(C_1, C) \subseteq P(C_2, C)$ is a nontrivial inclusion in the fiber. This implies that the basic theorems of Balmer [Bal16b, Theorem 1.5] on the behaviour of finite étale morphisms do not hold for the morphisms $p_n^* : SH(S^1) \to SH(S^1)$.

5.5. Lemma. Consider a diagram of coproduct-preserving ($N$-)tensor-triangulated functors between rigidly-compactly generated ($N$-)tensor-triangulated categories

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{g^*} & \mathcal{D} \\
\downarrow{k^*} & & \downarrow{k^*} \\
\mathcal{C}' & \xrightarrow{f^*} & \mathcal{D}'
\end{array}$$

which commutes up to natural isomorphism of symmetric monoidal functors. Denote the right adjoints by $f^* \dashv f_*$ and $g^* \dashv g_*$ and suppose that the Beck–Chevalley comparison map

$$h^* g_* \to f_* k^*$$
is a natural isomorphism of lax symmetric monoidal functors. If \( g^* \) is finite étale and \( f_* \) is conservative then \( f^* \) is finite étale.

**Proof.** The natural isomorphism \( h^* g_* \simeq f_* k^* \) provides an isomorphism of commutative algebras \( h^* g_*(\mathbb{1}_\mathcal{D}) \simeq f_*(\mathbb{1}_{\mathcal{D}'}) \). By assumption, \( g_*(\mathbb{1}_\mathcal{D}) \) is a compact commutative separable algebra in \( \mathcal{C} \), hence \( f_*(\mathbb{1}_{\mathcal{D}'}) \) is a compact commutative separable algebra in \( \mathcal{C}' \). The \( f^* \dashv f_* \) adjunction satisfies the projection formula (see [BDS16, Prop. 2.15]) and \( f_* \) is conservative by hypothesis. Hence, Proposition 3.8 provides the result. \( \square \)

5.6. **Example.** If \( \mathcal{C} \) is a presentably symmetric monoidal stable \( \infty \)-category and \( A \in \text{CAlg}(\mathcal{C}) \) is a commutative algebra in \( \mathcal{C} \), then we can consider the presentably symmetric monoidal stable \( \infty \)-category \( A \text{-Mod}_\mathcal{C} \) of \( A \)-modules. If \( \mathcal{C} \) is rigidly-compactly generated then so is \( A \text{-Mod}_\mathcal{C} \) (see [PSW21, Remark 3.11], for example). At the level of homotopy categories, the extension-of-scalars \( \text{Ho}(\mathcal{C}) \to \text{Ho}(A \text{-Mod}_\mathcal{C}) \) is then a geometric functor of rigidly-compactly generated \( \infty \)-tensor-triangulated categories whose right adjoint is conservative.

5.7. **Example.** Let \( \mathcal{C} \) be a presentably symmetric monoidal stable \( \infty \)-category and let \( A, B \in \text{CAlg}(\mathcal{C}) \) be commutative algebras in \( \mathcal{C} \). We then have

\[
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & B \text{-Mod}_\mathcal{C} \\
\downarrow & & \downarrow \\
A \text{-Mod}_\mathcal{C} & \longrightarrow & (A \otimes B) \text{-Mod}_\mathcal{C}
\end{array}
\]

where all four functors are extension-of-scalars. This is an example where the Beck-Chevalley property holds (at the level of the underlying stable \( \infty \)-categories). In particular, the induced diagram of \( \infty \)-tensor-triangulated categories

\[
\begin{array}{ccc}
\text{Ho}(\mathcal{C}) & \longrightarrow & \text{Ho}(B \text{-Mod}_\mathcal{C}) \\
\downarrow & & \downarrow \\
\text{Ho}(A \text{-Mod}_\mathcal{C}) & \longrightarrow & \text{Ho}((A \otimes B) \text{-Mod}_\mathcal{C})
\end{array}
\]

satisfies the first hypothesis of Lemma 5.5. Moreover, the right adjoints are all conservative (Example 5.6). Thus, if the top horizontal functor is finite étale (i.e. if \( B \) is a compact separable commutative algebra in \( \text{Ho}(\mathcal{C}) \)) then the bottom horizontal functor is also finite étale.

5.8. **Example.** Let \( G \) be a compact Lie group and let \( \text{Sp}_G \) denote the symmetric monoidal stable \( \infty \)-category of \( G \)-spectra (see [GM20, Appendix C]). Let \( \text{triv}_G : \text{Sp} \to \text{Sp}_G \) denote the unique colimit-preserving symmetric monoidal functor from the \( \infty \)-category of spectra. Since \( \text{res}_H^G \circ \text{triv}_G \simeq \text{triv}_H \) for any \( H \leq G \), we have a commutative diagram

\[
\begin{array}{ccc}
\text{Ho}(\text{Sp}_G) & \longrightarrow & \text{Ho}(\text{Sp}_H) \\
\downarrow & & \downarrow \\
\text{Ho}(\text{triv}_G(\mathcal{E}) \text{-Mod}_{\text{Sp}_G}) & \longrightarrow & \text{Ho}(\text{triv}_H(\mathcal{E}) \text{-Mod}_{\text{Sp}_H})
\end{array}
\]

for any \( \mathcal{E} \in \text{CAlg}(\text{Sp}) \). If \( H \leq G \) has finite index then the top horizontal functor is finite étale (Example 5.1) and hence the bottom horizontal functor is finite étale.
Taking $\mathcal{E} = \mathcal{H} \mathcal{Z}$, we obtain that the restriction functor

$$D(HZ_G) \to D(HZ_H)$$

between categories of derived Mackey functors studied in [PSW21] is finite étale. This will be utilized in the forthcoming [BHS21] which will classify the localizing tensor-ideals of these categories.

5.9. Example. A version of Example 5.7 holds purely at the level of triangulated categories if one assumes the two algebras are separable. More precisely, let $A$ and $B$ be two commutative separable algebras in a rigidly-compactly generated $\mathcal{N}$-tensor-triangulated category $\mathcal{T}$. Iterated extension-of-scalars behaves as one expects (see [Pau17, Proposition 1.14]) and we have a diagram of rigidly-compactly generated $\mathcal{N}$-tensor-triangulated categories

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{F_A} & A \text{-Mod}_{\mathcal{T}} \\
\downarrow{F_B} & & \downarrow \\
B \text{-Mod}_{\mathcal{T}} & \xrightarrow{(A \otimes B) \text{-Mod}_{\mathcal{T}}} & (A \otimes B) \text{-Mod}_{\mathcal{T}}
\end{array}$$

which commutes up to isomorphism. Lemma 5.5 implies that if the top functor is finite étale then so is the bottom functor.

5.10. Remark. Let $F : \mathcal{D} \to \mathcal{C}$ be a geometric functor of rigidly-compactly generated tensor-triangulated categories and let $\varphi : \text{Spc}(\mathcal{C}) \to \text{Spc}(\mathcal{D})$ be the induced map on spectra. For any Thomason subset $Y \subseteq \text{Spc}(\mathcal{D})$ with $V := \text{Spc}(\mathcal{D}) \setminus Y$, we have an induced functor $F|_V : \mathcal{D}(V) \to \mathcal{C}(\varphi^{-1}(V))$ on finite localizations such that

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{F} & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{D}(V) & \xrightarrow{F|_V} & \mathcal{C}(\varphi^{-1}(V))
\end{array}$$

commutes up to isomorphism. Moreover, on spectra

$$\varphi^{-1}(V) \cong \text{Spc}(\mathcal{C}(\varphi^{-1}(V))) \xrightarrow{\text{Spc}(F|_V)} \text{Spc}(\mathcal{D}) \cong V$$

is just the restriction $\varphi|_V : \varphi^{-1}(V) \to V$.

5.12. Example (Restriction in the target). If $F : \mathcal{D} \to \mathcal{C}$ is finite étale then the induced “restriction” functor

$$F|_V : \mathcal{D}(V) \to \mathcal{C}(\varphi^{-1}(V))$$

of Remark 5.10 is also finite étale. Here $V \subseteq \text{Spc}(\mathcal{D})$ is the complement of a Thomason subset. For example, $V$ could be a quasi-compact open subset. Indeed this is just a special case of Example 5.9 with $B = f_V$ the idempotent algebra for the finite localization $\mathcal{D} \to \mathcal{D}(V)$.

5.13. Remark. Additional equivariant examples are featured in the work of Balmer and Dell’Ambrogio on Mackey 2-motives [BD20, Del21]. On the other hand, the following basic example relates the tensor-triangular notion of finite étale with the ordinary scheme-theoretic notion:

5.14. Theorem (Balmer). If $f : X \to Y$ is a finite étale morphism of quasi-compact and quasi-separated schemes then the derived functor $Lf^* : D_{qc}(Y) \to D_{qc}(X)$ is a finite étale morphism in the sense of Definition 4.1.
Proof. This is provided by [Bal16a, Theorem 3.5]; see also [Nee18, Example 0.3]. □

5.15. Remark. The proof of the above theorem works verbatim for other tensor-triangulated categories $\mathcal{T}(X)$ fibered over a category of schemes, provided the pseudofunctor $X \mapsto \mathcal{T}(X)$ satisfies flat base change. Many motivic examples of such pseudofunctors are discussed in [CD19]. We just mention:

5.16. Example. Let $L/k$ be a finite separable extension of fields whose characteristic (if positive) is invertible in the ring $R$. The induced functor $\text{SH}(k;R) \to \text{SH}(L;R)$ between motivic stable homotopy categories (with coefficients in $R$) is a finite étale morphism in the sense of Definition 4.1. The same is true of the induced functor $\text{DM}(k;R) \to \text{DM}(L;R)$ between derived categories of motives. See [CD19] and [Tot18] for more information about these categories.

5.17. Remark. The author thinks it is interesting to have an “intrinsic” characterization of finite étale morphisms in tensor triangular geometry as expressed in Theorem 4.7. Nevertheless, actually classifying the finite étale extensions of a given category $\mathcal{T}$ amounts to classifying the rigid (strongly) separable commutative algebras in $\mathcal{T}$. For the equivariant stable homotopy category $\mathcal{T} = \text{SH}(G)$, this classification will be studied in forthcoming work with Balmer. The analogous problem for the stable module category $\mathcal{T} = \text{StMod}(kG)$ has been studied in [BC18] and is surprisingly subtle. It is currently only understood when $G$ is cyclic.

5.18. Remark. For the derived category $\mathcal{T} = \text{D}_{qc}(X)$ of a noetherian scheme, Neeman [Nee18] has obtained a very satisfactory classification of the (not necessarily compact) commutative separable algebras. His work shows that the tensor-triangular analogue of étale morphism (a.k.a. extension by a commutative separable algebra) lies somewhere between the classical étale morphisms of schemes and the pro-étale morphisms of Bhatt–Scholze [BS15]. His results also show that there are no exotic étale extensions of derived categories of schemes: An étale extension of a derived category of a scheme is another derived category of a scheme. We’ll state this result precisely in the case of finite étale extensions:

5.19. Theorem (Neeman). Let $X$ be a noetherian scheme. If $F : \text{D}_{qc}(X) \to S$ is a finite étale morphism (Def. 4.1) then there exists a finite étale morphism of schemes $f : U \to X$ and a tensor-triangulated equivalence $S \cong \text{D}_{qc}(U)$. With this identification, $F$ is naturally isomorphic to $L f^* : \text{D}_{qc}(X) \to \text{D}_{qc}(U)$.

Proof. Let $G$ denote the right adjoint of $F$. By definition, $F$ is extension-of-scalars with respect to the compact commutative separable algebra $G(1) \in \text{D}_{qc}(X)$. Neeman [Nee18, Theorem 7.10] establishes that there is a separated finite-type étale map of schemes $g : V \to X$ and a generalization-closed subset $U \subset V$ such that $G(1) \cong R f_*(\mathcal{O}_U)$ where $f : U \to X$ denotes the composite $U \hookrightarrow V \xrightarrow{g} X$. It then follows from Proposition 3.8 that $S \cong \text{D}_{qc}(U)$ with $F \cong L f^*$. Now, since $G(1)$ is compact, the argument in [Nee18, Remark 0.6] shows that $U \subset V$ is actually an open subset. (Take $L = 0$, $K \coloneqq f_*(K)$ and the identity map $K \to f_*(K)$ in loc. cit.) Thus, $f : U \to X$ is a separated finite-type étale map. It is also proper since $L f^* \cong F$ satisfies GN-duality (by [LN07]; see also [San19, Section 7] and [Lip09, Section 4.3]). This completes the proof since an étale map is proper if and only if it is finite. □

5.20. Remark. For the purpose of classifying the finite étale extensions of a given tensor-triangulated category, the results of Section 2 are worth keeping in mind.
They clarify that the compact/rigid commutative separable algebras that provide finite étale extensions are necessarily self-dual. This puts limits on the role finite étale morphisms can play in equivariant contexts over non-finite groups. Stated differently, Theorem 4.7 shows that the relative dualizing object $\omega_f$ for a finite étale morphism $f^*$ must be trivial. It is natural to wonder if there is a reasonable generalization of “finite étale” in tensor triangular geometry which shares some of its good properties (e.g., the results of [Bal16a, Bal16b]) and yet covers examples having non-trivial dualizing objects (e.g., the examples which arise in [Rog08]).

References


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