Economics 217 **Name:**________________________________

Exam #3

Instructions. Closed book and notes, 180 minutes. Please directly answer on the exam paper. Partial credit will be granted for brief, relevant remarks and for partial results, but not unrelated equations and text from memory. There are 8 questions, 110 points in total.

Problem 1 - AR processes

Please write down the equation for the AR(1) process. In terms of variance, please <u>derive</u> the key necessary condition for stationarity, specifying precisely what happens when this condition is not met. (10 points)

Answer

The basic AR(1) process is written as:

$$Y_t = \phi Y_{t-1} + u_t$$

The key (necessary) condition for stationarity is $|\phi| < 1$. To see this, solve for the variance of Y_t and simplify:

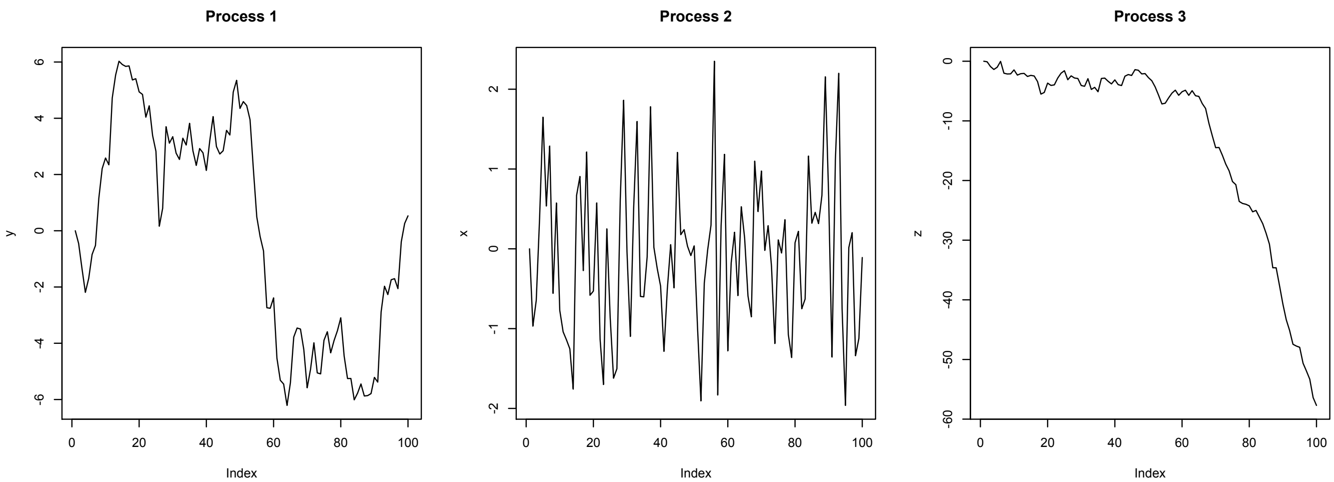
$$\begin{aligned} Var(Y_t) &= Var(\phi Y_{t-1}) + Var(u_t) \\ &= \phi^2 Var(Y_{t-1}) + Var(u_t) \\ &= \phi^2 Var(Y_t) + Var(u_t) \\ \Rightarrow \quad Var(Y_t) &= \frac{Var(u_t)}{(1-\phi^2)} \end{aligned}$$

The key condition is that $\phi^2 < 1$. Two problems arise when this is not satisfied. First, as ϕ^2 approaches 1, the variance of Y_t explodes. Second, when $\phi^2 > 1$ variance becomes negative, which violates the basic properties of variance.

4 points for necessary condition and 3 for deriving the expression of $Var(Y_t)$ in terms of $Var(u_t)$. 2 point for discussing each case of violation ($\phi^2 \to 1$ and $\phi^2 > 1$). Max is 10.

Problem 2 - AR processes

Below, I've illustrated three AR1 processes. Please briefly discuss the properties of each and what this means for the likely value of ϕ (compared with $\phi = 1$). (10 points)



Answer

In process #1, it does not look stationary, but not explosive. It is reminiscent of an random walk, so this process is likely generated by something around $\phi = 1$

In process #2, this is likely a stationary process with $\phi < 1$. The reason is that current values do not seem to predict future values very well, suggesting a lower value of ϕ.

Process #3 is clearly explosive with $\phi > 1$. The direction of the series is heavily dependent on the first value, and it moves away from zero at an exponential rate.

4 points for each graph (2 for value of ϕ and 2 for proper explanation). Max is 10.

Problem 3 - MA processes

Suppose that we have the following (restricted) MA(2) model:

$$y_t = u_t + \theta_2 u_{t-2}$$

Please derive the theoretical correlation between y_t and the first three lags, y_{t-1}, y_{t-2} and y_{t-3}. (20 points)

Answer

For a stationary process, the ACF function is written as:

$$ACF = \frac{Cov\left(Y_t, Y_{t-k}\right)}{Var\left(Y_t\right)}$$

The variance of Y_t is straightforward

$$\begin{aligned} Var\left(Y_t\right) &= Var\left(u_t\right) + Var\left(\theta u_{t-1}\right) \\ &= Var\left(u_t\right) + \theta_2^2 Var\left(u_t - 2\right) \\ &= Var\left(u_t\right)\left(1 + \theta_2^2\right) \end{aligned}$$

The tough part of this question is the covariance. First, let's do the covariance of Y_t and Y_{t-1}.

$$Cov\left(Y_t, Y_{t-1}\right) = E\Big[\left(Y_t - E(Y_t)\right)\left(Y_{t-1} - E(Y_{t-1})\right)\Big]$$

Since our white noise processes are mean zero, so are $E(Y_t)$'s

$$Cov\left(Y_t, Y_{t-1}\right) = E\left[Y_t Y_{t-1}\right]$$

Substituting using the MA(2) equation

$$\begin{aligned} Cov\left(Y_t, Y_{t-1}\right) &= E\Big[\left(u_t + \theta_2 u_{t-2}\right)\left(u_{t-1} + \theta_2 u_{t-3}\right)\Big] \\ &= E\left[u_t u_{t-1} + \theta_2 u_{t-2} u_{t-1} + \theta_2 u_t u_{t-2} + \theta_2^2 u_{t-2} u_{t-3}\right] \end{aligned}$$

Since shocks in different periods are uncorrelated, we have

$$Cov\left(Y_t, Y_{t-1}\right) = 0 \Rightarrow ACF(1) = 0$$

For the covariance of Y_t and Y_{t-2}

$$\begin{aligned} Cov\left(Y_t, Y_{t-2}\right) &= E\Big[\left(u_t + \theta_2 u_{t-2}\right)\left(u_{t-2} + \theta_2 u_{t-4}\right)\Big] \\ &= E\left[u_t u_{t-2} + \theta_2 u_t u_{t-4} + \theta_2 u_{t-2} u_{t-2} + \theta_2^2 u_{t-2} u_{t-4}\right] \end{aligned}$$

Getting rid of uncorrelated shocks, we have

$$Cov\left(Y_t, Y_{t-2}\right) = E\left[\theta_2 u_{t-2} u_{t-2}\right] = \theta_2 Var(u) \Rightarrow ACF(2) = \frac{\theta_2}{1 + \theta_2^2}$$

Finally, for the covariance of Y_t and Y_{t-3}

$$\begin{aligned} Cov\,(Y_t, Y_{t-3}) &= E\Big[\,(u_t + \theta_2 u_{t-2})\,(u_{t-3} + \theta_2 u_{t-5})\,\Big] \\ &= E\Big[u_t u_{t-3} + \theta_2 u_t u_{t-5} + \theta_2 u_{t-2} u_{t-3} + \theta_2^2 u_{t-2} u_{t-5}\Big] \end{aligned}$$

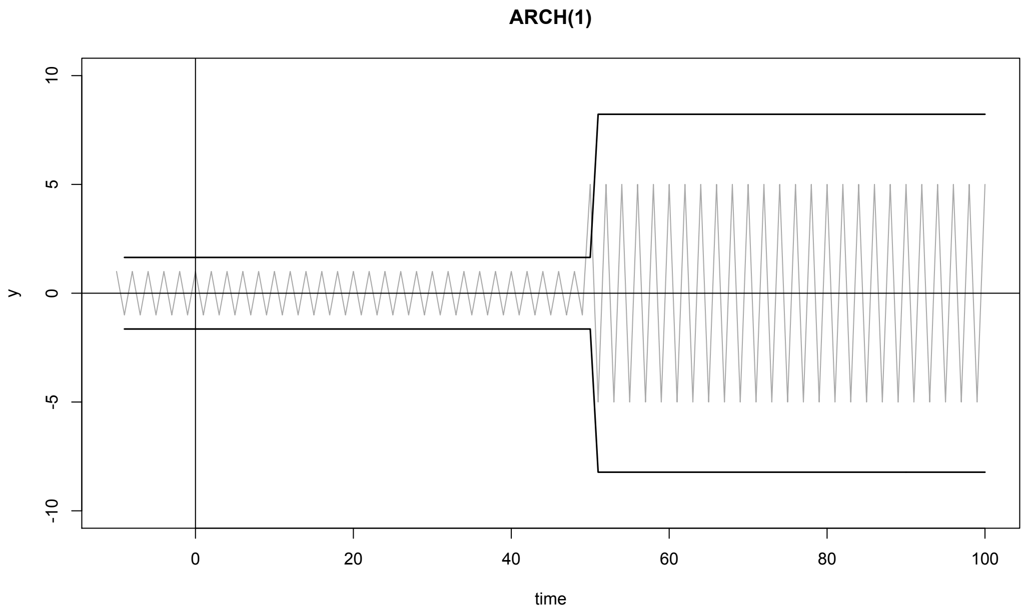
Again, since uncorrelated shocks are zero

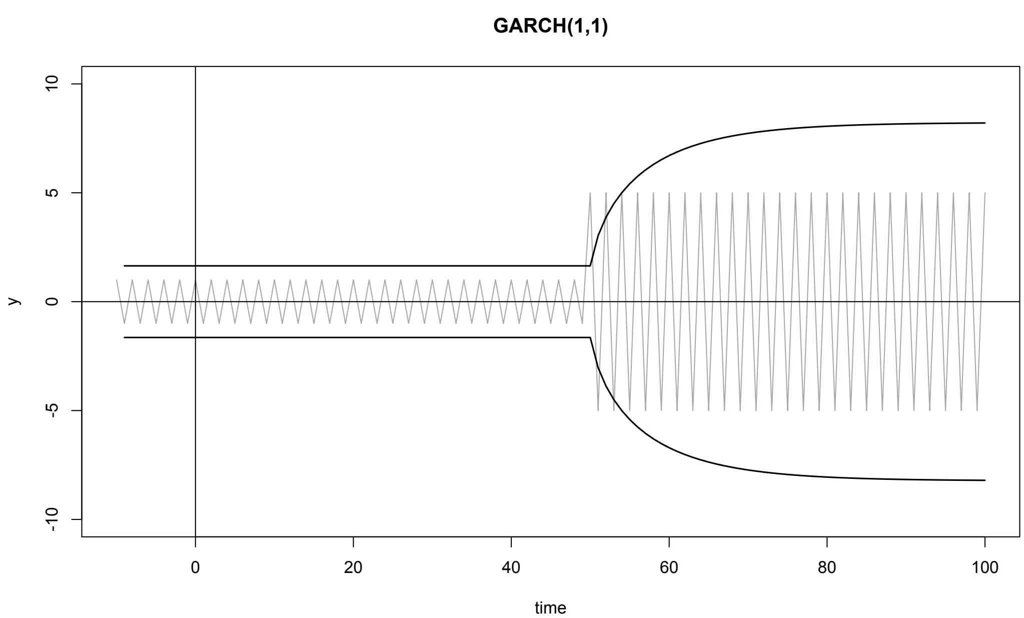
$$Cov\,(Y_t, Y_{t-3}) \;=\; 0 \Rightarrow ACF(3) = 0$$

5 points for $Var(Y_t)$, then, 7 points for γ_2, 4 points for γ_1 and 4 for γ_3.

Problem 4 - ARCH/GARCH

Below, I've plotted the same time series twice. In the first, I would like you to (qualitatively) draw the prediction bounds if this were modeled as an ARCH(1) process. In the second, I would like you to draw the prediction bounds if this were modeled as a GARCH(1,1) process with equal weights on each component of the process. (10 points)





4 for ARCH and 6 for GARCH. The key point is showing ARCH(1) has an immediate jump after the changes in residuals, but GARCH(1,1) has a gradual increase. Note: 2 points in GARCH accounts for the concave shape of bound.

Problem 5 - VAR

Prices, quantities, and wages are all linked through equilibrium conditions. Below is a 3 equation VAR for quantity, q_t, price, p_t, and wages, w_t.

$$\begin{aligned} q_t &= \beta_0 + \beta_1 q_{t-1} + \beta_2 p_{t-1} + \beta_3 p_t + \beta_4 w_t + u_t \\ p_t &= \alpha_0 + \alpha_1 p_{t-1} + \alpha_2 q_{t-1} + \alpha_3 q_t + \alpha_4 w_t + e_t \\ w_t &= \delta_0 + \delta_1 w_{t-1} + \delta_2 p_t + \delta_3 q_t + \epsilon_t \end{aligned}$$

Please derive the reduced-form VAR. (10 Points - you may leave matrix inverses in general terms: eg. B^{-1})

Answer

First, move all variables with t to the left-hand side.

$$\begin{aligned} q_t - \beta_3 p_t - \beta_4 w_t &= \beta_0 + \beta_1 q_{t-1} + \beta_2 p_{t-1} + u_t \\ -\alpha_3 q_t + p_t - \alpha_4 w_t &= \alpha_0 + \alpha_1 p_{t-1} + \alpha_2 q_{t-1} + e_t \\ -\delta_3 q_t - \delta_2 p_t + w_t &= \delta_0 + \delta_1 w_{t-1} + \epsilon_t \end{aligned}$$

Arrange in matrix form, being careful to include zeros where necessary

$$\begin{bmatrix} 1 & -\beta_3 & -\beta_4 \\ -\alpha_3 & 1 & -\alpha_4 \\ -\delta_3 & -\delta_2 & 1 \end{bmatrix} \begin{bmatrix} q_t \\ p_t \\ w_t \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \delta_0 \end{bmatrix} + \begin{bmatrix} \beta_1 & \beta_2 & 0 \\ \alpha_2 & \alpha_1 & 0 \\ 0 & 0 & \delta 1 \end{bmatrix} \begin{bmatrix} q_{t-1} \\ p_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ e_t \\ \epsilon_t \end{bmatrix}$$

Finally, multiplying by he inverse of the LHS matrix, we have the reduced form VAR:

$$\begin{bmatrix} q_t \\ p_t \\ w_t \end{bmatrix} = \begin{bmatrix} 1 & -\beta_3 & -\beta_4 \\ -\alpha_3 & 1 & -\alpha_4 \\ -\delta_3 & -\delta_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta_0 \\ \alpha_0 \\ \delta_0 \end{bmatrix} + \begin{bmatrix} 1 & -\beta_3 & -\beta_4 \\ -\alpha_3 & 1 & -\alpha_4 \\ -\delta_3 & -\delta_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 & \beta_2 & 0 \\ \alpha_2 & \alpha_1 & 0 \\ 0 & 0 & \delta 1 \end{bmatrix} \begin{bmatrix} q_{t-1} \\ p_{t-1} \\ w_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ e_t \\ \epsilon_t \end{bmatrix}$$

6 for writing the equations into correct matrix form and 4 for taking inverse (you don't need to calculate that).

Problem 6 - Granger Tests

For the question below, x is a time series data frame consisting of two variables for the US, real GDP (rgdp) and real capital stock (cap), for the period 1950-2012.

```
> granger.test(x,p=2)

              F-statistic  p-value
cap -> rgdp      2.22       0.118
rgdp -> cap     14.55       0.000
```

Please interpret the results in this table. Further, please write down the equations estimated for Granger test in the second row, and precisely indicate the hypothesis being tested. (20 points)

Answer

At the 10% level of significance, real GDP Granger cause the real capital stock. That is, information in the time series GDP is informative for real capital stock. In contrast, the opposite is not true (though close).

For the second row, we are running a Granger Causality test with two lags. Specifically, we first run the "unrestricted model":

$$cap_t = \beta_0 + \beta_1 cap_{t-1} + \beta_2 cap_{t-2} + \gamma_1 rgdp_{t-1} + \gamma_2 rgdp_{t-2}$$

Then, we run the restricted model by setting $\gamma_1 = \gamma_2 = 0$

$$cap_t = \beta_0 + \beta_1 cap_{t-1} + \beta_2 cap_{t-2}$$

Then, we run an F-test to see if we can reject the restrictions. The F-stat and p-value above suggest that if we reject the restrictions in favor of some combination of them being included in the model, then the probability of being wrong in doing so is extremely small (0.000).

6 for explaining the results (3 for each part). 8 for the equation (you only need to do the second one, but must correctly specify the independent variable to be cap) and 6 for hypothesis test. It is ok not to write in the formal way (unrestricted v.s. restricted, H_0 v.s. H_a) but you must specify the two parameters of interest and the joint test (F-test).

Problem 7 - Co-integrated Series

Suppose that you have a series of price and quantity, p_t and q_t, each of which have been determined to be I(1). Please detail the next steps you would take, including the code required for those steps, to determine whether it is sensible to evaluate a relationship between these variables, and how you would go about doing so if estimating the relationship were sensible. (20 points)

Answer

In the next step, we regress p_t on q_t,

$$p_t = a_0 + b_1 q_t + u_t$$

and test whether u_t is $I(0)$. The following code accomplishes this task

```
coint <- lm(p~q)
summary(coint)

beta<-coint$coef

resid <- p - (beta[1] + beta[2]*q)

adf.test(resid)
```

If the results of the Dickey-Fuller test suggest a non-stationary residual, then moving forward is not sensible. If the residual is stationary, meaning that it is I(0), then we can run an error-correction model to evaluate the short-run relationship and long-run adjustment process between p_t and q_t.

$$\Delta p_t = a_0 + b_1 \Delta q_t + \pi \widehat{u}_{t-1} + e_t$$

The code to run this ECM is the following:

```
dp<-p-lag(p.spy,1)
dq<-q-lag(q,1)

lag.resid<-resid-lag(resid,1)

ecm <-lm(dp~dq+lag.resid)

summary(ecm)
```

5 for the model and 5 for the (pseudo-) code for testing cointegration. Then, 5 for the model and 5 for the (pseudo-) code for ECM.

Problem 8 - Correcting my mistakes

Consider the following AR(1) model with drift.

$$y_t = \delta + \phi y_{t-1} + u_t$$

In my notes, I mistakenly labeled this as "non-stationary", where under certain conditions it is stationary in the long-run. Assuming $\phi \in (0,1)$ and $\delta > 0$, starting from an initial value of $y_0 = 0$, please derive the expected long-run value of y_t (10 points. Hint: iterate and take expectations. This might remind you of finance).

Answer

First, though this is not the answer to the question, if we assume a stationary process, we can find the expected value of the process quite easily. That is assuming $\mathbf{E}(y_t) = \mathbf{E}(y_{t-1})$, we have

$$\begin{aligned} \mathbf{E}(y_t) &= \delta + \phi \mathbf{E}(y_t) \\ \Rightarrow \mathbf{E}(y_t) &= \frac{\delta}{1-\phi} \end{aligned}$$

To derive that the process converges on this value, use the equation to iterate backward from t to 0.

$$\begin{aligned} y_t &= \delta + \phi y_{t-1} + u_t \\ &= \delta + \phi(\delta + \phi y_{t-2} + u_t) + u_t \\ &= \delta + \phi\delta + \phi^2 y_{t-2} + \phi u_{t-1} + u_t \\ &= \delta + \phi\delta + \phi^2(\delta + \phi y_{t-3} + u_{t-2}) + \phi u_{t-1} + u_t \\ &= \delta + \phi\delta + \phi^2\delta + \phi^3 y_{t-3} + \phi^2 u_{t-2} + \phi u_{t-1} + u_t \\ &\vdots \\ &= \sum_{k=1}^{t-1} \phi^{k-1}\delta + \phi^{t-1}\phi y_0 + \sum_{k=1}^{t-1} \phi^i u_t \delta \end{aligned}$$

Impose the initial value and take expectations:

$$y_t = \sum_{k=1}^{t-1} \phi^{k-1}\delta + \sum_{k=1}^{t-1} \phi^{k-1}\mathbf{E}[u_t]$$

Since $\mathbf{E}[u_t] = 0$

$$y_t = \sum_{k=1}^{t-1} \phi^{k-1}\delta$$

For the long run, take $t-1$ to infinity. This becomes an infinite series of δ, with past values discounted by ϕ. Precisely

$$y_t = \delta + \phi\delta + \phi^2\delta + \phi^3 + \cdots = \frac{\delta}{1-\phi}$$

10 points. You only get half if not using the iteration to show it.