1 Contractible Manifolds

(a) If \( f : X \to Y \) is homotopic to a constant map, show that \( I_2(f, Z) = 0 \) for all complementary dimensional closed \( Z \) in \( Y \), except if \( \text{dim}(X) = 0 \).

(b) Prove that intersection theory is vacuous in contractible manifolds: if \( Y \) is contractible and \( \text{dim} Y > 0 \), then \( I_2(f, Z) = 0 \) for every \( f : X \to Y \), \( X \) compact and \( Z \) closed, \( \text{dim} X + \text{dim} Z = \text{dim} Y \).

(c) Prove that no closed manifold (compact without boundary) - other than the 1-point space - is contractible.

2 Cobordism and Intersection Number

Prove that if \( X \) and \( Z \) are cobordant in \( Y \), then for every compact manifold \( C \) in \( Y \) with dimension complementary to \( X \) and \( Z \), the intersection number \( I_2(X, C) = I_2(Z, C) \).

3 Intersections in \( S^2 \times S^2 \)

In \( S^2 \times S^2 \), let \( X = S^2 \times \{a\} \) and \( Z = \{a\} \times S^2 \), where \( a \) is a given point in \( S^2 \). Are \( X \) and \( Z \) cobordant? Are their inclusion maps (with domain identified with \( S^2 \) in the obvious way) homotopic? Prove or disprove.

4 Ordered Bases

Let \( \beta = \{v_1, ..., v_k\} \) be an ordered basis for \( V \). Show that

(a) replacing one \( v_i \) by a multiple \( cv_i \) yields an equivalently oriented ordered basis if \( c > 0 \), and an oppositely oriented one if \( c < 0 \);

(b) transposing two elements (i.e. interchanging the places of \( v_i \) and \( v_j \), \( i \neq j \)) yields an oppositely oriented basis;

(c) subtracting from one \( v_i \) a linear combination of the others yields an equivalently oriented ordered basis.
5 Orientations on Half-Space

$H^k$ is oriented by the standard orientation of $\mathbb{R}^k$. Thus $\partial H^k$ acquires a boundary orientation. But $\partial H^k$ may be identified with $\mathbb{R}^{k-1}$. Show that the boundary orientation agrees with the standard orientation of $\mathbb{R}^{k-1}$ if and only if $k$ is even.

6 Derivations

Here we explain an important alternative perspective on tangent spaces. A vector $v \in \mathbb{R}^n$ determines a directional derivative $f \mapsto D_x f(v)$. The idea is that we can identify operators satisfying the Leibniz rule with tangent vectors by thinking of them as directions in which we can differentiate a function.

**Definition 6.1** The space of derivations a/t $x$, $\operatorname{Der}_x(\mathcal{O}_{M,x})$, is the $\mathbb{R}$-vector space of $\mathbb{R}$-linear maps $\delta : \mathcal{O}_{M,x} \to \mathbb{R}$ satisfying the Leibniz rule

$$\delta(fg) = (\delta f)g(x) + f(x)(\delta g)$$

Note also that this definition makes perfect sense even for modules over rings. It has the following concrete interpretation over $\mathbb{R}$; given $f$ a smooth function on $\mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, we have the derivation

$$\delta f = \sum_i v_i \frac{\partial f}{\partial x_i}(x) = \langle v, \nabla_x f \rangle.$$ 

6.1

Recall that we defined the cotangent space at $x \in M$ to be $m_x/m_x^2$. For $\delta$ a derivation, show that $\delta(m_x^2) = 0$, and thus $\delta$ descends to a linear map $\overline{\delta}$ on $T^*_x(M)$.

6.2

Show that for any linear map $T^*_x(M) \to \mathbb{R}$, the extension to $\mathcal{O}_{M,x}/m_x^2 \to \mathbb{R}$ given by $\delta(f) = \overline{\delta}(f - f(x))$ defines a derivation.

6.3

Conclude that the space of derivations is canonically isomorphic to $T_x M$ (defined as the space of arcs modulo the tangency relation) by constructing an explicit isomorphism.

Extra Problems

7 Sections of the Normal Bundle

Suppose that $Z$ is a compact submanifold of $Y$ with $\dim Z = \frac{1}{2} \dim Y$. Prove that if $Z$ is globally definable by independent functions, then $I_2(Z, Z) = 0$. 

2
8 Neighborhoods of $\mathbb{C}P^1$

Show there is no neighborhood of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ where $\mathbb{C}P^1$ is globally cut out by independent functions. What do you conclude about the normal bundle?