1 \( S^2 \cong \mathbb{C}P^1 \)

Construct a diffeomorphism from \( S^2 \) to \( \mathbb{C}P^1 \).

2 Inclusions

(i) Show that the standard embedding \( i : \mathbb{R}^k \to \mathbb{R}^{k+1} \) given by \( i(x_0, ..., x_{k-1}) = (x_0, ..., x_{k-1}, 0) \) descends to a well-defined function (of sets) \( \iota : \mathbb{R}P^{k-1} \to \mathbb{R}P^k \). Show that \( \mathbb{R}P^k - \iota(\mathbb{R}P^{k-1}) \) is diffeomorphic to \( \mathbb{R}^k \).

(ii) Prove that \( \iota \) is a smooth embedding.

3 Construction of Projective Space

(i) Prove that the canonical map \( \pi : \mathbb{R}^{k+1} - \{0\} \to \mathbb{R}P^k \) is a submersion. Find a basis for \( \text{Ker}(d\pi_{x_0, ..., x_k}) \) and interpret the answer geometrically.

(ii) Prove that \( \pi|_{S^k} : S^k \to \mathbb{R}P^k \) is a local diffeomorphism.

(iii) Given that \( \pi : \mathbb{C}^{k+1} - \{0\} \to \mathbb{C}P^k \) is a submersion, prove that \( \pi|_{S^{2k+1}} : S^{2k+1} \to \mathbb{C}P^k \) is a submersion.

4 Projective Varieties

A polynomial \( p : \mathbb{R}^{k+1} \to \mathbb{R}^k \) is called homogenous if there is a positive integer \( d \) such that for all \( x \in \mathbb{R}^{k+1} \) and \( \lambda \in \mathbb{R} \) we have \( p(\lambda x) = \lambda^d p(x) \).

(i) For \( p \) homogenous, show that the equation \( p(x) = 0 \) specifies a well-defined subset of \( \mathbb{R}P^k \).

(ii) For fixed \( a \in [0, 1] \), let \( V_a \subset \mathbb{R}P^2 \) be the variety determined by the homogenous polynomial \( p_a(x, y, z) = x^2 + ay^2 - z^2 \). Draw \( V_1 \) in each of the three standard local charts on \( \mathbb{R}P^2 \). Show that \( V_1 \) is a manifold. To what familiar manifold is it diffeomorphic?

(iii) What happens to \( V_a \) as \( a \) decreases from 1 to 0? (Draw this in the local charts.) Identify \( V_0 \) as an immersed manifold.
5 Solids in $\mathbb{R}^3$

(a) Show that the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ is a manifold with boundary ($a > 0$).

(b) For which values of $a$ is the intersection of the solid hyperboloid $x^2 + y^2 - z^2 \leq a$ and the unit sphere $x^2 + y^2 + z^2 = 1$ is a manifold with boundary? What does it look like?

6 Half-Space

Suppose that $X$ is a manifold with boundary and $x \in \partial X$. Let $\phi : U \to X$ be a local parametrization with $\phi(0) = x$, where $U$ is an open subset of $H^k$. Then $d\phi_0 : \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the upper half space $H_x(X)$ in $T_x(X)$ to be the image of $H^k$ under $d\phi_0$, $H_x(X) = d\phi_0(H^k)$. Prove that $H_x(X)$ does not depend on the choice of the local parametrization.