1 Surfaces

Compute the de Rham cohomology of $\Sigma_g$ the closed surface of genus $g$.

2 Spheres

Compute the de Rham cohomology of the sphere $S^n$.

3 Projective Spaces

Compute the de Rham cohomology of complex projective space $\mathbb{CP}^n$. (Hint: use the open subsets $U = \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ where $\mathbb{CP}^{n-1} = \{[0; z_1; \ldots; z_n]\}$, and $V = \mathbb{CP}^n \setminus \{[1; 0; \ldots; 0]\}$. You will need to use the result of the previous problem.)

4 Leray-Hirsch

Show that the map $S^3 \to S^2 = \mathbb{CP}^1$ taking a unit vector in $\mathbb{C}^2$ to its equivalence class in $\mathbb{CP}^1$ is a fiber bundle with fiber $S^1$. This is called the Hopf fibration. Verify that both the hypothesis and the conclusion of the Leray-Hirsch theorem are false.

5 Cup Product

Show that the cohomology ring of the torus $T^n$ is the exterior algebra on $n$ generators.

6 Poincaré Duality

On 4-manifolds, the Poincaré duality pairing gives an isomorphism between $H^2$ and $H_2$ (equivalently, a non-degenerate way pairing on elements of $H_2$). Calculate this pairing as a matrix for suitable bases of $H^*(\mathbb{CP}^2)$ and $H^*(S^2 \times S^2)$. 