1 Derivations

Here we explain an important alternative perspective on tangent spaces. A vector \( v \in \mathbb{R}^n \) determines a directional derivative \( f \mapsto D_x f(v) \). The idea is that we can identify operators satisfying the Leibniz rule with tangent vectors by thinking of them as directions in which we can differentiate a function.

**Definition 1.1** The space of derivations a/t \( x \), \( \text{Der}_x (\mathcal{O}_{M,x}) \), is the \( \mathbb{R} \)-vector space of \( \mathbb{R} \)-linear maps \( \delta : \mathcal{O}_{M,x} \to \mathbb{R} \) satisfying the Leibniz rule

\[
\delta(fg) = (\delta f)g(x) + f(x)(\delta g)
\]

Note also that this definition makes perfect sense even for modules over rings. It has the following concrete interpretation over \( \mathbb{R} \); given \( f \) a smooth function on \( \mathbb{R}^n \) and a vector \( v \in \mathbb{R}^n \), we have the derivation

\[
\delta f = \sum_i v_i \frac{\partial f}{\partial x_i}(x) = \langle v, \nabla_x f \rangle.
\]

1.1

Recall that we defined the cotangent space at \( x \in M \) to be \( \mathfrak{m}_x/\mathfrak{m}_x^2 \). For \( \delta \) a derivation, show that \( \delta(\mathfrak{m}_x^2) = 0 \), and thus \( \delta \) descends to a linear map \( \overline{\delta} \) on \( T^*_x(M) \).

1.2

Show that for any linear map \( T^*_x(M) \to \mathbb{R} \), the extension to \( \mathcal{O}_{M,x}/\mathfrak{m}_x^2 \to \mathbb{R} \) given by \( \delta(f) = \overline{\delta}(f - f(x)) \) defines a derivation.

1.3

Conclude that the space of derivations is canonically isomorphic to \( T_x M \) (defined as the space of arcs modulo the tangency relation) by constructing an explicit isomorphism.
2 New Bundles from Old

2.1
For a complex vector bundle $E$ over $X$, construct its conjugate bundle $E$ concretely: given a description of $E$ by transition functions, give a description of $\overline{E}$ by transition functions.

2.2
Do the same for its (complex, not Hermitian) dual bundle $E^*$; construct this concretely given a description of $E$ by transition functions.

2.3
A slight extension of this discussion allows us to define the direct sum $E \oplus F$ and tensor product $E \otimes F$ of two vector bundles. Construct them; how are the transition functions of $E \oplus F$ and $E \otimes F$ related to those of $E$ and $F$?

2.4
In all of the preceding we used $C^\infty$ complex vector bundles. We can similarly consider finite-dimensional holomorphic vector bundles over a complex manifold $X$. If $E$ and $F$ are holomorphic, which of $E^*, \overline{E}, E \oplus F$ carry natural holomorphic structures?

3 Trivializations
Show that a rank $N$ vector bundle is trivial if and only if it has at $N$ linearly independent nonvanishing sections.

4 Pullback Bundles
Construct the pullback vector bundle using transition functions. What is the pullback of the infinite M"{o}bius band (thought of as a bundle over $\mathbb{R}P^1$) via the covering map $S^1 \to \mathbb{R}P^1$, where $S^1 \subset \mathbb{R}^2$? How do you know?

5 Submanifolds and Regular Values
Deduce from the last exercise, and earlier results, that if $f : M \to N$ is a smooth map of manifolds such that $D_x f$ is always surjective, then the sub-bundle $\ker df : \{(x, v) \in TM \mid df(v) = 0\}$ is a vector bundle over $M$. Now say $f : M \to N$ is smooth, and $y \in N$ is a regular value. Verify that $M_y := f^{-1}(y)$ is a manifold and that

$$T_y N = (\ker Df)|_{M_y}.$$ 

Show also that $T^*_y(M_y) = T^*_y(M)/(Df^*(T_y(N))$. 
6 Quotient Bundles

Let $F \rightarrow M$ be a vector subbundle of a vector bundle $E \rightarrow M$. Prove that the quotient bundle

$$E/F := \sqcup_{q \in M} E_q/F_q$$

is a vector bundle over $M$ as well. Hints: Show first that given any vector $v$ in an $n$-dimensional fiber $E_q$ of $E \rightarrow M$ there is a local section $\tilde{v}$ defined on some neighborhood $W$ of $q$ with $\tilde{v}(q) = v$. Next suppose $s_1, ..., s_k : U \rightarrow F$ is a basis for each fiber of $F$ over some open subset $U \subset M$. Fix $q \in U$ and extend $\{s_i\}$ to a basis $\{e_j\}$ of $E_q$, then extend this basis to a local basis $\{s_i\}_{i=1}^n$ of fibers of $E$ over some subset $W \subset U$ so that the first $k$ local sections are the original local basis of $F$. Use the last $n - k$ sections to produce a local frame of $E/F \rightarrow M$.

7 Partitions of Unity

Given $\{U_\alpha\}$ an open cover of a paracompact space, show that there exists a countable open cover $\{V_i\}$ and a partition of unity $\{\rho_i\}$ such that each $V_i$ is contained in some $U_\alpha$ and $\text{supp}\rho_i \subset V_i$. 