Orientations, Integration, Stokes' Thm

Last time: needed coherent choice of sign for det (change of coordinates). This is given by an orientation.

Def A diffeom. \( \phi: U \to V \) of open sets in \( \mathbb{R}^n \) is oriented-preserving if \( \det(d\phi) > 0 \) \( \forall x \in U \). An atlas on a space \( M \) is oriented if the charts always map to the same linear space when domains overlap, and transition forms preserve orientation. Two oriented atlases are equivalent if their union is an oriented atlas.

Def An orientation on a smooth manifold \( M \) is an equivalence class of oriented smooth atlas.

This can be unwieldy to work with; we can use language of exterior algebra to define

Def A volume form for an \( n \)-manifold \( M \) is a top form \( \omega \in \Lambda^n(M) \) such that for each \( x \in M \), \( \omega_x \) is nonzero in \( \det T^*x \).

Ex: In \( \mathbb{R}^n \), \( f(x)dx_1 \wedge \cdots \wedge dx_n \) is a volume form if \( f \neq 0 \) smooth.
Prop: A volume form $\omega$ determines an orientation $O(\omega)$ for $\mathbf{x}$. Conversely, every orientation arises from a volume form.

\( \Rightarrow \): Suppose $\omega$ is a volume form, and let $\phi: \mathbb{R}^n \rightarrow U \subseteq M$ be a chart. Note $\phi^* \omega = f(x) \, dx_1 \wedge \cdots \wedge dx_n$ for some $f \neq 0$ on $\mathbb{R}^n$. If $f < 0$, replace $\phi$ by composition with the map $(x_1, \ldots, x_n) \mapsto (x_2, x_1, \ldots, x_n)$. Then $\phi$ is an oriented chart. Repeat for all charts to get an oriented atlas.

\( \Leftarrow \): If $M$ oriented, take p.d.u. subordinate to atlas $\{ U_i, \phi_i \}$ and let $w_i = \phi_i^*(dx_1 \wedge \cdots \wedge dx_n)$. Transitions have $\det(d\phi_i) > 0 \equiv \phi_j^* \left( \phi_i^*(dx_1 \wedge \cdots \wedge dx_n) \right)$ still positive, so $\rho_i w_i$ always positive.

Ex: Consider $\mathbb{RP}^n$ for $n$ even, thought of as the quotient of $S^n$ by the antipodal map.

\[
\begin{array}{ccc}
\mathbb{R}^{n+1} & \xrightarrow{\pi} & \mathbb{RP}^n \\
(x_1, \ldots, x_n, 1) & \mapsto & [x_1 : \ldots : x_n] \mapsto \frac{1}{\sqrt{n+1}} x
\end{array}
\]

$\pi(p) = \frac{1}{\sqrt{n+1}} p$.

If $\mathbb{RP}^n$ were orientable, suppose we oriented it so $\pi$ preserves orientation. Then if $\tau(x)$ is reflection through the origin, $\pi \circ \tau(x) = \pi(x)$. But $\tau$ is orientation-reversing on $\mathbb{R}^{n+1}$! So $\mathbb{RP}^n$ not orientable for $n$ even.
Once we have this, and $S$ is compact (or we work with compactly supported forms) then we may define the integral of $\Theta$ on $S$ as follows. Cover $S$ by coord. $\{\Phi_i, \psi_i\}$, subordinate to $1\rho_j^3$ partition of unity, for which the charts $\psi_i$ preserve orientation. Then $\Theta = \Sigma \rho_i \Theta_i$ and for $i$ we have

$$\int_S \rho_i \Theta_i = \int_{R^d} \rho_i \psi_i^*(\Theta) = \int_{R^d} \rho_i f(y) \, dx_1, \ldots, dx_d$$

Different choices of coordinates give the same integral, and different choices of partition of unity do not change the answer; if $\tilde{\Sigma} \rho_j^3$ is another,

$$\Sigma \int_S \rho_j \Theta = \Sigma \tilde{\Sigma} \int_S \rho_j \rho_i \Theta$$

Finite sum in any supp $\rho_i$

$$= \Sigma \int_S \rho_i \Theta$$

Ex: Volume $V$ of $B^+ \subseteq R^4$, radius $R$. 

\[
\begin{align*}
0 \leq \varphi &\leq \pi \\
0 \leq \theta &\leq 2\pi \\
0 \leq \sigma &\leq R
\end{align*}
\]
Use spherical coord. \((\sigma, \Psi, \phi, \Theta)\), \(\sigma\) = distance to \(O\), \(\Psi\) is angle to \(t\)-axis in Euclidean \((x,y,z,t)\). Then \[ t = \sigma \cos \Psi \]
\[ \rho = \sigma \sin \Psi, \rho = \text{distance of } (x,y,z) \text{ to } O \in \mathbb{R}^3 \]
With \(\phi, \Theta\) remaining spherical coord.,
\[ x = \rho \sin \phi \cos \Theta = \sigma \sin \Psi \sin \phi \cos \Theta \]
\[ y = \rho \sin \phi \sin \Theta = \sigma \sin \Psi \sin \phi \sin \Theta \]
\[ z = \rho \cos \phi = \sigma \sin \Psi \cos \phi \]

Exercise:
\[ \text{d}x \wedge \text{d}y \wedge \text{d}z \wedge \text{d}t = \sigma^3 \sin \Psi \sin \phi \, d\sigma \wedge d\Psi \wedge d\phi \]
So \[ \int_{S_R^4} \text{d}x \wedge \text{d}y \wedge \text{d}z \wedge \text{d}t = \int_0^R \sigma^3 \sin \Psi \sin \phi \, d\sigma \int_0^\pi \sin \Psi \sin \phi \, d\phi \int_0^{2\pi} \sin \phi \, d\Theta = \frac{1}{4} R^4 (\frac{\pi}{2}) 2\pi \cdot 2 \]
\[ = \frac{1}{4} \pi^2 R^4 \]

**Stokes' Theorem**

FTC says we can compute integrals by evaluating antiderivative on \(\partial\) ("integrate" over \(\partial\), \(\nu\) point measured). This is true in great generality.

Theorem: Let \(M\) be an \(n\)-manifold \(M\) boundary, and \(\omega\) a volume form on \(M\) inducing an orientation on \(\partial M\) (volume form) by \(\omega_{\partial M} = \nu_{\partial M} \omega\), where \(\nu\) is outward normal. Then for \(\alpha \in \Omega^n_c(M)\), compactly supported form on \(M\),
$S_m \, d\alpha = S_m \, d\Omega$

pf: Choose an atlas $\{ U_i, \phi_i \}$ where each $\phi_i$ has compact closure in $M$, and a partition of unity subordinate to this cover. As such, each $U_i$ is a manifold (possibly $0/\emptyset$, if $U_i \cap \emptyset \neq 0$).

Now $\alpha = \sum \phi_i \alpha$ where $\phi_i$ is partition of unity and the integral and $d$ are $R$-linear, so

$S_m \, d\alpha = \sum S_{U_i} \, d(\phi_i \alpha)$

$S_m \, d\Omega = \sum S_{U_i} \, \phi_i \, d\Omega$

thus it suffices to prove for compactly supported forms in $U_i$.

So, assume $\alpha$ is a compactly supported form in $R_0 \times R^m$ and WLOG $\alpha$ supported in $[0,1) \times (0,1)^m$. Then $\alpha$ looks like

$\alpha = \sum \alpha_i \, dx_1 \wedge \ldots \wedge dx_i \ldots \wedge dx_n$

where $\wedge$ indicates an omitted term. So then

$d\alpha = (-1)^i \frac{\partial \alpha_i}{\partial x_i} \, dx_1 \wedge \ldots \wedge dx_n$

By Fubini's theorem in $R^n$, we can switch the order of integration

$S_{[0,1]^m} \, d(\alpha_i) \, dx_1 \wedge \ldots \wedge dx_n = S_{[0,1]^m} \left( \int_0^1 \frac{\partial \alpha_i}{\partial x_i} \, dx_i \right) \, dx_1 \wedge \ldots \wedge dx_n$
Now by FTC and the compact support assumption,
\[ \int_0^1 \frac{\partial x_i}{\partial x_i}(x_1, \ldots, x_n) \, dx_i = \begin{cases} 0 & \text{if } i > 1 \\ -\alpha_1(0, x_2, \ldots, x_n) & \text{if } i = 1 \end{cases} \]

Thus
\[ \int_0^1 \frac{d}{dx_i} \left( x_1, \ldots, x_n \right) \, dx_1 \Lambda \cdots \Lambda dx_n = -\int_0^1 x_1, \ldots, x_n \, dx_1 \Lambda \cdots \Lambda dx_n \]

OTOH with our orientation convention, \( S_{03} \times \mathbb{R}^m \) is oriented by
\[ (-1)^m \left( dx_1 \Lambda \cdots \Lambda dx_n \right) = -dx_2 \Lambda \cdots \Lambda dx_n \]
so
\[ \int_0^1 \frac{d}{dx_i} \left( x_1, \ldots, x_n \right) \, dx_1 \Lambda \cdots \Lambda dx_n = -\int_0^1 x_1, \ldots, x_n \, dx_1 \Lambda \cdots \Lambda dx_n \]

which coincides with \( \ast \) \[ \int_0^1 \frac{d}{dx_i} \left( x_1, \ldots, x_n \right) \, dx_1 \Lambda \cdots \Lambda dx_n \]

As a corollary, suppose some \( \alpha \in \Omega^k(M^n) \) satisfies \( \alpha = d\beta \) for \( \beta \in \Omega^{k-1}(M) \). If \( M^n \) closed,
\[ \int_M \alpha = \int_{\partial M} \beta = 0 \]

Similarly, if \( M \) has boundary,
\[ \int_M \alpha = \int_{\partial M} \beta \]

**Ex:** Compute \( \text{Vol}(S^3) = \int_{S^3} \text{Vol}(B^4) \)
\[ \text{Vol}(B^4) = d\left( \frac{2}{3} x_1 \Lambda \cdots \Lambda x_n \right) \]
So \( \int_{S^3} \text{Vol}(B^4) = \int_{B^4} 4 \cdot dx_1 \Lambda dx_2 \Lambda dx_3 \Lambda dx_4 \)
\[ = 2\pi^2 \text{ using formula for volume of } B^4. \]
The Moser Trick

Given 2 volume forms, only connected component of $\mathcal{S}^\text{vol}(M)$ matters for orientation. In loc. coord. if 2 volume forms $\sigma, \tau$ induce the same orientation,

$$\sigma = g(x) \, dx_1 \wedge \ldots \wedge dx_n, \quad \tau = f \sigma$$

for $f, g > 0$

Also, note that by change of coord. if $\phi: M \to M$ and $\phi \in \text{Diff}^+(M)$,

$$\int_M \phi^* (\sigma) = \int_M \sigma$$

So if $\sigma = \phi^* (\tau)$, then total volume is identical.

Thm (Moser) If $\{ \sigma_t \}$ are a family of volume elements on a compact, connected manifold (smooth map $[0,1] \to \mathcal{S}^\text{vol}(M) \setminus \{0\}$) and $\int_M \sigma_0 = \int_M \sigma_t$ for all $t$, then there exists a diffeomorphism $\phi_t$ such that $\phi_t^* \sigma_t = \sigma_0$ and $\phi_0 = \text{Id}$.

Thus if $\sigma, \tau$ give same total volume, then $\exists \phi \in \text{Diff}^+(M)$ pulling back one to another (family is $t \sigma + (1-t) \tau$).

pE: We want to find a vector field $V_t$ so that its flow carries $\sigma_0$ to $\sigma_t$. To do this we differentiate the req'd condition w.r.t. $t$. 
\[
\frac{d}{dt}\left( \phi^* (\sigma_t) \right) = \phi^* \left( \frac{d}{dt} \nu_t (\sigma_t) + \frac{d}{dt} \sigma_t \right)
= \phi^* \left( \nu_t (d\sigma_t) + d\nu_t (\sigma_t) + \frac{d}{dt} \sigma_t \right)
= 0 = \frac{d}{dt} \sigma_0
\]

But \(\sigma\) is top-dimensional form; \(d\sigma = 0\). Note that \(\frac{d}{dt}\) hits the family \(\{\sigma_t\}\) as well so we end up with
\[
0 = \phi^* \left( d\nu_t (\sigma_t) + \frac{d}{dt} \sigma_t \right)
\]

Now for a big theorem:

\textbf{Thm (Hodge)} One can split subspace of closed \(k\)-forms
\[
\mathcal{Z}^k(M) = \bigoplus_{\alpha \in \mathcal{Z}^k(M)} d\alpha = 0
\]

into summands \(\mathcal{Z}^k(M) = \mathcal{H}^k(M) \oplus d(\mathcal{S}^{k-1}(M))\) such that \(\mathcal{H}^k\) is finite-dimil ("Harmonic forms").

Now since \(\frac{d}{dt}\int_M \sigma_t = 0\), writing \(\sigma_t = h_t + d\alpha_t\) for \(h_t\) harmonic and \(\alpha_t \in \mathcal{S}^{k-1}(M)\), we have \(\frac{d}{dt}(h_t) = 0\). Thus we can rewrite \(\frac{d}{dt}(\sigma_t) = d\alpha_t\) for some \(\alpha_t \in \mathcal{S}^{k-1}(M)\).

\[
0 = \phi^* \left( d\nu_t (\sigma_t) + d\alpha_t \right)
\]

so it suffices to find \(\nu_t\) so \(\nu_t (\sigma_t) = \alpha_t\).

Recall that \(\sigma_t\) is a volume form for \(M\), so at every \(p \in M\), \((\sigma_t)_p\) is nondegenerate (induces isom \(\Lambda^m(T^*_p M) \cong T_p^* M\))
Algebraically this means we can recover \( V_t \) from \( \phi_t \).

**Ex:** (\( \mathbb{R}^2 \), \( dx \wedge dy \)) Volume (area) form, \( \omega = f_x dx + f_y dy \). To recover \( V = (V_x, V_y) \), we need

\[
\begin{align*}
 f_x dx + f_y dy &= (V (dx \wedge dy)) \\
 &= dx(V) dy - dy(V) dx
\end{align*}
\]

So \( V_x = f_y \), \( V_y = -f_x \) and \( V = f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y} \).

Thus we get \( V_t \) on \( M \), which by compactness has globally defined flow \( \phi_t \) which satisfies desired property.

**Cor** Suppose two surfaces with area forms \( (\Sigma_1, d\sigma_1) \) and \( (\Sigma_2, d\sigma_2) \) have the same genus and total area. Then there is an area-preserving diffeom. \( \phi: \Sigma_1 \to \Sigma_2 \), \( \phi^*(d\sigma_2) = d\sigma_1 \).

Can also extend to compact manifolds with boundary using "doubling" trick:

\[
\begin{align*}
M \cup_{\partial M} \tilde{M} \cup_{\partial M} M
\end{align*}
\]

Get closed mFlod on which Moser applies. Versions for noncompact mFolds with prescribed behavior on ends also exist.
Another corollary: let $\text{Diff}(M)$ be diffeom. of $M$ smooth, $\text{Diff}(M, \omega)$ volume-preserving diffeom. of $M$. Then $\text{Diff}(M)$ deformation retracts onto $\text{Diff}(M, \omega)$.

Also possible to prove "relative" versions; given 2 families of volume forms indexed by compact manifold $(\omega_0, \omega)$ & fiber-preserving diffeom. relating them iff each fiber has same total volume. Can extend to non-compact if "ends" have same volume (Greene-Shiohama).