Def. Let $M$ be closed (compact, no $\partial$). The Euler characteristic $\chi(M)$ is $I(\Delta, \Delta)$, $\Delta$ diagonal in $M \times M$.

Note $\chi(M)$ is diffeomorphism invariant; orientation on $M$ orients $M \times M$ and $\Delta$ canonically, and if $\phi: M \to M$ diffeo changes orientation on $M$, get same relative orientations on $\Delta$, $M \times M$ (that is, $I(\Delta, \Delta)$ same).

Euler characteristic is our first diffeomorphism (really, homeomorphism) inv't of manifolds.

Proof. If $\dim X$ odd, $\chi(M) = 0$.

pf: $I(\Delta_1, \Delta_2) = \sum_{i=0}^{\dim X} (-1)^i I(\Delta_2, \Delta_i) = -I(\Delta_1, \Delta_2) \in \mathbb{Z}$

so $I(\Delta_1, \Delta_2) = 0$. \( \Box \)

Ex: $\chi(S^2) = I(\Delta, \Delta)$ where $\Delta \subset S^2 \times S^2$.
Use stereographic projection to get chart $\Psi: \mathbb{R}^4 \to S^2 \times S^2$, and $\Psi^{-1}(pt \times S^2) = \mathbb{R}^2 \times S^0$, $\Psi^{-1}(S^2 \times pt) = S^3 \times \mathbb{R}^2$.
These intersect transversely at $0$, disjoint otherwise. Same argument at $\infty$ gives another intersection, so $\chi(S^2)$. 

Lefschetz Number, Euler Characteristic
Applications of intersection number.
OTOH $X(t^2) = 0$, as we can define homotopy of $\Delta$ as $f_t(x_1, x_2, y_1, y_2) = (x_1, x_2, y_1 + \frac{t}{2}, y_2 + \frac{t}{2})$. $f_0 \cap f_1 = \emptyset$ so $X(t^2) = 0$. 

**Lefschetz Number**

Re-interpret $X(M) = I(\Delta, \Delta)$ and $\Delta = \text{Graph}(\text{Id})$ in $M \times M$. Let $\Gamma(f)$ be the graph of $f:M \to M$ in $M \times M$.

**Def** Given $f:M \to M$ smooth, the **Lefschetz number** of $f$ is $I(\Gamma(f), \Delta)$.

Thus $L(\text{Id}) = X(M)$, and note $(x,x) \in \Gamma(f) \cap \Delta$ iff $f(x) = x$, fixed point of $f$.

- If $L(f) \neq 0$, then $f$ has a fixed point.
- If $X(M) \neq 0$ then every $f \geq \text{Id}$ has a fixed point.

From now on we assume $M$ compact w/o boundary.
Def We say \( f : M \rightarrow M \) is Lefschetz if \( \Gamma(f, \Delta) \neq \emptyset \).

Then \( f \) has finitely many isolated fixed points.

Prop Every \( f : M \rightarrow M \) smooth is homotopic to a Lefschetz map.

pf: Let \( F : M \times \mathbb{B}^n \rightarrow M, M \subseteq \mathbb{R}^n \) so \( \forall x, F_x : \mathbb{B}^n \rightarrow M \) is a submersion for almost all \( v \in \mathbb{B}^n \). Take \( G : M \times \mathbb{B}^n \rightarrow M \times M, G(x, v) = (x, F(x, v)) \). Then \( G \) maps for almost every \( v \), by Trans. Htpy, so \( \exists v \in \mathbb{B}^n \) with \( G_v(v) = (x, F_s(x)) \) to \( \Delta \) and homotopic to \( f \).

Hence for \( f \) Lefschetz, \( (x, x) \in \Delta \cap \Gamma(f) \),
\[
T(x, x) \Delta \oplus T(x, x) \Gamma(f) = T_x M \times T_x M
\]
\[d f_x(T_x M)\]

which in turn holds if \( d f_x \) has no nonzero fixed point, i.e. \( \lambda \) not an eigenvalue of \( d f_x \).

Def A fixed point \( x \) of \( f : M \rightarrow M \) is Lefschetz if \( d f_x - \text{Id} \) is an isomorphism.

Thus \( f \) Lefschetz \( \Leftrightarrow \) all fixed points are Lefschetz.

Def The local Lefschetz number \( L_v(f) \) for
$x \in \text{Fix}(f)$ is the orientation number of $(x \times) \Delta \Pi \Psi(y)$.

So if $f$ Lefschetz,

$$L(f) = \prod_{x \in \text{Fix}(f)} \lambda(f) = \prod_{x \in \text{Fix}(f)} L_x(f)$$

Prop. If $x \in \text{Fix}(f)$ Lefschetz, $L_x(f) = \text{sign} (\det (df_x - \text{Id}))$.

pf. Let $(V_1, \ldots, V_k)$ be pos. oriented basis for $T_x \mathbb{M}$. Then

$$L_x(f) = \text{sign} \left( \langle V_1, V_1 \rangle, \ldots, \langle V_k, V_k \rangle, \langle V_1, df_x(V_1) \rangle, \ldots, \langle V_k, df_x(V_k) \rangle \right) + \text{for } \Delta$$

and we can subtract vectors, preserving orientation:

$$= \text{sign} \left( \langle V_1, V_1 \rangle, \ldots, \langle V_k, V_k \rangle, (0 \cdot df_x - \text{Id})(V_1), \ldots, (0 \cdot df_x - \text{Id})(V_k) \right) + \text{for } \Delta$$

Now $\text{Id} \cdot (df_x - \text{Id}) \cdot (V_i)$ still basis of $T_x \mathbb{M}$ iff $1$ not eigenvalue of $df_x$, so altogether still basis of $T_x \mathbb{M} \times T_x \mathbb{M}$.

Since first $k$ are positive for $T_x \mathbb{M}$, by product orientation this reduces to

$$= \text{sign} \left( (df_x - \text{I})(V_1), \ldots, (df_x - \text{I})(V_k) \right)$$
Ex: (ℝ^k) Suppose \( f: \mathbb{R}^k \rightarrow \mathbb{R}^k \) has 0 ∈ Fix(\( f \)). Then if \( df \) symmetric, \( f \) basis of \( \mathbb{R}^k \) so

\[
d d f _0 = \begin{bmatrix} \alpha _1 & \cdots & \cdots \alpha _k \end{bmatrix}
\]

Suppose \( \alpha _i > 1 \) \( \forall i \). Then \( \det (df - I) > 0 \) so \( L_0(\( f \)) = 1 \) and to first order, map looks like

(Expanding)

If exactly one \( \alpha _i < 1 \) then \( L_0(\( f \)) = -1 \) and we see

If all \( \alpha _i \in (-\infty , 1) \) then \( L_0(\( f \)) = (-1)^k \)
Vector Fields, ODEs and Flow

Local Lefschetz is $\mathbb{Z}_2$-valued. Can promote to $\mathbb{Z}$-valued invariant of any isolated fixed point.

Observation: find nbhd of fixed point so $f: X \to X$ sends this to (some subset of) itself. Shrinking so it lies in a chart, $f: \mathbb{R}^n \to \mathbb{R}^n$ determines a vector field by $x \mapsto f(x)$. Note if $f(x_0) = x_0$ then vector field at $x_0$ is 0.

This amounts to differentiating $f$ at fixed point. In fact, these give rise to deformation of smooth mfd's.

Def A time-dependent vector field $V$ is an open interval $I$ and a smooth map $V: I \times M \to TM$ s.t. $V(k,x) \in T_x M$.

For $V$ a time-dependent vector field, an integral curve of $V$ is a map $\gamma: J \to M$ such that $\gamma_x(t) = V(k,x), J \leq I$.

Ex: In $\mathbb{R}^n$, $V(k,x) = \sum V_i(k,x) \cdot x(e_i)$. Then $\gamma$ is an integral curve iff it solves the ODE system $\sum \dot{\gamma}_x^i(t) = V_i(k,x)$ for $i=1, \ldots, n$. Solvability of this system follows from

Thm (Picard) For any time-dependent vector field $V$ on $M$, and $(t_0, x_0) \in I \times M$, there is at most 1 integral curve $\gamma: J \to M$ with $t_0 \in J$ and $\gamma(t_0) = x_0$. Moreover, $J \cup x_0$ open in $M$. 
such that for any $y_0 \in U$, there is an open interval $J$ and an integral curve $\gamma: J \rightarrow M$ with $\gamma(t_0) = y_0$. The function $F: J \times U \rightarrow M$, $F(t, y) = \gamma_y(t)$ is smooth.

One generally sees proofs on $\mathbb{R}^n$ but this generalizes readily to manifolds.

Thus we have solvability for small intervals, but long-time existence may fail. Consider $\mathbb{R}$ as smooth manifold, $V = (e^t)^2$, so that $V$ smooth, time independent. Then flow is $\gamma(t) = \tan(t)$ which blows up in finite time.

Note however that we may normalize the vector field, so that integral curves are parametrized by arclength: $V = \frac{1}{2} \frac{d}{dt}$ is complete on $\mathbb{R}$ (its flow exists for all time).

More generally, compactness guarantees long-time existence.

The (Long-Time Existence) If $V$ is a compactly supported vector field, i.e. $V(t, x) = 0$ outside some compact subset of $M$, then for any $(t_0, x_0) \in I \times M$, there is an integral curve $\gamma$ st. $\gamma(t) = x_0$.

Anywhere on $M$, pull back $V$ w/ chart to $\mathbb{R}^n$, find integral curve. Find some small time so that $\gamma(x, t) \in \mathbb{R}^n$. 
$\Gamma_x(t) \neq \Gamma_x(t')$ if $x \neq x'$ by local uniqueness.

The map $x \mapsto \Gamma_x(t)$ is called the time-$t$ flow, $\Phi_t$, of $V$. By defn. $\frac{d}{dt}(\Phi_t) = V$. By above, $\Phi_t$ smooth (since $V$ smooth) and bijective, with smooth inverse $\Phi_{-t}$. Note $\Phi_0 = \text{Id}$.

Vector fields on manifolds can thus be thought of as deformations, by considering their flow. Moreover, can arrange for nice geometric properties by requiring that $V$ have certain linear algebraic conditions:

$\exists V \in Tz, \exists y$ submfd $\exists \Rightarrow \exists \Phi_v$ flow preserves $\exists$

$\exists V \in Tz \Rightarrow \exists \Phi_v$ pushes $V$ off itself, when $\neq 0$

Also, vector fields are easy to work with; if $\rho$ is cutoff function supported in $U$ open, $\rho V$ only nonzero in $U$, so $\Phi_{\rho V}$ flow is $\text{Id}$ outside $U$!

Key thm: if $V$ is nonzero at $x \in X$, then $3$ local coordinates about $x$ where $V$ is constant. ("Straightening")

Near zeroes of $V$, question is more subtle (e.g. normal forms near stationary points). Reduces to linear algebra (linearize $V$ to get matrix). E.g., if all real eigenvalues, called Hartman-Grobman Linearization.
Back to Lefschetz number: wanted more refined method for computing $X(M)$ Euler characteristic.

Def Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth, $x$ a fixed point. If $B \ni x$ is closed ball that does not contain any other fixed point, there is a map $F : \partial B \to S^{n-1}$

$$z \mapsto \frac{f(z) - z}{\|f(z) - z\|}$$

defined if no other fixed points. Its (oriented) degree is also called the local Lefschetz number of $f$ at $x$.

Prop At Lefschetz fixed points, this defn. and the defn. with intersection number agree.