Oriented Intersection Number, Degree, Applications.

Last time: technical lemmas about boundary and orientation.

In let $F : X \to Y$ be smooth, $Z \subset Y$, $2Y = \emptyset \neq \mathbb{Z}$, and $X, Y, Z$ oriented. Letting $S = f^{-1}(Z)$,
$$\exists S = (-1)^{\dim Z} (2F)^{-1}(Z).$$

Def We say the data $(F, X, Y, Z)$ above are suitable for oriented intersection theory if $X$ compact, $Z$ closed, and $\dim X + \dim Z = \dim Y$.

In this case $\dim F^{-1}(Z) = 0$ and $\exists F'(Z)$ consists of points, which have two associated orientations: for $x \in \exists S$, have

- Boundary orientation vs Preimage orientation
  - (using outward normal)
  - (using $\exists F''(Z)$)

as in the above. We say the orientation number of $x$ is $+1$ if these agree, and $-1$ otherwise.

Def If $F^{-1}(Z)$ in above setup, let
$$I(f, Z) = \sum \text{orientation} \#s \text{ of } f^{-1}(Z),$$
which is a finite sum as $X$ compact.
Thin let \( X = \partial W \) and \( W \) compact, oriented, with \( X \) having boundary orientation. If \( f: X \to Y \) extends to \( F: W \to Y \) and \( Z \subset Y \) with \((X,Y,Z,f)\) suitable for oriented intersection theory, then \( I(\xi, Z) = 0 \) \( \forall Z \).

pf: WLOG \( F(\xi) \subset Z \). Then by above thin

\[
\mathcal{E}(F'(\xi)) = (-1)^{\text{codim } Z} \mathcal{E}(F)(\xi)
\]

and \( I(\xi, Z) \) is signed count of RHS. But LHS is

\[
(\text{by defn of }) \quad I(\xi, Z) = (-1)^{\text{codim } Z} \left( \sum \text{ orientation#s of } F^{-1}(\xi) \right)
\]

Since \( \mathcal{E} \) of compact l-manolds has cancelling orientation #s.

\[ \square \]

Cor: If \( f_0 \subset f \), then \( I(f_0, Z) = I(f, Z) \).

pf: Given homotopy \( F: X \times I \to Y \) and \( \partial(X \times I) = X_1 - X_0 \),

\[
\mathcal{E}(F^{-1}(\xi)) = f^{-1}_1(\xi) - f^{-1}_0(\xi)
\]

so \( I(\mathcal{E}F, Z) = 0 \) as \( F \) is extension of \((f_0, f_1)\) to \( X \times I \). Hence \( I(f_1, Z) - I(f_0, Z) = I(\mathcal{E}F, Z) = 0 \). \( \square \)
Co-iff \( I(f, z) \) can be defined when \( f \) not transverse at \( z \) by above as trans. generic.

Ex: \( 2(I \times CP^2) = 1 \times CP^2 - 0 \times CP^2 \). Letting \( CP^2 \) have its canonical (complex) orientation, \( f \) any \( S \)-manifold with boundary \( CP^2 \sqcup CP^2 \).

Q: How to generalize degree? Want \( I(f, S^0) \).

When \((X, Y, f, y), f)\) suitable for oriented intersection theory, \( f^{-1}(y)\) is compact \( 0\)-manifold. For \( x \in f^{-1}(y)\), both \( T_x(x)\) and \( T_y Y\) are oriented vector spaces, so orientation \# is \( +1 \) if \( df_x \) preserves orientation, and \(-1\) if \( df_x \) reverses.

Ex: \( z \mapsto z^n \) on \( S' \subseteq \mathbb{C} \) has degree \( n \).

Proof: If \( f: X \to Y \) extends over \( W \) w/ \( \partial W = X \) w/ boundary orientation then \( \deg f = 0 \). (HW)
Case (FTA)

Showed before \( p_{1/1^{1/1}} \cong \frac{\mathbb{Z}^m}{\mathbb{Z}^m} \) when \( p \geq 2^m + \ldots \) and we restrict to suff. large disc \( D \) about 0. So both \( p_{1/1^{1/1}}, \frac{\mathbb{Z}^m}{\mathbb{Z}^m} \) have degree \( m \). If \( m > 0 \) and no zeroes in \( D \), then \( p_{1/1^{1/1}} \) extends but then has deg 0, impossible.

Special case of oriented intersection theory: submanifolds.

Def: If \( X, Z \subseteq Y \) suitable for oriented intersection and \( i : Z \hookrightarrow Y \) inclusion,

\[
I(X, Z) = I(i, Z)
\]

if \( X \cap Z \). We count \( X \cap Z \) (0-dim'l) with signs

- \( + \) iff \( T_x X \oplus T_x Z = T_x Y \) as oriented vs.
- \( - \) otherwise

Ex: Orient \( X, Z \subseteq Y = T^2 \) as below.

Here \( I(X, Z) = -2 \) since \( T_x X \oplus T_x Y \) has opposite orientation than \( \nu \). OTTOH note \( I(Z, X) = +2 \).

Move \( \nu \) to \( X \cap Z \) by orientation-preserving rotation to compare.
Thm \( I(z,x) = (-1)^{\dim Z \cdot \dim z} (x,z) \)

We prove this later by first developing intersection number of maps, then specializing to inclusion.

Def Given \( f: X \to Y \), \( g: Z \to Y \), we say \( f \cap g \) if for every \( y = f(x) = g(z) \), \( df_x(T_xX) + dg_z(T_zZ) = T_yY \).

If \( \dim X + \dim Z = \dim Y \) then \( df_x \), \( dg_y \) must be inj. and the sum is direct. So we can declare \( y = f(x) = g(z) \) to have intersection number \( \# \) if

\[ df_x(T_xX) \oplus dg_y(T_zZ) = T_yY \]

as oriented vector spaces.

Taking \( X, Y, Z \) all compact w/o boundary, get finite sum. When either \( f, g \) are inclusions this reduces to prov. case.

\[
\text{Def } I(f, g) = \sum_{(x,z)} \text{oriented intersection } \#
\]

\[
\text{st. } f(x) = g(z)
\]

Thm When \( f, g \), \( x, y, z \) suitable for oriented intersection theory, this sum is finite. Also,

\[
I(f, g) = (-1)^{\dim Z} I(f \circ g, \Delta) \leftarrow \text{previous defn of intersection of map w/ submfd}
\]
**Proof:** First note \( \dim g \Rightarrow \# \{ (x,z) \mid f(x) = g(z) \} \) is finite. Let \( \Delta = \{ (y,y) \in Y \times Y \} \) diagonal. Then \( f \circ g : X \times Z \to Y \times Y \) smooth and \( f(x) = g(z) \iff (x,z) \in (f \circ g)^{-1}(\Delta) \), finite.

**Claim:** \( \dim g \iff (f \circ g) \cap \Delta. \)

**Proof:** True if, on the level of tangent spaces, we have

\[
\text{Im} \, df_x \oplus \text{Im} \, dg_y = T_x Y \iff (\text{Im} \, df_x \oplus \text{Im} \, dg_y) \oplus T(\Delta) = T(x,y).
\]

But this is clear from linear algebra: if \( U, W \subseteq V \)
linear subspaces, \( U \cap V = 0 \iff (U \cap W) \cap \Delta = 0 \) in \( V \times V \).
So we have direct sums, need to check if they span. By dimension count

\[
\dim U + \dim W = \dim V \iff \\
\dim (U \cap W) + \dim \Delta = \dim \\
\dim (U \cap W) + \dim V = 2 \dim V \\
\dim (U \times W) = \dim V
\]

Now to show equation holds as oriented vector spaces, take \( \mathfrak{B}_v, \mathfrak{B}_w \) ordered bases for \( V, W \). Then the basis \( \mathfrak{B}_v \times \mathfrak{B}_w, \mathfrak{B}_v \times \mathfrak{B}_w \cap \Delta \) orients \( V \times W \), and \( \mathfrak{B}_v \times \mathfrak{B}_w \cap \Delta \) orients \( \Delta \) (all positively).

Hence for \( (U \times W) \cap \Delta \), the basis
is positive. Note that for ordered positive basis \( \{x_1, \ldots, x_n\} \)
the basis \( \{x_1, \ldots, x_i - x_j, \ldots, x_n\} \) remains positive \( \forall i, j \).

\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
2 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\]

So another pos. basis is
\[
\{ (u, 0), \ldots, (u_k, 0), (0, u_i), \ldots, (0, u_k) \}
\]
Performing \( l \cdot k \) transpositions, get
\[
\{ (u, 0), \ldots, (u_k, 0), (0, u_i), \ldots, (0, u_k), (0, w_1), \ldots, (0, w_k) \}
\]
and another \( l \cdot (l + k) \) transpositions gives
\[
\{ (u, 0), \ldots, (u_k, 0), (w_1, 0), \ldots, (w_k, 0), (0, u_i), \ldots, (0, u_k) \}
\]
which is positive for \( V \times V \). Hence we perform
\[
lk + l \cdot (l + k) = lk + l^2
\]
transpositions, so orientation only reverses if even.

Applying this to \( \Omega = \Omega^f_x(T_x X) \), \( \Omega = \Omega^g_x(T_x X) \) and
\( V = T_{x_1} Y \), result follows.
Upshot: Re-interpreted $I(f, g)$ in terms of old defn of intersection number, so previous results apply.

Def If $f$ not transverse to $g$, find homotopy of $(f \times g)$ so $(f \times g) \cap \Delta$, then $I(f, g) = (-1)^{\dim X} I(f \times g, \Delta)$.

Prop
1. If $f_0 = f$, $g_0 = g$, $I(f_0, g_0) = I(f, g)$.
2. If $g: Z \to Y$ inclusion, $I(f, g) = I(f, Z)$.
3. If $\dim X = \dim Y$, $I(f, g)$ indep. of $\gamma$.
4. $I(f, g) = (-1)^{\dim X + \dim Y} I(g, f)$
5. $I(\gamma, 2) = I(2, \gamma)$

Pf: 1. Use homotopy $(f_0 \times g_0) \simeq (f \times g)$ via $(F_t, s_t)$ and homotopy invariance.

2. Homotope $f = f'$ so $f' \cap \Delta$.

3. Use 2. and isotopy sending one point to another.

4. Lemma For $f, g$ as above, not nec. transverse, $f \cap f' \subset f$ so $f' \cap g$.

Pf: Extend $f$ to $F: X \times B^n \to Y$ as in proof of Transversality-Homotopy Thm. Then for any $x \in X$, $F_x: B^n \to Y$ is submersion, $X \subset \mathbb{R}^n$. 


Taking \( F \circ g : X \times S \times T \rightarrow Y \times Y \), we have 
\((F \circ g) \uparrow \Delta \) as \( F \circ g \uparrow (x, x) \) is a submersion. 
Thus \( F \circ g \uparrow \Delta \) for almost all \( S \). Choosing one such \( S \), \( F_S \uparrow \) and conclusion holds. 

Then (4) follows as we can assume \( F \uparrow g \) and use 

Thus proven above. (5) is special case.

Applications: Orientability and Self-Intersection

Note that if \( X \subseteq Y \) and \( \dim X \) is odd w/ \( \dim Y = 2 \dim X \),
then \( I(X, X) = -1 \cdot I(Y, Y) \in \mathbb{Z} \) so \( I(X, X) = 0 \). Also \( I_2(X, X) = I(X, X) \mod 2 \).

Cor. If \( X, Y \) as above and \( I_2(X, X) \neq 0 \) then either \( X \) or \( Y \) non-orientable.

Ex. \( \text{O}(\text{Möbius}) \)

Can show \( I_2(X, X) = 1 \) by explicit isotopy. \( X \subseteq S' \) is 
orientable, so Möbius is not.

More generally, \( \mathbb{R}P^{2k} \) non-orientable, \( \mathbb{R}P^{2k+1} \) is orientable. For 
\( \mathbb{R}P^{2k+1} \subseteq \mathbb{R}P^{4k+2} \), this is \( k = 0 \) and in fact \( I_2(\mathbb{R}P^{2k}, \mathbb{R}P^{2k+1}) \) 
is non-zero (consider subspaces in \( \mathbb{R}^{4k+3} \)).
Consider $\mathbb{C}P^2$ and smooth embeddings

1: $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$

$[z_0; z_1] \mapsto [0; z_0; z_1]$

2: $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$

$[z_0; z_1] \mapsto [z_0; z_1; 0]$

$\mathbb{C}P^2$ has canonical $\mathbb{C}x$ orientation, w.r.t which $I(i_0, i_1) = +1$.

However, $\overline{\mathbb{C}P^2}$ with reversed orientation has $I(i_0, i_1) = -1$. Note that orientation-preserving diffeom. preserves oriented intersection number. Hence if an orientation-reversing self-diffeom. of $\mathbb{C}P^2$, as it would not preserve $I(i_0, i_1)$. 