Morse Functions and Handles

Consider $T^2 = \mathbb{R}^2 \times \mathbb{R}$.

Let $f(x, y, z) = z$ and consider level sets

$$(T^2)^a = f^{-1}((0, a))$$

Ex:

$(a = 1)$

$(a = 0)$

$(a = -1)$

Get decay into simple pieces, i.e. homotopy type of sub-level sets changes as we pass through critical values. But we need a particular kind of function. Let $M^a = f^{-1}(00, a]$. 

Let $M^a$ be compact, $M^a$ closed. A smooth func $f: M \to \mathbb{R}$ is Morse when $f$ has only non-degenerate crit pts.

Recall that $p \in M$ critical if $df_p = 0$, and non-deg mean $\text{Hess}_p(f) = \text{non-degenerate}$:

$$H_f : T M \times T M \to \mathbb{R}$$

$$H_f (v, w) = v(\text{df}(w))$$

$\text{df}(v)$ smooth for $f$, $\forall$

or in local coord. $(x, \xi)$, say,

$$H_f(p) = (\frac{\partial^2}{\partial x \partial \xi})$$
Rmk: \( p \in \text{Crit } f \) for this to be well-defined.

Det the index of a critical point is defined as the dim. of the maximal subspace on which \( H_f(p) \) negative definite.

We'll also need to look at points where \( f \) is not critical.

A Riemannian metric on \( M \) is symmetric, positive definite, bilinear pairing \( \langle \cdot, \cdot \rangle : TM \times TM \to \mathbb{R} \).

Def: The gradient of \( f \) is a vector field \( \nabla f \in \Gamma(TM) \) such that \( g(\nabla f, -) = df(-) \).

Thm: For \( M, g, f \) as above, let \( a, b \in \mathbb{R} \) st. \( f^{-1}(a, b) \) is compact and contains no critical points. Then \( M^a \) is a deform. retract of \( M^b \), so \( M^a \to M^b \) is a homotopy equivalence.

Pt: Let \( \alpha : M \to \mathbb{R} \) be given by \( \alpha(p) = \| \nabla f_p \|^2 \). Then \( \frac{1}{\alpha} \) is defined on \( f^{-1}(a, b) \), so we may define

\[ Z = -\frac{1}{\alpha} \nabla f \]

on \( f^{-1}(a, b) \) and extend to rest of \( M \) as \( 0 \) (using partition of unity etc.). Let \( \phi^t \) be the time \( t \) flow along \( Z \), e.g. \( \frac{d}{dt} \phi^t(x) = \nabla f(\phi^t(x)) \),
Note that
\[ \mathcal{L}_z f = \langle \nabla f, \frac{1}{a} \nabla f \rangle = -1 \]
so in fact \( f \) decreases with unit speed along \( z \). Then \( \phi^{t - a} \) sends \( M^b \) to \( M^a \), and is the real \( d \)th retract. \( \Box \)

We also have normal form for such critical pts.

(The Morse Lemma) If \( p \in \text{crit} \( f \) \) and \( p \) non-deg., then \( \exists \) local coordinates so that (if ind \( f \); \( k \))
\[ f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 \]
Think of this as "diagonalizing" \( \nabla f \) near \( p \). To visualize gradient flow, choose chart \( t \) metric so
\[ f = -(x_1)^2 - \cdots - (x_k)^2 + (x_{k+1})^2 + \cdots + (x_n)^2 \]
\[ g = \sum_i (dx_i)^2 \]
\[ \Rightarrow \quad df = -2x_1 dx_1 - \cdots + 2x_{k+1} dx_{k+1} + \cdots \]
\[ \Rightarrow \quad \nabla f = 2x_1 \frac{d}{dx_1} + \cdots - 2x_{k+1} \frac{d}{dx_{k+1}} + \cdots \]
At \( p \in \text{crit}(f) \), define the ascending and descending manifolds

\[
A_p = \{ x \in M \mid \lim_{t \to +\infty} y_t(x) = p \} \quad \text{(stable, } f \text{ increases from } p \text{)}
\]

\[
D_p = \{ x \in M \mid \lim_{t \to -\infty} y_t(x) = p \} \quad \text{(unstable, } f \text{ decreases from } p \text{)}
\]

Note: Each \( x \) lies in exactly one \( A_p \).
Moreover if \( p \in \text{crit}(f) \), \( p \in A_p \cap D_p \).

Facts:

1. \( A_p \) and \( D_p \) are manifolds
2. \( \dim A_p = \text{ind } p \).

Thus \( D_p \) (resp. \( A_p \)) is diffeomorphic to a ball of dimension \( \text{ind}(p) \)
(resp. \( -\text{ind}(p) \)).

If \( E_+^p, E_-^p \) are \( +,- \) eigenspaces for \( \text{Hess}_p(f) \) w/r/t metric,
then \( T_p(D_p) \cong E_+^p, T_p(A_p) \cong E_-^p \).

\( \mathcal{F} \): For some Morse chart \( c^j \) w/ std metric, in fact \( E_+^p \subseteq A_p \) and \( E_-^p \subseteq D_p \). We may choose such a metric because the distinct eigenvectors (spans) are orthogonal. So linearization of \(-\nabla F\) is as before.
Application: Reeb sphere theorem.

Theorem: Suppose $M^n$ closed, $f: M \to \mathbb{R}$ Morse with exactly 2 critical points. Then $M$ is homeomorphic to a sphere $S^n$.

Proof: Let $p$ be max, $q$ min. We have $A_q$ and $D_p$ both identified with $B^n$ open.

For any $0 < \lambda < 1$, take $\lambda B^n \subseteq B^n$ closed and consider corresponding closed subsets $K_p, K_q$ of $A_q, D_p$. We may use gradient flow $\varphi^t$ to arrange that $M = K_p \cup K_q$ and $K_p \cap K_q \approx S^{n-1}$. So they only intersect along sphere boundaries.

Then $M = B^n \cup B^n$. Choosing any identification $g$ of $\partial K_p$ with $S^{n-1} = \partial B^n$, we have $g: \partial K_p \to S^{n-1}$. But any homeomorphism of $S^{n-1}$ extends to homeomorphism of $B^n - B^n$ is a cone on $S^{n-1}$ so define $h(tS^{n-1}) = tg(S^{n-1})$.

This determines a homeomorphism $S^n \to M$ by defining it on each $B^n$-hemisphere of $S^n$. 

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**Handles and Morse Functions**

**Key point:** For $M^n = \{ x \in \mathbb{R}^n | \alpha \in \text{Crit}(f) \}$, this is critical level where topology changes.

**Example:** $F(x,y,t) = xy + t$, $t \in (-1,1)$, $X = F^{-1}(0) \subseteq \mathbb{R}^3$. Morse function is projection to $t$-coordinate.

**Exercise:** If $f: X \to [a,b]$ has no critical pts, $X = M^n \times [a,b]$.

**Proof** If $f: X \to [a,b]$ has 1 critical point, $X = M^n \times [a,b] \cup B^{-\text{ind}(p)} \times D$.

Use exercise on $f^{-1}[c,\infty)$ w/ no crit values. On $f^{-1}((g,\infty))$, use Morse chart near critical point:

Take $B^{-\text{ind}(p)} \times B^{\text{ind}(p)}$ in Morse chart, gradient flow eventually enlarges it until it intersects level set $f^c(\cdot)$.
Def: For a manifold of dimension $n$, a k-handle is a

$$D^k \times D^{n-k}$$

along $S^{k-1} \times B^{n-k}$

Can reconstruct $M$ from these pieces starting from 0-handles attaching k-handle along $S^{k-1} \times S^{n-k}$ is boundary. If index of $p_{\text{crit}} = k$, then $M^{p_{\text{crit}}} = M^{p_{\text{crit}}-k} \cup (k\text{-handle})$.

Ex: Consider a genus 1 closed surface $T^2$ as before:

We start with a 0-handle $D^0 \times D^0$, attached to nothing ($\emptyset$).

Attach the 1-handles $D' \times D'$ along $\partial(D') \times D' = S^0 \times D'$.

Now what remains has $S^1$ boundary, so we may cap off with a 2-handle $D^2 \times D^0$, glued along its $\partial(D^2) = S^1$.
Anatomy of a k-handle: a 1-handle in a 3-manifold:

$D^1 \times D^2$ has $\partial = D^2 \times S^0 \sqcup S^1 \times D^1$

2-handle in a 3-mfld: $D^2 \times D^1$ has $\partial = S^1 \times D^1 \sqcup D^2 \times S^0$

3-handle: cap off w/ $B^3$. 
Ex: $T^3$. Recall that we may depict $T^3$ as $[0,1]^3$.

Consider the handle decomposition given by the following Morse function (determined by its level sets):

$1 \times \mathbb{D}^2$ (attached)

$S^2 \times D^2$ (3 total)
(2-1n) attached along $S' \times D'$

(3 total)

End result looks like $T^3 \setminus \text{null } ((0,0,0))$:

Cap off $D/3-1n = B^3 \times B^0$.

Note: $(0-1n) \cup (1-1n) = \text{Genus 3 handlebody.}$

$(2-1n) \cup (3-1n) =$ \text{Nbdts of edges of } [0,1]^3

Also genus 3 handlebody. This decomposition is called a Heegaard splitting.
Closing: 4-manifolds.
As before, start p1 D-1. Attach 1-h = $D^1 \times D^3$ along $S^0 \times D^3$. Attach 2-h along $S^1 \times D^2 = \text{thickened knot}$.

Ex $T^4$. Start p1 $D^4$; attach to $2D^4 = S^4$. Use stereographic projection to picture this as $\mathbb{R}^3$.

Attach 1-h along $D^3 \subset \mathbb{R}^3$:

(plus extra $\infty$ at $\infty$)

Now attach 2-h along $S^1 \times D^2$.
Slight subtlety: when attaching handles along \( S^{n-1} \times D^{n-k} \) normal bundle to embedded \( S^{n-1} \) cones is framing, i.e. trivialization of normal bundle (rank \( n-k \) vector bundle) to \( S^{n-1} \). This is torus for \( \pi_{k-1}(GL(n,k)) = \pi_{k-1}(O(n-k)) \).

For \( n = 2,3 \) not needed, but for \( n = 4 \):

1. \( \pi_0(O(3)) = 1 \)
2. \( \pi_1(O(2)) = \mathbb{Z}, \) so 3 \( \mathbb{Z} \)'s worth of choices.