**Symplectic Reduction**

$\mathbb{R}$ acts on symplectic manifolds via Ham. flow. Other groups are possible, and useful to consider. First, preliminaries on Lie groups.

**Def.** A linear action of a Lie group $G$ on a vector space is a smooth map $\rho: G \to \text{SL}(V)$ such that

1. $\rho(e) = \text{Id}$
2. $\rho(gh)(x) = \rho(g) \circ \rho(h)(x)$

Differentiating $AG \to \text{SL}(V)$ at $\text{Id}$, we get $\text{dA}: g \to \operatorname{End}(V)$. For $v \in V$, $x \in g$ let $\alpha_x(v) = \text{dA}(x)(v)$. This gives a vector field on $V$ for every $x \in g$.

The **dual action** of $G$ on $V^*$ is characterized by

$\varrho^*: V^*(x) = V^*(g \cdot x)$

This gives rise to the coadjoint actions of $G, g$ on the dual to the Lie algebra $g^*$.

Define $\text{Ad}^*: G \times g^* \to g^*$ using above and $\text{Ad}: G \times g \to g$. Then since $\text{Ad}$ is linear action, differentiating at the identity gives a map $g \to g$ and there is an analogous action $\alpha^*: g \times g^* \to g^*$; if $f \in g^*$ and $u, v \in g$,

$$\langle \alpha_u(v), u \rangle = \langle f, \text{ad}_u(v) \rangle = \langle f, -[u, v] \rangle$$
Ex. For $G = U(n)$, recall $g = \{\text{skew-Hermitian matrices}\}$. In fact $\mathfrak{g} = \{\text{Hermitian matrices}\}$ using $-Tr(iAX)$. For $A$ skew-Hermitian, $X$ Hermitian, then computation $\Leftrightarrow \text{ad}^*_A(x) = [A, x]$.

Suppose $G = SO(3)$, $g = \{\text{skew-symmetric $3 \times 3$ matrices}\}$. Then (note $\mathfrak{g} = \mathbb{R}^3$, bracket is cross product) $\text{Ad}^*$ and $\text{Ad}$ are both given by applying $A \in SO(3)$ to $v \in \mathbb{R}^3 \cong g$.

For manifolds, not much differs. We already saw detn. for Grassmannians.

Def. Suppose $G$ acts on $X$, so $G \times X \rightarrow X$ smooth. Then the stabilizer $\text{Stab}_p(G) = \{g \in G | g.p = p \}$, and if $\text{Stab}_p(G)$ is discrete for all $p \in X$ we say the action is locally free.

Ex. $U(1) \subset S^1 \subset \mathbb{C}^2$ via $e^{i\theta} \cdot (v, w) = (e^{i\theta}v, e^{i\theta}w)$. This is a free action, no fixed points.

$U(1) \subset S^3 \subset \mathbb{C}^2$ via $e^{i\theta} \cdot (v, w) = (e^{i\theta}v, e^{2i\theta} w)$ is only locally free: along $2\pi \mathbb{Z}$, $\text{Stab}_p(U(1)) = \{e^{i\pi}\}$.

As with vector spaces, $G$ induces vector fields on $X$ (coming from homomorphism $G \rightarrow \text{Diff}(X)$), denoted by $\hat{\mathfrak{g}}$ for $\mathfrak{g}$ for $G$, and vector at $p \in X$ is $\hat{\mathfrak{g}}_p$ the Lie algebra stabilizer $q \subset \mathfrak{g}$ is the subalgebra of $G$ s.t. $\langle \hat{\mathfrak{g}}_p = \mathfrak{g} \rangle$. 
This Suppose $G \not\subseteq X$ freely. Then $\varphi: G \to T_x X$ is injective for all $x$, and if $G$ is compact, then $X/G$ is a manifold with $T_x (X/G) = T_x X / G$.

Ex: For free $U(1) \not\subseteq S^3$, $S^3 / U(1) \cong S^2$.

Prop Suppose $G \not\subseteq X$ transitively, so $\forall x,y \in X \exists g \in G$ s.t. $g \cdot x = y$. Then for any $x \in X$, $X$ is $G$-equivariantly diffeomorph to $G / \text{stab}_x(G)$.

Idea: Define $G / \text{stab}_x(G) \to X$ as $g \cdot \text{stab}_x(G) \to g \cdot x$. $\text{stab}_x(G)$ acts on $G$ freely, so quotient has manifold structure. This is clearly equivariant bijection; smoothness follows from $T_x X \cong G / G_x$.

**Moment Maps**

Using symplectic form to encode group action.

Let $G \not\subseteq X$ via Hamiltonian diffeomorph $G \to \text{Ham}(M,w)$. The derivative at $g \in G$ is $Dg \to \Gamma(TX)$, with image $\chi_H$ for $H: M \to \mathbb{R}$. In fact, if $\{H_i\}$ basis of $g$, get $H = \sum \alpha_i H_i$ for $\alpha_i$ generating image $\chi_i$ of $g_i$.

These fit together into map $\chi: M \to \mathfrak{g}^*$; if $x \in M$, $u \in \mathfrak{g}$ with $u = \sum \alpha_i u_i$, then $\langle \chi(x), u \rangle = \sum \alpha_i \langle H_i(x), u_i \rangle$. 
Example $(M,\omega)$, $\mathbb{R}^M$ via flow along $X_H$ Hamiltonian. 
Since $\mathfrak{g}^* = \mathbb{R}$, $\langle \mu(x), t \rangle = t H(x)$, $\mu(x) = H(x)$.

Therefore, $U(1) \subset \mathbb{C}P^1$ via $e^{i \theta} [x_1 : x_2] \mapsto [e^{i \theta} x_1 : x_2]$. Note this action is Hamiltonian. In a chart $[1: z]^2$, we get

$$
\omega_{fs} = \frac{-\frac{1}{2} \frac{d x_1 d x_2}{\sqrt{\|x_1^2 + x_2^2\|^2}}}{(x_1^2 + 1)(x_2^2 + 1)^2}.
$$

Now compute

$$
\frac{d}{d \theta} \left( \frac{1}{2} \frac{|z|^2}{|z|^2 + 1} \right) = \frac{2 z}{(z^2 + 1)^2} dz,
$$

and this is contraction of $\omega_{fs}$ with $2 \frac{\partial}{\partial \theta}$. So

$$
N(z) = \frac{1}{2} \frac{|z|^2}{|z|^2 + 1}
$$

is a formula for the moment map in local coordinates.

This vector field is Hamiltonian for "height function" on $\mathbb{R}$, $x$ as above.

In general, $U(n)$-action on $\mathbb{R}^n$ has moment map

$$
\langle \mathcal{M}(x), U \rangle = -\frac{i}{2} \frac{\langle U(v), v \rangle}{\|v\|^2} \quad U \in \mathfrak{g}
$$

Note some properties of these two examples:

1. For all $U \in \mathfrak{g}$, $\langle \mathcal{M}_x (-), U \rangle = N(x \cdot U, -)$
2. $\mathcal{M}$ is $G$-equivariant, where $G \subset \mathfrak{g}^*$ via the co-adjoint action.
Def A moment map for a Hamiltonian $G$-action is a smooth function $\mu : M \to g^*$ s.t. 1 and 2 hold.

Symplectic Reduction

Recall linear algebra lemma: if $S \subset (V,w)$ coisotropic then $(S/_{s^*}, w)$ is symplectic. We quotient by vector fields $a_p$.

Suppose $G \subset (M, w)$ with moment map $\mu : M \to g^*$. By equivariance of $\mu$, $\mu^{-1}(0)$ is an orbit of the $G$-action.

Prop For all $p \in M$, the Lie algebra stabilizer $g_p$ is dual to $\frac{g^*}{d\mu(p)}$.

pf: At $p \in M$, $\langle d\mu(v), U \rangle = 0$ $\forall U \in g$, $v \in T_pM$

$\iff d\mu_p(v) = 0$ $\forall v$

$\iff w(X_u, v) = 0$ where $X_u$ is vector field generated by $U$

$\iff X_u = 0$ $\forall U \in g_p$.

Then any element of $g^*/\text{Im}d\mu$ is well-def on $g_p$. $G$ acts on both $g_p$, $g^*/\text{Im}d\mu$, and $\langle -,- \rangle$ is equivariant. Then $U \mapsto \langle U, - \rangle$ is the real isomorphism. $\square$
Clearly $\text{Im}(d\mu_p) = 0$ if $p$ is a fixed point. We also have:

**Corollary** $0 \not\in g^*$ is a regular value of $\mu$ iff $g_0 \cap \mu^{-1}(0)$ is locally free, i.e. $g_0$ is the Lie algebra stabilizer is trivial. If in addition $G$ acts freely, $\mu^{-1}(0)/G$ is a manifold of dimension $\dim M - 2\dim G$.

**pf:** By proposition, $\text{Im}(d\mu_p)$ surjective (regular value) iff $g_p = 0$ for $p \in M$. □

Key property of symplectic reductions: reduced space has symplectic form.

**Thm** Suppose $G$ acts freely on $\mu^{-1}(0)$, so $\mu^{-1}(0)/G$ smooth. There exists a unique 2-form $\omega_{\text{red}}$ on $\mu^{-1}(0)/G$ such that $\omega_{\text{red}}$ coincides with pullback of $\omega$ to $\mu^{-1}(0)$. This form is symplectic on $\mu^{-1}(0)/G = X$.

**pf:** Since $G$ acts freely, map $\mu^{-1}(0) \to X$ is submersion so pullback of forms injective. Thus if $\omega_{\text{red}}$ exists, it is unique. Moreover, $d$ commutes w/ pullback so if $\omega_{\text{red}} = \iota^*(\omega)$, it is closed. So we show $\omega_{\text{red}}$ exists and is non-deg. By construction if $p \in \mu^{-1}(0)$ then $\omega_p(G \to M)$ is action at $p \in M$, $T_p(X) \cong T_p(\mu^{-1}(0))/\text{Im } d\mu_p$. We show that $T_p(\mu^{-1}(0))$ lies in $(\text{Im } (d\alpha))^\perp$, so $\omega$ is well-defined on $X$. For $U \in T_pX$, $\nu \in \text{ker } d\mu_p$.
\( \omega = \omega_{\text{red}} \) \( \Rightarrow \) moment map condition
\[
\omega_p(d\alpha(u),v) = \langle d\mu_p(v),u \rangle = \langle 0, u \rangle = 0
\]
So \( \omega \) descends to \( \omega_{\text{red}} \) and is nondegenerate. \( \triangleleft \)

**Ex.** Consider diagonal \( U(1) \)-action on \( (\mathbb{C}^n, \omega_{\text{Fub}}) \) with
\[
M(z) = -\frac{1}{2}(1z_1^2 + \ldots + 1z_n^2) + c
\]
for \( c \in \mathbb{R} \). Then \( M^{-1}(0) = S^{2n-1} \), sphere of radius \( \sqrt{2c} \).

\( U(1) \) action rotates great circles. Quotient is \( \mathbb{C}P^{n-1} \).

\( \triangleright \) Recall \( SO(3) \circlearrowleft \mathbb{R}^3 \cong S^2 \). In fact coadjoint orbits are spheres. If \( S^2_{r_i} \) is sphere of radius \( r_i \), consider diagonal action
\[
SO(3) \circlearrowleft S^2_{r_1} \times \ldots \times S^2_{r_k} \subset (\mathbb{R}^3)^k
\]
with induced area forms.

Can check that for \( x \in S^2_{r_i} \), inclusion into \( \mathbb{R}^3 \) is moment map (tangentially equivariant), and
\[
M(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i
\]
is moment map for diagonal action.

Reducing at \( M^{-1}(0) \) means we consider
\[
\{ (x_1, \ldots, x_k) | \sum x_i = 0, \|x_i\| = r_i \}\]
Thinking of these as segments in $\mathbb{R}^3$, any $(x_1, \ldots, x_k)$ gives a polygon in $\mathbb{R}^3$ with edge lengths $(x_i)$. $SO(3)$ acts by rotations (congruences). Reduced space is polygon space $Pol(r_1, \ldots, r_n)$ embedded polygons in $\mathbb{R}^3$.

Prop: Given $(r_i)$, suppose $J$ any $S \subset \{1, \ldots, n\}$ such that
\[
\sum_{i \in S} r_i = \sum_{i \notin S} r_i.
\]
Then $Pol(r_1, \ldots, r_n)$ is a compact manifold of dim $2(n-1)$.

Idea: Just check $SO(3)$ acts freely. If $J S$, $S^c$ partition of $(r_i)$ with above property, then vectors $\frac{1}{2} r_i$, $\frac{1}{2} r_i$ are collinear.

For $n = 3$,

\[
V_1 \rightarrow V_2
\]
Degenerate triangles; $U(1)$ stabilizer

For $n = 4$,

\[
V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4
\]
Rhombus; $D_{2,2}$ stabilizer

If such $S$ do not exist, free action.

Note that we need not reduce at $O_\infty$; any orbit of $O$-adjoint action will do.
Natural question: Does symplectomorphism type change as we change coadjoint orbit?

Ex. (Guillemin-Sternberg) Let $U(1)$ act on $\mathbb{C}^{n+1}$ by

$$e^{it} \cdot z = (e^{it} z_0, e^{it} z_1, \ldots, e^{it} z_n)$$

with moment map $\mathcal{M}(z) = -1/2 |z_0|^2 + \frac{1}{2} \sum_{i=1}^{n} |z_i|^2$. This map has isolated critical value. We compute its symplectic reductions at coadjoint orbits $\pm \mathcal{E}$, $\mathcal{E} > 0$ small.

$$(\mu(\mathcal{E}) = -\mathcal{E}) \quad \mathcal{M}^{-1}(\mathcal{E}) = \left\{ -1/2 |z_0|^2 + \frac{1}{2} \sum_{i=1}^{n} |z_i|^2 = -\mathcal{E} \right\}$$

so $|z_0|^2 = \mathcal{E} + \frac{1}{2} |z_1|^2$. $U(1)$ acts as above; take $\left\{ \Re(z_0) > 0, \Im(z_0) = 0 \right\}$ as a slice.

But this slice can be identified w/ $\mathbb{C}^n$ via

$$\mathbb{C}^n(\chi_1, \ldots, \chi_n) \to (\sqrt{\mathcal{E} + \sum_{i=1}^{n} |\chi_i|^2}, \chi_1, \ldots, \chi_n)$$

and restricted Kähler form from $\mathbb{C}^{n+1}$ gives $\omega_{std}$ on $\mathbb{C}^n$.

$$(\mathcal{E} + \mathcal{E}) \text{ Then } -1/2 |z_0|^2 + \sum_{i=1}^{n} |z_i|^2 = 2, \text{ so } \sum_{i=1}^{n} |z_i|^2 = 2 + 1/2 |z_0|^2.$$ Change coordinates to $w_0 = z_0$, $w_i = (\mathcal{E} + 1/2 |z_0|^2)^{-1/2} z_i$.

Then, $\mathcal{M}'(\pm \mathcal{E})$ in $w$-coord. is $S^{2n-1} \times \mathbb{E}$, $S^{2n-1}$ is unit sphere in $w$-coord. $U(1)$ action in $w$-coord. is unchanged.
\[ e^{it} w_i = e^{it} w_o \]

\[(i = 1, \ldots, n) \quad e^{it} \cdot w_i = e^{it} w_i \]

We deal with first factor of \( S^{2n-1} \times \mathbb{C} \) first. In \( S^{2n-1} \), this is Hopf fibration of previous example; reduced space is \( \mathbb{CP}^{n-1} \).

Projection \( S^{2n-1} \times \mathbb{C} \to S^{2n-1} \) induces a map \( S^{2n-1} \times \mathbb{C} / \{w_i\} \to \mathbb{CP}^{n-1} \). Fiber? Fix \( \lambda \in \mathbb{CP}^{n-1} \) represented by some \((\lambda, \ldots, 0) \in \mathbb{C}^n\). We get \( |z_0|^2 + |\lambda|^2 = \varepsilon \), so \( |z_0|^2 = |\lambda|^2 + \varepsilon \).

Running over all representatives \( \lambda \), we get bijection \( \lambda \leftrightarrow z_0 \), and fiber is a copy of \( \mathbb{C} \). So total space is line bundle over \( \mathbb{CP}^{n-1} \), i.e. fact the tautological line bundle \( O_{\mathbb{CP}^{n-1}}(-1) \). Total space is blow-up of \( \mathbb{C}^n \) at the origin: birationally equivalent to \( \mathbb{C}^n \).