Manifolds With Boundary, Transversality
Modeled on half-space $H^k = \{ x_1, \ldots, x_k \in \mathbb{R}^k \mid x_k \geq 0 \}$

Def A subset $X \subseteq \mathbb{R}^n$ is a manifold with boundary if $\forall x \in X$ there is a set $U \subseteq H^k$ open and $\phi : U \to X$ s.t. $\phi : U \to \phi(u)$ is a diffeomorphism (i.e. $\exists$ extension to open sets in $\mathbb{R}^k$, $\mathbb{R}^n$ so map is smooth).

Def We say that the boundary $\partial X$ of $X$ consists of all $\forall x \in X \mid (U, \phi)$ loc. coord near $x$, $\phi^{-1}(x) \in \mathbb{R}^k \times \{0\}$

Def The interior of $X$ is $X - \partial X$.

Given $\phi : U \to X$ define $d\phi_x$ for $x \in U \cap \mathbb{R}^n$ by extending $U$ locally to a neighborhood in $\mathbb{R}^k$ as in defn of manifold. The map $d\phi_x$ on the boundary is indep of this extension by continuity of $\frac{\partial \phi}{\partial x_i}$.

Then $T_xX := d\phi_x(\mathbb{R}^k)$ “Tangent space at $x$ point is limit of tangent spaces of interior points”.

Q: Why is defn. of $\partial X$ indep. of chart?

Let If $x \in \partial X$ for chart $(U, \phi)$ then for every chart $(V, \chi)$ from $V \supseteq X$, $V \subseteq H^k$ and $\chi(a) = x$, $a \in \mathbb{R}^{k-1} \times \{0\}$. 
Consider the alternative:

Define \( h = \varphi^{-1} \circ \varphi \), a diffeomorphism (can shrink the nbhd of \( x \)) of the "half-open" nbhd \( U \) to the open nbhd \( V \subset \mathbb{R}^k \). Suppose \( u \in U \) satisfies \( h(u) \in \text{Int}(V) \). Then \( h(u) \) lies in an open nbhd \( W \) in \( \mathbb{R}^k \).

Since \( dh_u \) is bijective, by Inverse Function Theorem, there is a diffeom. \( h_u \) defined on open subset of \( W \) to a nbhd of \( u \), also open in \( \mathbb{R}^k \). But then \( u \notin \mathbb{R}^{k-1} \), contradicting assumption 3.

(ax) \( \exists x \) is a manifold, \( \dim \partial X = k-1 \), \( \partial \partial X = \emptyset \).
(b) \text{Int } X \text{ is a manifold.}
(c) \( T_x (\partial X) \) is a codim 1 subspace of \( T_x X \).
(d) If \( f : X \to Y \) smooth denote \( \partial f = f \mid \partial X \). Then \( d(\partial f)_x = df_x \mid T_x (\partial X) \).
Note: If $X, Y$ manifolds with boundary, $X \times Y$ has corners (not manifold with boundary). But if $\partial X = \emptyset$, then $X \times Y$ is manifold with boundary.

\[ \text{Corners (IxI)} \]
\[ \text{No corners (S' x I)} \]

Lemma: Suppose $S$ is manifold and $\pi: S \to \mathbb{R}$ is smooth with regular value $a$. Then $\pi^{-1}(-\infty, a]$ is a manifold with boundary and $\partial(\pi^{-1}(-\infty, a]) = \pi^{-1}(a)$.

Proof: $\pi^{-1}(-\infty, a]$ is a manifold by previously shown thus. Now suppose $x \in \pi^{-1}(a)$. Local Submersion $\Rightarrow$ near $x$, $\pi$ is canonical submersion.

So subhd looks like half-space by Local Submersion, as needed. \[ \blacksquare \]
Ex: Closed unit ball $B^k \subset \mathbb{R}^k$.

Let $\pi: \mathbb{R}^k \to \mathbb{R}$ be $\pi(x) = -\sum x_i^2$ and take
$\pi^{-1}(\mathbb{C}^0, \infty)$. $0$ is a regular value, so $B^k$ is
mead w/ boundary.
Transversality and Neat Submanifolds

We define transversality, immersions, etc. as before. But in generalizing PreImage Thm, get extra property on $\mathcal{E}$.

Def Let $M^n$ be a manifold with boundary. $N \subset M$ a submanifold is neat if for every $p \in N$, $\exists$ a chart $(U, \varphi)$ of $M$ s.t. $N \cap U = \varphi(R^n_{<0})$, where $R^n_{<0}$ means

\[ \{ (0, \ldots, 0, x_{n+1}, \ldots, x_{m}) \mid x_{m+1} < 0 \} \]

Neat

Not neat

This guarantees:
1. $N \cap \partial M = \partial N$
2. $N - \partial M$

Thm Given $f: X \to Y$, $\exists Y \subset Y$, $\exists Y = \partial Y$. Suppose $f(\partial Y)$ and $f(\partial Y)$ (recall $f\partial = f\partial_\partial$). Then $f(\partial)$ is a neat submanifold of $X$.

pf: Interior of $X$ is manifold $\partial / \partial$, and by hyp. $f(\text{Int}(X)) = \partial Y$. Thus $f^{-1}(Y) \cap \partial \text{Int}(X)$ is a manifold with codim same as codim $Y$, by previous Preimage Thm.
So, suffices to look at $x \in \partial X \cap f^{-1}(z)$. We reduce to case of reg. values by locally cutting out $f^{-1}(z)$ near $f(x)$ by $\phi : V_{x,z} \to \mathbb{R}^l$, $l = \text{codim} \, z$.

Let $h$ be local parametrization with $h(0) = x$, so $g^{-1}(0) = h^{-1}(f^{-1}(z))$.

Now it suffices to work locally to prove neatness. Since $\exists f \in \mathcal{Z} \implies 0$ is reg. value of $g$, e.g. Since $g$ smooth at 0, it extends to a smooth map $\tilde{g}$ on an open subset $\tilde{U}$ to $\mathbb{R}^l$. Since $d \tilde{g}_0 = dg_0$, we see $d \tilde{g}_0$ is surjective so $\tilde{g}$ is a submersion at 0.

Thus $S = \tilde{g}^{-1}(0)$ is a submanifold of $\tilde{U}$ with codim $l$ and $\tilde{g}^{-1}(0) \cap \tilde{U} = S \cap H^k$. 

We claim $S \cap R^{k-1} \times \{0\} \subseteq H^k$. As $d(g|_{R^{k-1}\times \{0\}})$ and $d(g_o)$ are both surjective, there is some $v \in \ker d(g_o)$ but not in $R^{k-1} \times \{0\}$ (in kernel of restriction).

Need $\ker(d(g_o) + R^{k-1} \times \{0\}) = T_o(H^k)$. Since $R^{k-1}$ is codim 1, $\text{span}(v) + T(R^{k-1} \times \{0\})$ spans $R^k$ as needed.

For $g^{(1)}(0, \exists x = q(g^{(0)})$, let $\pi : S \to R$ be projection to the $k^{th}$ coordinate. Then 0 is regular value of $S$, as we found $V \notin R^{k-1} \times \{0\}$ but in $\ker d(g)$. By Lemma shown previously, $\pi^{-1}(0) = S \cap H^k = g^{(0)}(\bar{U})$.

We also have genericity of regular values.

**Thm (Sard)** Given $f : X \to Y$ where $\partial Y = \emptyset$, almost every $y \in Y$ is a regular value of $f$ and $\partial f$.

**Pr:** $y \in Y$ is a regular value of both $f$ and $\partial f$ unless it is a crit. value of either $f$ or $\partial f$. So $x$ is regular for $\partial f \iff x$ regular for $f$. Since $\partial X$, int($X$) are manifolds without boundary, Sard applies.
Then Every compact, connected 1-manifold is diffeomorphic to $[0,1]$ or $S^1$.

pf: Milnor, etc.

Cor Any compact 1-manifold has an even number of boundary points.

Then If $X$ is a compact 1-manifold with boundary then if any smooth retraction $g: X \to \partial X$ (meaning $g|_{\partial X}$ cannot be inclusion).

pf: Suppose such a $g$ exists, and let $x \in \partial X$ be a regular value of both $g, \partial g$. What is $g^{-1}(x)$? Has to be 1-manifold since $\partial X$ is codim 1. Also, $g^{-1}(x)$ compact so it even # of 2 points.

On the other hand, $\partial g^{-1}(x) = \partial g^{-1}(x) = x$ as $\partial g \subseteq \text{Id}_{\partial X}$. But this is not even.

Then (Brouwer) Every smooth $f: B^n \to B^n$ has a fixed point.

pf: Suppose $f$ without fixed points, so $f(x) \neq x$ for any $x \in B^n$. We construct a retraction $g: B^n \to \partial B^n$ contradicting the above thin.
If \( f(x) \neq x \), there is a ray pointing from \( f(x) \) to \( x \) that intersects \( \partial B^n \) once:

![Diagram](image)

Call this intersection point \( g(x) \). If \( x \in \partial B^n \) then let \( g(x) = x \), so \( g \mid_{\partial B^n} = \text{Id}_{\partial B^n} \).

Why is \( g \) smooth? Note \( x \in \text{span} \left( g(x) - f(x) \right) \), so \( g(x) - f(x) = t (x - f(x)) \). Moreover, \( x \) is between \( f(x) \) and \( g(x) \) so \( t \geq 1 \) and

\[
g(x) = tx + (1 - t)f(x)
\]

If \( t \) depends smoothly on \( x \), done. Take the norm of both sides (suppose WLOG \( \| g(x) \| = 1 \)) so

\[
0 = t^2 \| x - f(x) \|^2 + 2 t \langle x - f(x) \mid f(x) \rangle + \| f(x) \|^2 - 1
\]

This is quadratic in \( t \) with a unique positive root and hence depends smoothly on \( x \). \( \square \)

**Def:** A smooth family \( f_s : X \to Y \) parameterized by \( S \) is a smooth map \( F : X \times S \to Y \) s.t. \( f_s \in S, F(x,s) = f_s(x) \)
The (Transversality) Given \( F : X \times S \to Y \), with \( dS = \phi = dY \) suppose \( F, \partial F \pitchfork Z \subseteq Y \). Then for almost every \( s \) in \( S \), \( F_s \) and \( \partial F_s \pitchfork Z \).

Claim is that almost all vertical slices are transverse to \( Z \).

pf: Let \( W = F^{-1}(Z) \times S \).

Then \( W \) is a submanifold with boundary, and
\[
\partial W = W \cap (\partial X \times S)
\]

Need to show that \( \circ \ F_s \pitchfork Z \) when \( (\pi : X \times S \to S)_W \) has \( s \in S \) as a reg. value, and \( \circ \partial F_s \pitchfork Z \) when
\( \pi |_{\omega w} \) has \( s \) as a reg. value. Then by Sard, done.
Also note (1) \( \Rightarrow \) (2) using case of boundanless \( \alpha x \) and map \( \exists F. \ 2x \times s \Rightarrow y \).

So to show \( f_s \subseteq \mathbb{Z} \), let \( s \) be reg. value of \( \pi |_{\omega w} \).
Let \( x \in f_s^{-1} (z) \), \( z \in f_s (x) \). Transverse iff
\[ d(f_s)_x [T_x X] + T_{f_s (z)} (z) = T_{f_s (z)} Y \]

Now, given that \( dF (x, s) \) satisfies this condition, then
for any \( a \in T_{f_s (x)} Y \), \( \exists b \in T_{s, y} (x \times s) \) with
\[ dF_{s, y} (b) - a = e T_{f_s (x)} Z \]

We wish to show that \( \exists \nu \in T_s (X) \) with
\[ dF_s (\nu) - a = e T_{s, x} \]

Since \( T_{s, y} (x \times s) = (T_x X) \times (T_s S) \), write \( b = (w, e) \)
for \( w \in T_x X \) and \( e \in T_s S \). If \( e = 0 \), done since
\[ d(f_s) = d \neq d(pr_s) \]

If not, since \( d(pr_s) \) surjective, can find \((u, e)\) in \( (T_x X) \times (T_s S) \) with \( d(pr_s) (u, e) = e \). Moreover
\((u, e)\) can be taken to lie in \( TW \) since we assumed
\( s \) is reg. value of
Applying $dF_{(x,e)}(u,e)$ we get element of $T_{f,x}(Z)$ since $F: W \rightarrow Z$ by construction. Thus

$$T_{x,X} \exists V : = W - U \leftarrow \text{anything } \pi_1|_U \text{ sends to } e$$

from $b = (w,e)$

$$T_{(x,e)}(x,x)$$

is our required vector in $T_{x,X}$, as

$$dF_{(x,e)}(v) - a = dF_{(x,e)}[w,e] - (u,e] - a$$

$$= \left[ dF_{(x,e)}(w,e) - a \right] - dF_{x}(u,e)$$

both in $T_{f,x}(Z)$ by construction. \[\square\]