Manifolds I

Allee 150 @ UCSC
MW 11:40 AM - 1:15 PM
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Office hours

Focus of course: smooth manifolds.

A topological space $X$ is a smooth manifold if it is second countable, Hausdorff, and (smoothly) locally Euclidean: $\forall x \in X, \exists U \ni x$ with $\varphi: U \to \mathbb{R}^n$ smooth for some $n$.

Objects which locally behave according to our intuition in space. But what does smooth mean for abstract (loc. Euclid.) space?

Single variable calculus: Graph of a function $\varphi: \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}^2$. 

\[ f: \mathbb{R} \to \mathbb{R} \text{ in } \mathbb{R}^2. \]
Can linearize $f$ at $(a, f(a))$: $y = f(a) + f'(a)(x-a)$

This is affine (not linear) but can forget $f(a)$ and keep the slope component: $y = f'(a) \cdot x$ still captures linear approx. of $f$ at $x=a$.

For $f: \mathbb{R} \to \mathbb{R}$ as above, let $df_a: \mathbb{R} \to \mathbb{R}$, the derivative of $f$ at $x=a$, be $df_a(x) = f'(a) \cdot x$, linear map.

Higher dimensions? Consider $f: \mathbb{R} \to \mathbb{R}^m$, some parametrized curve.

Again throw away constant term to get

$$df_a(t) = t \cdot f'(a) = \begin{bmatrix} f'_1(a) \\ f'_2(a) \\ \vdots \\ f'_m(a) \end{bmatrix} t$$

Dually, for $f: \mathbb{R}^n \to \mathbb{R}$, linearization corresponds to the gradient.
\[ \nabla f(a) = \frac{\partial f}{\partial x_1}(a) + \ldots + \frac{\partial f}{\partial x_n}(a) \]

or by Taylor expansion \( f(x) = f(a) + \nabla f(x-a) + \ldots \) and by throwing away quadratic & higher order, get
\[
\begin{bmatrix}
\frac{\partial f}{\partial x_1}(a) \\
\vdots \\
\frac{\partial f}{\partial x_n}(a)
\end{bmatrix}
\]

So here \( df_a : \mathbb{R}^n \rightarrow \mathbb{R} \) linear approximation.

More generally for \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \), at \( a \in \mathbb{R}^n \) get linear transformation
\[
\begin{bmatrix}
\frac{\partial f}{\partial x_1}(a) & \ldots & \frac{\partial f}{\partial x_n}(a) \\
\vdots & \ddots & \vdots \\
\frac{\partial f}{\partial x_1}(a) & \ldots & \frac{\partial f}{\partial x_n}(a)
\end{bmatrix}
\]

where we drop constants to ensure linearity.

Def Given an open set \( U \subset \mathbb{R}^n \), consider \( f : U \rightarrow \mathbb{R}^m \) with \( a \in U \). We say \( f \) is differentiable at \( a \) if there exists a linear transformation \( T_a : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that
\[
\lim_{x \rightarrow 0} \frac{f(a+x) - f(a) - T(x)}{\|x\|} = 0
\]
Prop: If \( T_a \) exists, it is unique.

Proof: Let \( x = h e_i \), where \( e_i \) is the standard basis vector in \( \mathbb{R}^n \) and \( h \in \mathbb{R}^+ \). As \( h \to 0 \),

\[
\lim_{h \to 0^+} \frac{f(he_i + a) - f(a) - Ta(he_i)}{h} = 0 \text{ by def. of } T_a
\]

but otherwise

\[
= \lim_{h \to 0} \frac{f(he_i) - f(a)}{h} - \lim_{h \to 0} \frac{Ta(he_i)}{h}
\]

\[
= \frac{\partial f}{\partial x_i}(a) - Ta(e_i)
\]

\[
= 0 \text{ so } Ta(e_i) = \frac{\partial f}{\partial x_i}(a). \text{ Since } \frac{\partial f}{\partial x_i}(a)
\]

is uniquely determined by \( f \) for every \( i \) and any \( T_a \), is also uniquely determined.

We denote this unique \( T_a \) by \( df_a \) when it exists; it is the "best possible" linear approx. of \( f \) at \( a \).

Def: A function \( f: \mathbb{R}^n \to \mathbb{R}^m \) is smooth if

\[
\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \text{ exist and are continuous for all } i,j
\]

Prop: If \( f \) smooth, \( f \) is differentiable (\( df_a \) exists at \( a \)).
Prop. If $f : \mathbb{R}^n \to \mathbb{R}^m$ linear then $df_a = f.$

pf: Since $f$ smooth $df_a$ exists, so let $x, a \in \mathbb{R}^n.$

\[
\lim_{h \to 0} \frac{f(hx + a) - f(a) - df_a(hx)}{h} = \lim_{h \to 0} \frac{f(hx) - df_a(hx)}{h} = f(x) - df_a(x) = 0
\]

Since $x$ arbitrary, $f = df_a.$ \(\square\)

Another key fact: let $f, g$ be smooth with

$U \overset{g}{\to} V \overset{f}{\to} \mathbb{R}^m$

Prop (Chain Rule) $d(f \circ g)_a = df_{g(a)} \circ dg_a$ for $a \in U.$

Idea: Start with single variable:

$\mathbb{R} \overset{g}{\to} \mathbb{R} \overset{f}{\to} \mathbb{R}, \quad \frac{d}{dx} (f(g(x))) = f'(g(x)) \cdot g'(x)$

$\mathbb{R} \overset{g}{\to} \mathbb{R}^n \overset{f}{\to} \mathbb{R}, \quad \frac{d}{dx} (f(g(x)))$

$= \sum \frac{df}{dg} \cdot \frac{dg}{dx}$

$= \nabla f_{g(x)} \cdot g'(x)$

$= \begin{bmatrix} \frac{dg_1}{dx} & \ldots & \frac{dg_n}{dx} \end{bmatrix} \begin{bmatrix} \frac{df_1}{dg_1} & \ldots & \frac{df_m}{dg_n} \end{bmatrix}$
\[ d(f \circ g)(x) = df_x \cdot dg_x \]

In the general case, reduce to particular row/column of \( df_x \) and \( dg_x \), identical to \( \odot \) that is, \( \odot \) says how to multiply matrices.

Intuitively: doesn't matter if we linearize before or after composing functions.

**Def:** For \( U \subset \mathbb{R}^n \), \( V \subset \mathbb{R}^m \) open, \( f: U \to V \) is a **diffeomorphism** if it is bijective & smooth, and \( f^{-1} \) is also smooth.

**Ex:** \( f(x) = x \) Identity is diffeo, but \( g(x) = x^3 \) is smooth and bijective while \( g'(x) = 3x^2 \) is not smooth; \( d(g^{-1}) \) doesn't exist.

Given a diffeo \( f: U \to V \), \( f \circ f^{-1} = \text{Id}_V \) and \( f^{-1} \circ f = \text{Id}_U \) so by the chain rule,

\[
\begin{align*}
\text{d}(f \circ f^{-1}) &= df \circ df^{-1} = \text{Id}_V \\
\text{d}(f^{-1} \circ f) &= df^{-1} \circ df = \text{Id}_U 
\end{align*}
\]

So \( d(f^{-1}) f(a) = (df(a))^{-1} \) as matrices.
Conversely, if $f$ is a diffeomorphism, then $df_a$ is invertible for all $a$ in the domain of $f$.

Converse to (b)?

**Def.** For a smooth function $f: U \to \mathbb{R}^m$, $U$ open, $f$ is a **local diffeomorphism** at $a \in U$ if $f$ is a homeomorphism $V \ni a \to f(V)$ open in $\mathbb{R}^m$ and $f'_V$ is a diffeomorphism.

Note $f$ is a local diffeomorphism if $df_a$ is an isomorphism.

**Thm.** (Inverse Function Theorem) If $f: U \to \mathbb{R}^m$ is smooth and $df_a$ at $a$ is an isomorphism, then $f$ is a local diffeomorphism at $a$.

Some preliminaries: let $V$ be a vector space. Recall $V^* = \text{linear functionals on } V$, vector space. A **norm** on $V$ is a function $||\cdot||: V \to \mathbb{R}^+$ s.t.

1. $||0|| = 0$
2. $||x+y|| \leq ||x|| + ||y||$ for $x, y \in V$
3. $||x\lambda|| = ||x|| ||\lambda||$ for $x \in V$ and $\lambda \in \mathbb{R}$

If $V, W$ normed $V$-spaces, set norm on $\text{Hom}(V, W)$ by $||T|| = \sup_{||v||=1} ||Tv||_W$. 

$\text{Hom}(V, W)$ is a vector space. 

$(\text{Hom}(V, W), ||\cdot||)$ is a normed vector space. 

A function $f: U \to \mathbb{R}^m$ is **smooth** if each $f_i$ is smooth.
Prop Let \( V \) be finite dim'ed normed vector space \((V, \| \cdot \|)\). Then given \( f: [a, b] \to V \) differentiable, we may define \( S_a^b \int f \ dt \in V \), satisfying
\[
\begin{align*}
0 & \leq \int_a^b \| f(t) \| \ dt \\
& \leq \int_a^b \| f(t) \| V \ dt.
\end{align*}
\]
\[ S_a^b \int f \ dt = f(b) - f(a) \]

Prop Work in 'components'; for any \( \eta \in V^* \), note \( I(\eta) \) given by \( \eta \mapsto S_a^b \eta(f(t)) \ dt \in \mathbb{R} \)

is linear on \( V^* \), so lies in \( V^{**} \equiv V \). Then define \( S_a^b \int f \ dt = I \).

With this \( \circ \) is straightforward. For \( \circ \) take any \( \eta \in V^* \) with \( \| \eta \| = \sup_{\| v \| = 1} | \eta(v) | = 1 \) and \( I(\eta) = \| \eta \| \). This exists by
\[
(HW) \quad \| \eta \| = \sup_{\| v \| = 1} | \eta(v) | \\
\| \eta \| = 1
\]

Then \( \| S_a^b \int f \ dt \| \| \eta \| = I(\eta) = S_a^b \eta(f(t)) \ dt \)
\[
\leq S_a^b \| \eta(f(t)) \| \ dt \\
\leq S_a^b \| \eta \| \| f(t) \| \ dt
\]
pf: Let \( \gamma(t) = tx + (1-t)y \), and \( g(t) = \ell(\gamma(t)) \).

Then \( \ell(y) - \ell(x) = g(1) - g(0) = \int_0^1 \frac{dg}{dt} \, dt \)
\[ = \int_0^1 dg_t \left( \frac{\gamma'}{dt} \right) \, dt \]
\[ = \int_0^1 d\ell_{\gamma(t)} \circ dg_t \left( \gamma'(t) \right) \, dt \]
\[ = \int_0^1 d\ell_{\gamma(t)} (x-y) \, dt \]

So then
\[
\| \ell(x) - \ell(y) \| \leq \int_0^1 |d\ell_{\gamma(t)} (x-y)| \, dt
\]
\[
\leq C \| x-y \| \quad [3]
\]

Thm (Contraction Mapping) If \( (X,d) \) complete metric space, \( \ell: X \to X \), and \( 0 < C < 1 \) with \( d(\ell(x), \ell(y)) \leq C \cdot d(x,y) \) for all \( x, y \) then \( \ell \) has a unique fixed point.

Thm Suppose \( U \subset \mathbb{R}^n, p \in U \) with \( \ell: U \to \mathbb{R}^n \) smooth and \( d\ell_p \) bijective. Then \( \exists \tilde{U} \subseteq U \) open containing \( p \) so \( \ell \) is a local diffeomorphism at \( p \).

pf: First compose \( \ell \) with \( d\ell_p^{-1} \) to reduce to the case \( d\ell_p = \text{Id} \). Also WLOG \( p = 0 = \ell(0) \).
Let $\phi(\xi) = \xi - \xi(\xi), \text{Id} - f$. Then note that 
$\phi(0) = 0$ and $d\phi_{\xi} = 0 = \text{Id} - \text{Id}$. By continuity of entries of $d\phi$, we can find $\varepsilon > 0$ so that if $\|v\| < \varepsilon$, $\|d\phi_{\xi}v\| < \frac{1}{2}$.

Then $\|\phi(\xi)\| < \frac{\varepsilon}{2}$ for $\|\xi\| < \varepsilon$, i.e. we have 
$\phi: B_{\varepsilon} \rightarrow B_{\varepsilon}$.

Now for $\eta \in B_{\varepsilon/2}$ we show $f(\xi) = \eta$ has a unique solution in $B_{\varepsilon/2}$. We do this by iteration; approximating $\xi$ by the identity, and perturbing $\phi$.

That is, consider $\phi_{\eta}(\xi) = \eta + \xi - \xi(\xi)$ so that $\phi_{\eta}(\xi) = \xi$ iff $\xi(\xi) = \xi$, which is what we want.

Claims: ① $\phi_{\eta}: B_{\varepsilon/2}$ when $\|\eta\| \leq \frac{\varepsilon}{2}$.

Proof: $\|\phi_{\eta}(\xi)\| = \|\eta + \xi - \xi(\xi)\|$
$\leq \|\eta\| + \|\phi(\xi)\|$
$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

② $\|\phi_{\eta}(\xi) - \phi_{\eta}(\xi_2)\|$
$= \|\eta + \xi - \xi(\xi) - \eta - \xi_2 + \xi(\xi_2)\|$
$= \|\xi - \xi_2 - (\xi(\xi) - \xi(\xi_2))\|$
\[ \| \phi'(s) - \phi'(r') \| \]

But if \( s_1, s_2 \in B_\varepsilon \) then \( \phi(s_1), \phi(s_2) \in B_{\varepsilon/2} \)
so their difference is no greater than \( \varepsilon \) in length:
\[ \| \phi(s_1) - \phi(s_2) \| \leq \frac{\varepsilon}{2} \| s_1 - s_2 \| \]
Thus \( \phi \) is a contraction on \( \overline{B_\varepsilon} \), complete metric space. So \( \exists! \; \eta \) so \( \phi(\eta) = \eta \), and so \( f(\eta) = \gamma \) is unique soln.

Doing this \( \forall \gamma \in B_{\varepsilon/2} \), let \( \tilde{U} = B_\varepsilon \cap f^{-1}(B_{\varepsilon/2}) \)
so that in \( \tilde{U} \) we have \( g : f(\tilde{U}) \to \tilde{U} \) with \( f \circ g = \text{Id} \).

Now why is \( g \) smooth?

Take \( \gamma_1, \gamma_2 \in B_{\varepsilon/2} \) and \( s_1, s_2 \) with \( g(\gamma_1) = s_1 \).
By construction \( s_1 \) is a fixed point:
\[ \| g(\gamma_2) - g(\gamma_1) \| = \| s_2 - s_1 \| \]
\[ = \| y_2 + s_2 - f(s_2) - \gamma_1 - s_1 + f(s_1) \| \]
\[ \leq \| y_2 - \gamma_1 \| + \| f(s_2) - f(s_1) \| \]
\[ \leq \| y_2 - \gamma_1 \| + \frac{\varepsilon}{2} \| s_2 - s_1 \| \]
So \( \| s_2 - s_1 \| \leq 2 \| y_2 - \gamma_1 \| \) and \( g \) is Lipshitz.
To see that $g$ is differentiable, use the linear map $dF_p$.

$$
\| g(\gamma_2) - g(\gamma_1) - dF_p(\gamma_2 - \gamma_1) \|
= \| F(\gamma_2) - F(\gamma_1) - dF_p(\gamma_2 - \gamma_1) \|
\leq \| dF_p \| \cdot \| F(\gamma_2) - F(\gamma_1) - dF_p (\gamma_2 - \gamma_1) \| \quad (\star)
$$

Now $f$ smooth $\Rightarrow$ once differentiable so given $\varepsilon > 0$ (actually take $\frac{\varepsilon}{2\|dF_p\|}$) then $\exists \delta > 0$ so
if $\|\gamma_2 - \gamma_1\| < \delta$,

$$
\| f(\gamma_2) - f(\gamma_1) - dF_p (\gamma_2 - \gamma_1) \|
\leq \|\gamma_2 - \gamma_1\| \cdot \frac{\varepsilon}{2\|dF_p\|} \quad (**)
$$

So if $\|\gamma_2 - \gamma_1\| < \delta/2$ then since $\gamma_i = g(\delta_i)$, $\|\delta_2 - \delta_1\| < \delta$. Finally, then

$$
\| g(\gamma_2) - g(\gamma_1) - dF_p (\gamma_2 - \gamma_1) \|
\leq \| dF_p \| \cdot \| f(\gamma_2) - f(\gamma_1) - dF_p (\gamma_2 - \gamma_1) \| \text{ by } (\star)
\leq \frac{\varepsilon}{2} \text{ by } (**)
$$

So the derivative of $g$ at $p$ is $dF_p$.

Smoothness follows from smoothness of $f$ and the map $T \mapsto T'$ (Cramer’s Rule).