**Invariance of Morse Homology, Gelfand Theory**

Recall: In definition of Morse homology, need to choose smooth Morse function $f$, Riemannian metric $g$. The pair $(f, g)$ must be Morse-Smale. To recover cellular homology, want self-inducing.

Q: How common are Morse functions?

Ex: For $T^2 \subseteq \mathbb{R}^3$, consider $S^2$-family of embeddings

$$v \in S^2 \subseteq \mathbb{R}^3$$

Outside a "small" subset of $S^2$, all critical points are Morse.

Thus If $f: M \rightarrow \mathbb{R}$ smooth, $M \subseteq \mathbb{R}^N$ embedded, $\partial M$ thin then for almost all $(v; i) \in \mathbb{R}^N$, $f_v(x) = f(x) - i$; $v; x$ is Morse.

Pf: Consider $g = (\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_n}(x))$, so $df = g - a$.

Sand \Rightarrow Almost every $a \in \mathbb{R}^n$ is a regular value of $g$, i.e. $dg$ is an isomorphism. But $dg = \text{Hess}_a(\epsilon)$. \qed
(Genericity)

"Almost all" makes no sense in function spaces; no measure.

**Def**: Let $X$ be a top. space, $P(X)$ a statement for each $x \in X$ (which could be true or false). We say $P(x)$ is true for generic $x \in X$ if the set $\{ x \in X | P(x) \text{ true} \}$ contains a countable intersection of open dense sets.

**Ex**: Irrationals are generic in $\mathbb{R}$; complement of $\mathbb{Q}$ (which is dense)

Polynomials not generic in $C^\infty(\mathbb{R};\mathbb{R})$

We rephrase Morse condition as the transversality of a certain section of a vector bundle to the zero section.

**Thm**: (Transversality) Let $X, Y$ be separable Banach manifolds, $E \to Y \to Z$ a Banach space bundle, and $s: Y \times Z \to E$ a smooth section. Suppose that $s^{-1}(0) \neq \emptyset$.

1. $D_\gamma| s: T(Y \times Z) \to TE$ is surjective,
2. The restriction $D_\gamma| s: T_\gamma \to TE$ has finite kernel, ok.

Then for generic $y \in Y$, the set $\{ z \in Z | s(y, z) = 0 \}$ is a manifold of dimension ($\text{Ker } D_\gamma - \text{Coker } (D_\gamma)$), and $D_\gamma$ is surjective on $T_\gamma$.

With this we can show Morse functions are generic:
For $z$ closed in $\mathbb{R}^d$, $k \geq 2$ an integer, a generic $C^k$ func $f: \mathbb{Z} \to \mathbb{R}$ is Morse.

Let $Y = C^k(\mathbb{Z}, \mathbb{R})$, $\pi: Y \times \mathbb{Z} \to \mathbb{Z}$ is projection. Define $E = \pi^*(T^*\mathbb{Z}) \to Y \times \mathbb{Z}$, pullback of cotangent bundle, with canonical section $\sigma: Y \times \mathbb{Z} \to E$, $\sigma(f, z) = (df_z, f, z)$.

If $\sigma(f, z) = 0$, $f \in \mathbb{Z}$ arbitrary, and $v \in T_z(\mathbb{Z})$ of dim $\mathbb{Z}$ note that $z \in \text{Crit}(f)$, since $df = 0$. So $\sigma^{-1}(0) \cap \{f\} \times \mathbb{Z}$ corresponds to $\text{Crit}(f)$, and we must verify that they are non deg.

Claim: $\nabla \sigma = df_i(v) + \nabla v(df_i)$, $\nabla \sigma|_{T_z \mathbb{Z}}: T_z \mathbb{Z} \to T^*_z \mathbb{Z} \cong \text{Hess}(f)$

The two terms of comm. come from $d_z$ and $d_Y$ applied to the section $\sum \frac{\partial f}{\partial x_i} dx_i \in T^*_z(\mathbb{Z})$.

$d_z(df: \mathbb{Z} \to T^*_z \mathbb{Z}) = \sum \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j \cong \text{Hess}(f)$ at $z \in \text{Crit}(f)$ in the sense that it gives the $(i, j)$-th entry.

Specifically, need a map $T_z \mathbb{Z} \to T^*_z \mathbb{Z}$. Sections already map into $T^*$, so lets look at $\nabla_i \sigma$, fixed $i$. $\nabla_i \sigma = \nabla_i df(z)$.

$df(z)$ is in $\mathbb{R}^d$, apply $\nabla_i$ to get vector $(\frac{\partial^2 f}{\partial x_1 \partial x_i}(z), \frac{\partial^2 f}{\partial x_2 \partial x_i}(z), \ldots, \frac{\partial^2 f}{\partial x_d \partial x_i}(z))$.

$i$-th row Hes$(f)$. In vectors, $\sum \frac{\partial^2 f}{\partial x_j \partial x_i} e_j dx_i$ where $\{e_j\}$ span $E_z$.

For $dy$, along zero section we can apply standard total derivative. But note $\nabla_y$ linear $\Rightarrow d_0 y = \nabla_y$ so $\nabla y = df_i$. //
Now the Transversality Theorem applies: \( \forall \sigma: T(Y \times Z) \rightarrow T^*Z \) is surjective since \( T(C^k(Z, \mathbb{R})) = Y \) and we may choose any \( f, g \in TY \), and \( \forall \sigma: T^*_Z \rightarrow T^*_Y \) is Fredholm since finite dim. But \( \text{Hess}(\varepsilon) \) only surjective if \( \text{Crit} \# \text{p} = 0 \) (Ondey). \( \square \)

To show that a generic metric pairs \( w \) any Morse function to make a Morse-Smale pair, use same Transversality Theorem, \( w \)

\[ Y = \text{space of metrics} \]

\[ Z = \text{space of smooth functions} \]

\[ E \rightarrow Y \times Z \text{ bundle, } E((g, \tau)) = \nabla (-\varepsilon \cdot (w \cdot (TX))) \text{ vector fields} \]

\[ \xi((g, \tau)) = \nabla'^{-1} - \nabla \text{ always } \gamma \]

Rmk: Note also that Morse functions aren't generically self-indexing; critical values shouldn't generically coincide. But given \( f, f_i \) differ only by crossing critical values, homotopy can be effected locally near critical points by changing handle decomp.

Such functions \( w \) distinct critical values are called excellent. (Thom).
Q: To what extent does Morse homology $M^k$ depend on choices? Specifically, given any two Morse-Smale pairs $(f_0, g_0)$ and $(f_1, g_1)$, is there a 1-parameter family of Morse-Smale pairs between them?

Consider $(f_t, g_t)$ as function, metric on $[0,1] \times X$:

$$\text{Crit}(f_t)$$

What can happen to handles? Consider $M \times [0,1]$, 1-parameter family $(f, g)$. Then we have families $A_p$, $D_q$.

Consider intersection with level set: let $A_p := A_p \cap f^{-1}(t)$, $D_q := D_q \cap f^{-1}(t)$. Also $\text{ind}(p) = k$, $\text{ind}(q) = l$, so $\dim(A_p) = n-k+l$, $\dim(D_q) = l!$. Then $\dim A^+_p = n-k$, $\dim D^-_q = l$, both in $[0,1] \times f^{-1}(t)$.

Moreover, for any isotopy of $A^+_p$, $D^-_q$ in $f^{-1}(t) \times [0,1]$, we may extend to ambient isotopy in $f^{-1}(t) \times [0,1]$. That is, restrict to class of time-preserving isotopies, and in $S^3 \times M$, alter $\nabla f$ in $f^{-1}(t-\varepsilon, t+\varepsilon)$ to extend to $M$ and cut off. All this keeps $S^2(0,1)$ fixed.

This suffices to assume $A^+_p$ transverse to $D^-_q$, so need only count dimensions.
\[ \text{codim } A^*_p \cap D^+_q = k + n - l, \text{ so } \dim = l - k. \]

Case 1: If \( l < k \), \( A^*_p \cap D^+_q = \emptyset \) generically in \( 1 \)-parameter families.

Ex: In a 3-diag,

Attaching sphere doesn't intersect belt sphere 'in generic homotopies.'

Con: Let \( M' = (M \cup \{H_a\}) \cup H_b \) with \( H_a, H_b \) handles of index \( a \leq b \). Then \( M' \) can be obtained by attaching \( H_a \) and then \( H_b \). That is, handles can be attached in weakly increasing order of index.

Case 2: \( A^*_p \cap D^+_q = \text{points}, \ l = k \). So at isolated times \( st(\alpha) \) could have \( A^*_p \cap D^+_q \). These are called \underline{handleslides}.

Slide attaching region over belt region.
In gradient flow picture, change metric in $f^{-1}(\text{null of critical value})$.

At critical time, fails to be Morse-Smale.

Case 3: $l-k=1$, $l=k+1$

Consider the following local picture: $f(x,t) = x^3 - tx$.

Two critical points at $t=1$ merge and cancel at $t=0$, so at $t=-1$ no critical points. Note that at $t=0$, $f_t$ not Morse.

Ex: rotationally symmetric sphere in $\mathbb{R}^3$:

$\mathbf{v} = (1,0,0)$

$\mathbf{CM}(M, f, g) = \{a, b, c, d, e, f\}$
For 1-parameter family $f(t)$, can track paths of critical values

This is called a Cerf graphic. Note that at $t = \tau$, we have a "birth" of two new critical points. In reverse, a pair of critical points cancels: a "death".

In a 3-mfld: Consider $(1-h)$, $(2-h)$.

"Convertible sunroof": retract back down to cancel handle

Worse singularities are possible, but this is all we need for Morse theory.

Theorem (Cerf) In a generic family of Morse functions, all critical values are distinct except for finitely many times when crit. values may cross.
Thm (Thom–Mather) A generic 1-parameter family of functions is generically Morse, outside finitely many times when birth–deaths occur.

Thm (Kirby) In a generic $\mathbb{R}^2$-fan of pairs $(F_t, g_t)$, all are Morse–Smale except at finitely many times when handleslides or birth–deaths occur.

Cor. Any two handle decompositions of $M$ are related by a sequence of handleslides, cancellations, and isotopy of attaching maps.

More generally: Can think of homotopy $H$ in terms of the vector field induced on $M \times [0,1]$ by $-\nabla f_t$ on each slice. $H$ counts index 0 flow lines asymptotic to critical points (necessarily of the same index!), i.e. map $\mathcal{D}: \mathbb{R} \to M \times [0,1]$ s.t.

1. $\mathcal{D}_t \left( \frac{d\mathcal{D}}{dt} \right) \geq 0$
2. $\lim_{s \to -\infty} \mathcal{D}(s) \in \text{Crit}(f_0), \lim_{s \to \infty} \mathcal{D}(s) \in \text{Crit}(f_1)$

Continuation map argument works for isotopies $\{f_t\}$ as well since these don't affect #/index of critical points.

Arbitrary homotopies? By Cerf, we only consider cases 1, 2, 3.
(Changing critical values) Doesn't affect $|\text{Crit}(f)|$, homotopy can be supported in neighborhoods of critical points.

(Handleslides) Consider attaching 2-h to $(1-h)U(0-h)$

Note homology class of $[\mathcal{D}(M) \cap D_q] \in H_1(M, 0-h)$ is the image of $q$ under differential. So here we get $\varphi(q) = b, \varphi(p) = a$.

Slide $H_p$ over $H_q$: isotope attaching region of $H_q$ over belt region of $H_p$.

Now $\varphi(p) = a$, but $\varphi(q) = a+b$. Taking change of basis $\langle a, b \rangle \mapsto \langle a, atb \rangle$, differential is same.

In fact, handleslides realize arbitrary changes of basis in $C_\ast(X, f, g)$ when $\ast > 0$.

(Reaching-Deaths) Note that pair of crit. points differs in index by 1. Want to show that $\varphi$ maps one to the other. Recall that $\dim(A^+_{a, b} \cap D_f^+) = 1$, so $A^+_{a, b} \cap D_f^+ = 2$. Both transverse to every $X \times S^3$, we get trajectory from $p$ to $q$. 
After accounting for these changes, suppose to show

Theorem: Given Morse-Smale pairs \((f_i, g_i)\) there is a canonical isomorphism

\[ \Phi^0 : \mathcal{H}_*(M, f, g) \to \mathcal{H}_*(M, f, g) \]

such that \(\Phi^0 \circ \Phi^0 = \text{Id}\), \(\Phi^0 \circ \Phi^1 = (\Phi^0)^{-1}\), and if \((f_2, g_2)\) Morse-Smale, \(\Phi^0 \circ \Phi^1 \circ \Phi^0 = \Phi^0 \circ \Phi^1 \circ \Phi^0 \).

Want to formulate something that works for infinite dimensions as well.

Let \((f_0, g_0), (f_i, g_i)\) be Morse-Smale pairs, \((C^i, \partial_i)\) be Morse complexes. For \(I\) generic path of Morse-Smale pairs there is a vector field on \([0,1] \times X\) given by

\[ V := \frac{\partial}{\partial t} (V_{f_0} - \frac{1}{2} t) \partial_t + V_t \]

where \(t \in [0,1]\) and \(V_{f_0} = -\nabla f_0\), so \(\partial_t\)-component nonzero on \((0,1)\). Then \(V_t\) has zeros at \(\text{Crit}(f_i)\), ascending/descending manifolds which are \(M\) for generic \(I\). Using \(V\) we may define \(M(p,q)\) as usual for \(p \in \text{Crit}(f_0), q \in \text{Crit}(f_i)\) also zeroes of \(V\).

We consider maps \(\gamma : \mathbb{R} \to [0,1] \times X\) st.

\[ \gamma = V(\gamma(t)) \]

\[ \lim_{s \to -\infty} \gamma(s) \in \text{Crit}(f_0), \lim_{s \to \infty} \gamma(s) \in \text{Crit}(f_i) \]
Then we have

$$\Phi^{0}_p : C^\infty(M, f, g) \to C^\infty(M, f, g)$$

where

$$\Phi^{0}_p (\rho) = \sum_{q \in \text{Crit}_0^f} \text{deg}_q \cdot (f, g) q.$$ 

$\Phi^{0}_p$ is called a continuation map.

Then given Morse-Smale pairs $(f, g)$, there is a canonical isomorphism

$$\Phi^{0}_* : H_* (M, f_0, g_0) \to H_* (M, f, g)$$

such that $\Phi^{0}_* = \text{Id}$, $\Phi^{0}_* = (\Phi^{0}_*)^{-1}$, and if $(f_2, g_2)$ Morse-Smale, $\Phi^{0}_* \Phi^{0}_2 = \Phi^{0}_2$. 