Manifolds II
Math 209, MW 9:50 - 11:25 AM
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Goal: Introduction to integration on manifolds, cohomology.

Consider \( \mathcal{O} \subset \mathbb{R}^3 \) compact. If \( V \) is smooth vector field on \( \mathcal{O} \), natural to ask whether \( V \) is conservative, i.e. \( V = \nabla \Phi \) for some \( \Phi: \mathcal{O} \to \mathbb{R} \).

A necessary condition certainly that \( \text{curl}(V) = 0 \) as \( \text{curl} \circ \nabla = 0 \). But this is not sufficient.

Locally, in \( \mathcal{O} \) at least, we can find an antiderivative for \( V \) by setting some initial value, and integrating along paths \( \gamma: [0,1] \to \mathcal{O}, \gamma(t) = x \) ("Gradient Theorem"),

\[
\Phi(x) = \int_{\gamma} \langle V, \gamma' \rangle
\]

As long as \( \text{curl}(V) = 0 \), one can define an antiderivative by attempting to patch these functions into a globally defined \( \Phi: \mathcal{O} \to \mathbb{R} \) runs into issues if \( \exists \) loops in \( \mathcal{O} \) not contractible to points, i.e. \( \pi_1(\mathcal{O}) \neq 0 \). So integration/differentiation see topology of \( \mathcal{O} \)!
But in order to extend this, need framework for doing calculus (integration) on manifolds. Vector fields can be integrated against paths/loops, get $\text{Hom}(\mathbb{H}, (\mathbb{S}^1, \mathbb{R}))$, the cohomology of $\mathbb{S}^1$. For higher dimensional information, we need objects to integrate against higher-dimensional cycles.

As usual, start o/ linear model.

**Linear Algebra**
Let $K$ be a field (for this construction any commutative ring with unit will do).

**Def** An associative $K$-algebra $A$ is a $K$-module equipped with an associative $K$-linear product $A \times A \to A$, denoted $(a, b) \mapsto a \cdot b$. $A$ is unital if $1 \in A$ such that $1 \cdot a = a \cdot 1 \forall a \in A$.

We say a $K$-algebra is graded if there is a given decomposition $A = \bigoplus_{n \in \mathbb{Z}} A^n$ into $K$-submodules $A^n$ with $A^n \cdot A^m \subseteq A^{n+m}$. The grading of $a \in A^n$ is denoted $|a| = n$. We say $a$ is homogeneous of degree $n$.

A 2-sided homogeneous ideal in $A^*$ is a subset $I$ such that if $x \in I$ and $x = \sum x_n \in A^n$ with $|x_n| = n$, then each $x_n \in I$. Any $I \subset A^n$ generates a homogeneous ideal.
Examples: 1. Polynomial rings $\mathbb{K}[x_1, \ldots, x_n]$. Homogeneous ideals are e.g. $\langle x_i^2 \rangle$, or $\langle x_1^2, x_1 x_2, x_3^5 \rangle$.

2. Tensor algebras: if $V$ is a $\mathbb{K}$-module, its tensor algebra
   $$T^n V := \bigoplus_{n \geq 0} V \otimes_n$$
   is a unital graded $\mathbb{K}$-algebra. Product is juxtaposition
   $$V_1 \otimes \cdots \otimes V_n \quad (V_{m_1} \otimes \cdots \otimes V_{m_n}) = V_1 \otimes \cdots \otimes V_{m_1 + \cdots + m_n}$$
   and for any $d \in \mathbb{Z}$, grading on $T^n V$ so $V \otimes_n$ has degree $nd$ for all $n$. Above product extends linearly to all of $T^n V$.

This algebra has the following universal property: there is a map of $\mathbb{K}$-modules $\iota: V \rightarrow T^n V$ such that for any graded, associative unital $\mathbb{K}$-algebra $A$ and any $\mathbb{K}$-linear map $f: V \rightarrow A^d$, there is a unique extension to $\tilde{f}$ such that $f = \tilde{f} \circ \iota$.

$$
\begin{array}{c}
\uparrow \\
T^n V \xrightarrow{\tilde{f}} A^* \\
V \xrightarrow{f} A^d
\end{array}
$$

The utility of this definition is that it provides a way to construct graded unital associative algebras as quotients of $T^n A$ by homogeneous ideals. In fact any graded unital assoc. $\mathbb{K}$-algebra generated by a set $S$ of elements of all the same degree $d$ is a quotient of $T^n(\mathbb{K}[S])$ by a
homogeneous ideal.

The Exterior Algebra

We now construct the appropriate generalization of vector fields that can be integrated against higher chain objects.

Def: A graded algebra $A^*$ is graded commutative if for $a, b \in A^*$, $b \cdot a = (-1)^{\|a\|\|b\|} a \cdot b$ for homogeneous $a, b$.

The exterior algebra on a $K$-module $V$

\[ \Lambda^* (V) = \bigoplus_{n \geq 0} \Lambda^n (V) \]

is a graded-commutative, associative $K$-algebra with a map $i : V \to \Lambda^1 (V)$ satisfying the following universal property: for any graded comm. assoc. $K$-algebra $A^*$ and $K$-linear map $f : V \to A^1$, $\exists !$ extension $\tilde{f} : \Lambda^* (V) \to A^*$ such that $\tilde{f} = f \circ i$.

Check: any two realizations of $\Lambda^* (V)$ are related by unique isomorphism. So suffices to show existence of some model.

Consider the quotient $T^* (V) / I$ where $I$ is the 2-sided homogeneous ideal generated by elements of the form $v \cdot v$ for $v \in V$. The product in $T^* V$ descends to a product in $\Lambda^* (V)$, we write $x \Lambda y$. 
In low degree: $\Lambda^0 = k$
$\Lambda^1 = V$ using map $\iota: V \rightarrow T^1(V)$

Note $x \otimes y + y \otimes x = 0$, since
$x \otimes y - y \otimes x = (x + y) \otimes (x + y) - x \otimes x - y \otimes y \in I$

Thus the product $\Lambda$ is graded-commutative. Moreover, there is a map $\iota: V \rightarrow \Lambda^0(V)$, the identity.

Prop $\Lambda^0(V)$ satisfies the universal property of the exterior algebra.

pf: Suppose $f: V \rightarrow A^1$ is a $k$-linear map. Consider the extension (defined on simple tensors, and extended linearly)
$\tilde{f}^0: T^0(V) \rightarrow A^1$
$\tilde{f}^0(v_1 \wedge \cdots \wedge v_n) = f(v_1) \cdots f(v_n)$

This clearly satisfies $\tilde{f}^0 \circ \iota = f$. To see uniqueness, if $\tilde{f}'$ is any other such lift, it must agree with $f$ and $\tilde{f}'$ on $\Lambda^0(V)$ and be a homomorphism of $k$-modules, so
$\tilde{f}'(v_1 \wedge \cdots \wedge v_n) = f'(v_1) \wedge \cdots \wedge f'(v_n) = \tilde{f}'(v_1 \wedge \cdots \wedge v_n)$.

This construction is also functorial: if $L: V \rightarrow W$ is a map of $k$-modules, it induces an obvious map of exterior algebras on simple elements of degree $n$,
$L^0(v_1 \wedge \cdots \wedge v_k) = L(v_1) \wedge \cdots \wedge L(v_k)$

Check this is well-defined.
One can also think of the exterior algebra as a submodule of the tensor algebra, as follows. This is helpful in calculations.

let $\text{alt}_n : T^n(V) \to T^n(V)$ be the linear map

$$\text{alt}_n (v_1 \otimes \cdots \otimes v_n) = \sum_{\sigma \in S_n} \text{sign} (\sigma) \, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

so that $\text{alt}_n \circ \text{alt}_n = n! \, \text{alt}_n$. We call the submodule $\text{im} (\text{alt}_n)$ the space of alternating tensors. Also define $\Lambda$ the wedge product

$$\Lambda^m(V) \otimes \Lambda^n(V) \to \Lambda^{m+n}(V), \quad \lambda_1 \wedge \lambda_2 := \frac{1}{m! \, n!} \, \text{alt}_{m+n} (\lambda_1 \otimes \lambda_2)$$

where the coeff. compensates for stabilizer of $\lambda_1 \otimes \lambda_2$. One can restrict to $\sum_{m+n} \Lambda^m(V) \Lambda^n(V)$ to remove this factor if $K$ does not contain $\Omega$.

The wedge product here is again associative, graded-commutative and by the universal property of $\Lambda^* V$, there is a unique map of graded unital algebras $q : \Lambda^* V \to A^* V$ that extends $V \to \Lambda^1(V)$.

**Bases, Dimension of $\Lambda^*(V)$**

Suppose $V$ is a free $K$-module, and $\{ e_i \}$ are a basis. Then $\{ e_i \otimes \cdots \otimes e_i \}$ are a basis for $T^n (V)$, and it is straightforward to check that their images $e_i \wedge \cdots \wedge e_i$ will span $\Lambda^n (V)$. WLOG, all $i_j$ must be distinct, and up to sign we may assume $i_1 < \cdots < i_n$. 
Prop. The elements \( \{ e_{i_1}, \ldots, e_{i_n} \} \) for \( i_1 < \cdots < i_n \) are a basis for \( \Lambda^n(V) \).

Cor. For \( V \) a free \( \mathbb{k} \)-module, the map \( \varphi : \Lambda^*V \to \Lambda^*V \) is an isomorphism. When \( V \) is finite dimensional over \( \mathbb{k} \), we have
\[
\dim_{\mathbb{k}}(\Lambda^n(V)) = \binom{d}{n} \quad d = \dim_{\mathbb{k}}(V)
\]

Examples

1. Suppose \( V = \mathbb{R}^2 \), \( \mathbb{R} \)-vector space. Then \( \Lambda^0 = \mathbb{R} \), \( \Lambda^1 = V \), and \( \Lambda^2 = \mathbb{R} \), where the first isomorphism is canonical (using the defn. of the exterior algebra) but the second is not.

Let \( A \in M^{2\times 2}(\mathbb{R}) \), \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) with \( a, b, c, d \in \mathbb{R} \). Then \( A \) acts naturally on \( \Lambda^2(V) \); if \( e_1 = (1,0) \) and \( e_2 = (0,1) \), then \( e_1 \wedge e_2 \) is an induced basis for \( \Lambda^2 \) and
\[
A(e_1 \wedge e_2) := A(e_1) \wedge A(e_2)
\]
\[
= (a_{11}, a_{12}) \wedge (a_{21}, a_{22})
\]
\[
= (a_{11} e_1 + a_{12} e_2) \wedge (a_{21} e_1 + a_{22} e_2)
\]
\[
= \left( a_{11} e_1 + a_{12} e_2 \right) \wedge \left( a_{21} e_1 + a_{22} e_2 \right)
\]
\[
+ \left( a_{11} a_{21} - a_{12} a_{22} \right) e_1 \wedge e_2
\]
\[
+ \left( a_{12} a_{21} - a_{11} a_{22} \right) e_2 \wedge e_1
\]
\[
= (aq_{21} - a_{21} q_{11}) e_1 \wedge e_2
\]
\[
+ (aq_{22} - a_{22} q_{11}) e_1 \wedge e_2
\]
\[
+ (aq_{12} - a_{12} q_{11}) e_2 \wedge e_1
\]
\[
+ (aq_{11} - a_{11} q_{11}) e_1 \wedge e_2
\]

\begin{align*}
  &+ (q_{12} q_{22} - q_{22} q_{12}) e_2 \otimes e_2 \\
  &= (q_{11} q_{22} - q_{12} q_{21}) e_1 \otimes e_2 \\
  &- (q_{11} q_{22} - q_{12} q_{21}) e_2 \otimes e_1 \\
  &= \det A \ e_1 \wedge e_2
\end{align*}

This is a key property of the exterior algebra. In fact one can interpret $\Lambda^2$ as unit of area (HW). Note that if we changed basis in $V$ using a transformation with determinant 1, "area" stays constant. This will allow us to define integrals over higher dual (smooth) cycles.

**Duality**

Formalizing idea of $\Lambda^2 \leftrightarrow \text{area}$.

Suppose $V$ is a free module, basis $\{e_i\}$. Recall that the dual to $V$, denoted $V^*$, consists of the set of $K$-linear homomorphisms $\text{Hom}_K(V, K)$. This has a natural $K$-mod structure.

We say $\{f_i\} \subseteq V^*$ is a dual basis to $\{e_i\}$ if $f_i(e_j) = \delta_{ij}$, Kronecker delta.

There is a natural isomorphism $F^* : \Lambda^*(V^*) \rightarrow (\Lambda^*(V))^*$ as follows. We have induced bases $\{f_i, \ldots, f_{ik}\}$ and $\{e_i, \ldots, e_{ik}\}$ for $\Lambda^k(V^*)$ and $\Lambda^k(V)$ respectively.
So we define $F^k$ on those basis vectors as

$$F^k(e_i, \ldots, e_i) = (e_i, \ldots, e_i)$$

meaning $(F^k(e_i, \ldots, e_i), e_i, \ldots, e_i) = 1$, and evaluates as 0 on rest of basis. Then extend linearly.

**Exercise:** Show that $F^k$ is independent of the choice of $e_i$.

One further observation is necessary. If $L : V \rightarrow W$ is a map of $k$-modules, there is a natural dual map

$L^* : W^* \rightarrow V^*$, $L^*(\varphi) = \varphi \circ L$. This extends to a map of

$\Lambda^k(W^*) \rightarrow \Lambda^k(V^*)$ via

$L^*(f_i, \ldots, f_i) = L^*(f_i) \wedge \ldots \wedge L^*(f_i)$

which is an algebra homomorphism. Note that if $L_1, L_2$ are as above, $(L_1 \circ L_2)^* = L_2^* \circ L_1^*$.

Now suppose $V = W$, and take $d = \dim(V)$. Using formula for $\dim(\Lambda^k(V))$, note $\Lambda^d(V)$ is $1$-dim. So induced map

$L^* : \Lambda^d(V) \rightarrow \Lambda^d(V)$ is multiplication by constant $\lambda \in k$. We claim $\lambda = \det(A)$ for any choice of basis $B : k^d \rightarrow V$.

For $T \in \Lambda^d(V)$, $L^* B^* (\det) = \lambda B^* (\det)$ so

$$(B^*)^{-1} L^* B^* (\det) = \lambda (B^*)^{-1} B^* (\det) = \lambda (\det)$$

and so $(BLB^*)^* (\det) = \lambda \cdot \det$, in $k^d$ where $d$ makes sense.
But evaluating both sides on basis \( \{ e_i \} \) determined by \( B \) (i.e. checking in \( \mathbb{K}^d \) as in \( V = \mathbb{R}^2 \) example) it is clear that \( \lambda = \det(L) \). Thus, if \( L \) is linear, \( \Lambda^d(T) \) for any \( T \in \Lambda^d(V) \) is simply \( \lambda \det(L) \) for \( \lambda \in \mathbb{K} \). Dually, on \( \Lambda^d(V^*) \), \( \Lambda^d \cdot f_i \Lambda^d \cdot f_{i'j} = \det(L) \cdot f_i \Lambda^d \cdot f_{i'j} \).

**Upshot:** suppose that \( SC \cdot M^n \) is a \( k \)-dim'l submanifold of \( M^n \) smooth. At \( x \in M \), let \( \varphi : \mathbb{R}^n \rightarrow M \) be local coord. centered at \( x \).

At each \( x \in M \), \( T_x S \) is a 2-dim'l subspace of \( T_x M \); think of as \( d \varphi \cdot (T_x S) \subseteq \mathbb{R}^3 \). Can "integrate" against some alternating tensor \( T \in \Lambda^2(\mathbb{R}^3) \). For different choices of coordinates, result differs by determinant:

\[
T(\cdot d(\varphi_2 \circ \varphi_1) \cdot d \varphi_1(T_x S)) = \det(\varphi_2 \circ \varphi_1) T(d \varphi_1(T_x S))
\]

But if we integrate, answer is unchanged (e.g. \( u \)-sub in \( \mathbb{R} \), \( \int_{a}^{b} \cdot d(u(x)) \cdot u'(x) \cdot dx \)).