Vector Bundles
Recall: definition of general vector bundles.

Let $M$ be a smooth manifold. A $C^\infty$ $\mathbb{R}$-vector bundle $(\mathcal{V}, \pi, \tau, \cdot)$ is a smooth manifold $\mathcal{V}$, a smooth surjective map $\pi: \mathcal{V} \to M$, and for each $x \in M$, maps

$$\tau_x: \pi^{-1}(x) \times \pi^{-1}(x) \to \pi^{-1}(x)$$

$$\cdot_x: \mathbb{R} \times \pi^{-1}(x) \to \pi^{-1}(x)$$

giving $\pi^{-1}(x)$ the structure of a finite-dimensional $\mathbb{R}$-vector space.

We also require that every $x \in \mathcal{B}$ has a neighborhood $U_x$ s.t.

exists a diffeom. $\tau: \pi^{-1}(U_x) \to U_x \times \pi^{-1}(x)$ called a local trivialization such that $\pi^{\tau}(U_x) = \pi\tau \circ \tau$, projection to the second coordinate, and $\tau$ restricted to any fiber is a linear isomorphism.

We could just as easily require that fibers be complex vector spaces and obtain smooth $\mathbb{C}$-vector bundles. If $\mathbb{B}$ happens to be a complex manifold, we could ask that the local trivializations be biholomorphic to obtain holomorphic vector bundles.

These objects are ubiquitous in topology and geometry. Some occur canonically (e.g. tangent bundle), others are harder to see.

Ex: $\mathbb{CP}^n \setminus \{1, 0, \ldots, 0\}$ is the total space of a complex line bundle over $\mathbb{CP}^n$.\"
To see this, note that the locus of points $\{0; z_1; \ldots ; z_n\}$ can be identified with $\mathbb{CP}^n$. This is the zero section, and since these are homogeneous coordinates, at least one other $z_i$ is nonzero. Fixing these $z_i$, the points $\{z_1; z_2; \ldots ; z_n\}$ are the $\mathbb{C}$-fiber of this rank $n$ bundle.

The idea of transition functions determining bundles is a crucial perspective and tool for understanding their structure.

**Ex:** (Mobius band) We use an alternate definition of the (infinite) Mobius band than given previously. Consider $S^1 \subseteq \mathbb{C}$ the unit circle and define

$$M = \{ (e^{i \theta}; r e^{i \phi}) | (e^{i \phi})^2 \in \mathbb{R} \times \mathbb{R} \subseteq \mathbb{R}^2 \}$$

We attempt to calculate the transition functions in local trivializations of $S^1$.

Consider $U = S^1 \setminus S + 3$, $W = S^1 \setminus S - 3$. The open subsets of $\mathbb{C}$ given by $\Pi^1(U), \Pi^1(V)$ are given by

$$\Pi^1(U) = \{ (e^{i \theta}; r e^{i \phi}) | \theta \in (0, 2 \pi) \}$$

$$\Pi^1(V) = \{ (e^{i \theta}; r e^{i \phi}) | \theta \in (\pi, 3 \pi) \}$$

We then have diffeomorphisms:

$$\tau_U: \Pi^1(U) \to U \times \mathbb{R}, \ (e^{i \theta}; re^{i \phi}) \mapsto (e^{i \theta}, r)$$

$$\tau_V: \Pi^1(V) \to V \times \mathbb{R}, \ (e^{i \theta}; re^{i \phi}) \mapsto (e^{i \theta}, r)$$

and $\tau_U, \tau_V$ identical formula on $\Pi^1(V)$.

Note that these are...
well-defined blc of domain of $\Theta$. These trivializations give rise to transition functions. Next, have to check that they're compatible.

First note $U \cap V = \{ (e^{i\theta}, re^{i\theta}) \mid \theta \in (0, \pi) \cup (\pi, 2\pi) \}$, so we compute transition in 2 different subsets.

If $\Theta \in (0, \pi)$ then
\[
\tau_v \circ \tau_u^{-1}(e^{i\theta}, r) = \tau_u(e^{i\theta}, re^{i\theta}) = \begin{cases} 
\tau_u(e^{i\theta}, re^{i\theta}) & \text{in domain of } \tau_u \\
(e^{i\theta}, -re^{i\theta}) & \text{of } \tau_u
\end{cases}
\]

So transition function $\phi_{uv}$ is $\forall r \mapsto -r^2 \in GL(1; \mathbb{R})$.

OTOH if $\Theta \in (\pi, 2\pi)$ then
\[
\tau_u \circ \tau_v^{-1}(e^{i\theta}, r) = \tau_u(e^{i\theta}, re^{i\theta}) = (e^{i\theta}, e^{i\theta}r)
\]

So transition $\phi_{vu}$ is $\forall r \mapsto +1 \in GL(1; \mathbb{R})$.

Thinking of vector bundles as families of vector spaces, operations on vector spaces generalize to operations on vector bundles.

**Ex:** (Hw) Dual bundle $E \to E^\star$ Conjugate $\bar{E}$

For $E, F$ vector bundles, can form tensor product $E \otimes F$, direct sum, etc.

Key example: If $E$ graded, can form $\Lambda^n E$ exterior algebra bundle!
Another key construction: pullbacks of vector bundles. If \( f: X \to Y \) is smooth, and \( V \to Y \) a vector bundle, let \( f^*(V) \to X \) be the pullback of \( V \), so the fiber over \( x \in X \) is the fiber of \( V \) over \( f(x) \): (full def. in HW)

**Ex:** \((Y = pt) \quad \text{Trivial bundle} \quad \downarrow \quad \downarrow \quad u \quad \quad \downarrow \quad f \quad \downarrow \quad \ast \)

Now as a consequence of defn, every vector bundle has local sections, i.e. \( \forall p \in M \exists \psi \in U_p \times \mathbb{R}^k \) trivializing neighborhood, \( \sigma_p : U_p \to V \) such that \( \sigma_i \circ \tau = \text{Id}_{U_p} \).

Fundamentally important question: when does a collection of local sections \( \{ \sigma_i \} \) glue up to give a global section?

In terms of transition functions: suppose we have trivial nhds \( \tilde{U} \cong U \times \mathbb{R}^k \), \( \tilde{V} \cong V \times \mathbb{R}^k \) for \( U, V \) coord. charts in base \( B \) of \( V \to B \).
Then local sections $\sigma_u$, $\sigma_v$ glue up to give global section iff $\sigma_{uv}(\sigma_u) = \sigma_v$, or $\sigma_{uv}(\sigma_v) = \sigma_u$. That is, transition functions should identify local sections with one another.

Anatomy of a rank $k$ vector bundle $V \to M$:

A section of a bundle is a map $\sigma : M \to V$ such that $\pi \circ \sigma = \text{Id}_M$.

Every vector bundle admits a section, the zero section, which sends $x \in M$ to the origin in $\pi^{-1}(x)$. Some vector bundles have many sections (e.g., the trivial vector bundle) but others have none other than the zero section.

**Ex:** Consider the bundle over $\mathbb{C}P^3$ given by

$$L = \{ (z, z) \in \mathbb{C}^2 \times \mathbb{C}^3 \mid z \in \mathbb{C}^3 \}$$

analogue of Mobius band but over $\mathbb{C}$.

In std local trivial $U_i = \{ z_i \in \mathbb{C}^3 \mid z_i \neq 0 \}^3$ we may trivialize $L$ using $\pi^{-1}(U_i) = \tilde{U}_i$. 

\( \tau_1 : \tilde{U}_1 \to \mathbb{C} \times \mathbb{C}, ([1, \frac{3i}{2}, 1], (z, \bar{z})) \mapsto (\frac{3i}{2}, \bar{z}) \)

Let \( v = \frac{3i}{2}, w = \frac{3i}{2} \), so \( \tau_1^{-1}(v, z, \bar{z}) = ([1; v], (z, \bar{z}, v)) = (\frac{3i}{2}, \bar{z}, v) = (v, z, \bar{z}) \)

So \( \tau_1 \circ \tau_1^{-1}(v, z, \bar{z}) = \tau_1([1; v], (z, \bar{z}, v)) = (\frac{3i}{2}, \bar{z}, v) = (v, z, \bar{z}) \)

Similarly \( \tau_2 : \tilde{U}_2 \to \mathbb{C} \times \mathbb{C}, ([\frac{3i}{2}, 1], (z, \bar{z})) \mapsto (\frac{3i}{2}, \bar{z}) \)

Thus any sections locally look like functions \( y_1(v) \) in \( U_1 \) and \( y_2(w) \) in \( U_2 \). Computing transition functions, we see

\[
\tau_2 \circ \tau_1^{-1}(v, z, \bar{z}) = \tau_2([1; v], (z, \bar{z}, v)) \\
= \tau_2(\frac{1}{v}, 1, (z, \bar{z}, v)) \\
= (\frac{1}{v}, z, \bar{z})
\]

Thus \( z, v \mapsto z, \) i.e. transition is \( y_1(v) \mapsto w y_2(v) \). These must match to give a global section, so \( y_1(v) = w y_2(v) \). But as \( w \to \infty \), then \( y_1 \) blows up! No holom. sections.

OTOH smooth v.b. always admit global nonzero (not nec. nonvanishing!) sections (HW).
Isomorphisms of Vector Bundles, Homotopy, Extra Structures

We now consider two related questions: ① when are two vector bundles isomorphic? ② How can we put extra structures on vector spaces (i.e. metrics, norms, etc) onto vector bundles?

We first require an important proposition.

Prop For vector bundles over base $V \times [0,1]$, there is canonical bundle isomorphism $V \times \{0\} \cong V \times \{1\}$.

Lemma Given $\{U_i\}$ open cover of a paracompact space, $\exists$ an open cover $\{V_i\}$ countable such that each $V_i$ is contained in some $U_\alpha$, and a partition of unity $\{\rho_i\}$ subordinate to the $\{V_i\}$, i.e. $\text{supp}(\rho_i) \subseteq V_i$.

pf: HW.

Lemma If $V \rightarrow B \times [0,1]$ is a vector bundle, $V$ is trivial iff $\exists$ some $c \in [0,1]$ st. $V|_{B \times [0,c]}$, $V|_{B \times [c,1]}$ are triv.

pf: $(\Rightarrow)$ obvious. For $(\Leftarrow)$, suppose $V_{c^-}$ and $V_{c^+}$ are trivial, with fixed trivializations $V_{c^-} \cong \mathbb{R}^k \times B \times [0,c^{-}]$ and similarly for $V_{c^+}$.

Then along $B \times \{c\}$, we have transition function denoted
\[ \tau_+ \circ \tau_- (v, x, c) = \left( g_+^-(x) \cdot v, x, c \right) \]

for \( g_+^- : \mathcal{B} \to \text{Gr}_L (\mathbb{R}, k) \). So letting \( \tau : V_c^+ \to \mathbb{R}^x \mathcal{B} \times [c, 1] \)

\[ \tau (v, x, t) = g_+^-(x) \cdot \tau_+ (v, x, t) \]

this agrees with \( \tau_- \) on the overlap. \( \square \)

Now since \( \mathcal{E} \) trivializing 

\[ V \to X \]

we may extend these to trivializing 

\[ \text{wlds on } U_i \times [0, 1] \]

by the above lemma. That is, any triv. 

\[ \text{wlds over } X \times [0, 1] \]

give open cover of \( X \times I \)

compact, so take finite subcover and use lemma.

Now, from the above we may show that \( V \mid_{X \times \mathbb{R}^3} \) is isom. to \( V \mid_{X \times \mathbb{R}^3} \). For \( U_i \) triv. 

\[ \text{wlds of } V \to X \]

each \( V \mid_{U_i \times [0, 1]} \) is trivial. Let \( \lambda_i = \rho_i + \ldots + \rho_i \), and let \( X_i \) in 

\[ X \times I \]

be the graph of \( \lambda_i \).

\[ \text{Fibers} \]

\[ \tilde{U}_i \]

\[ V \times \mathbb{R}^3 \]

\[ \text{Graph of } \rho_i \]

\[ V \times \mathbb{R}^3 \]

In each \( \tilde{U}_i \) triv. wld, the projection \( X_i \to X_{i-1} \) lifts to an isom. \( h_i : V \mid X_i \to V \mid X_{i-1} \). Extend to all of \( V \times [0, 1] \), call it \( h_i \).
Composition $h_1 \circ h_2 \circ \cdots$ is formally infinite, but only finitely many $h_i$ are supported at any $x \in X$, so only consider the $\exists U_i \cdots$ containing $x$. As trans. functions must satisfy cocycle condition, image is well defined. \[\Box\]

Cor If $F: X \times [0,1]$ is a homotopy of maps and $V \to X$ is a vector bundle, then $F^*_t(V) \to X \times \{t\}$ is isomorphic to $V$.

Cor Vector bundles over contractible bases are trivial.

Note also that we may consider homotopies of the trans. functions as well, by the same argument. This leads to the notion of

**Reductions of the Structure Group**

For general vector bundles the transition functions have values in $\mathcal{G}L(rk(V);\mathbb{F})$ for $\mathbb{F}$ a field. However, it is also useful/common to consider bundles with extra structure.

**Ex:** Suppose that $\exists U_i$, $\mathcal{T}_i: U_i \times \mathbb{R}^k \to \pi^i(U_i)$ are trivial bundles for $V \to X$. Then a priori $\mathcal{T}_i \circ \mathcal{T}_j /_{\pi^i | U_i \cap U_j} \in \mathcal{G}L(rk(V);\mathbb{F})$, but if in fact they take values in, say, $\mathcal{O}(rk(V))$, then the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^k$ is defined in each $U_i \times \mathbb{R}^k$, $U_j \times \mathbb{R}^k$, and if $V, W \in \mathbb{R}^k \times \mathbb{R}^k$,
\[ \langle v, w \rangle = \langle \tilde{v}_i \circ \tilde{e}_j, \tilde{w}_i \circ \tilde{e}_j \rangle_{\text{affine} V} \]

Thus, if all \( \tilde{v}_i \circ \tilde{e}_j \) take values in \( O(\text{rk}(V)) \), then \( \langle \cdot, \cdot \rangle \)
induces a well-defined inner product on \( V \).

Similarly, if \( V \) complex, could ask that they take values in \( U(n) \) \( \Rightarrow \) Hermitian metric, etc.

The Any vector bundle (over paracompact \( X \)) admits a Hermitian / Riemannian metric.

To fit this into previous discussion, consider the deformation retract \( p : G L(n, \mathbb{F}) \to O(n) \). Suppose \( V \to X \) has a metric; in any local trivialization \( U_i \times \mathbb{F}^k \), \( \exists \) local sections \( \tilde{s}_m, \tilde{\sigma}_3 \)
that span \( \mathbb{F}^k \times \mathbb{F}^k \) \( \forall x \in U_i \). Applying Graham-Schmidt to the \( \tilde{s}_m, \tilde{\sigma}_3 \) get an ONB. Similarly in \( U_j \) get \( \tilde{s}_m, \tilde{\sigma}_3 \), ONB \( \forall x \in U_i \cap U_j \).

Then change-of-basis from \( \sigma_m \to s_m \) is in \( O(n) \) or \( U(n) \) as needed, so we get transition functions. As \( \sigma \rightarrow s \)
is def. retract, \( \exists \) homotopy from old transition functions to new, and homotopy theorem \( \Rightarrow \) isomorphic bundles.