NO PERIODIC GEODESICS IN $J^k(\mathbb{R}, \mathbb{R}^n)$

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ABSTRACT. The space of $k$-jets of $n$ real function of one real variable $x$ admits the structure of a Carnot group, which then has an associated Hamiltonian geodesic flow. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does the space of $k$-jets have periodic geodesics? This study will demonstrate the integrability of sub-Riemannian geodesic flow, characterize and classify the sub-Riemannian geodesics in the space of $k$-jets, and show that they are never periodic.

1. Introduction

This paper is the generalization of [10, 8, 9]: In [10], the space of $k$-jets of real function of a single variable was presented as a sub-Riemannian manifold, the sub-Riemannian geodesic flow was defined and its integrability was verified. In [8], the sub-Riemannian geodesics were classified and some of their minimizing properties were studied. In [9], the non-existence of periodic geodesics on the space of $k$-jets of a real function of a single variable was proved.

The $k$-jets space of $n$ real functions of a single real variable, denoted here by $J^k(\mathbb{R}, \mathbb{R}^n)$ or $J^k$ for short, is a $(n(k + 1) + 1)$-dimensional manifold endowed with a canonical rank $n + 1$ distribution, i.e., a linear sub-bundle of its tangent bundle. This distribution is globally framed by $n$ vector fields, denoted by $X_1, \cdots, X_{n+1}$ in Section 2, whose iterated Lie brackets give $J^k(\mathbb{R}, \mathbb{R}^n)$ the structure of a stratified group. Declaring $X_1, \cdots, X_{n+1}$ to be orthonormal endows $J^k(\mathbb{R}, \mathbb{R}^n)$ with the structure of a sub-Riemannian manifold, which is left-invariant under the group multiplication. Like any sub-Riemannian structure, the geodesics are projection of the solution to a Hamiltonian system defined on $T^*J^k$, called the geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$.

This paper has four main goals, the following theorem is the first.

**Theorem A.** The sub-Riemannian geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$ is integrable.

The bijection between geodesics on $J^k(\mathbb{R}, \mathbb{R})$ and the pairs $(F, I)$ will be generalized, module translation $F(x) \rightarrow F(x - x_0)$, where $F(x)$ is a polynomial of degree $k$ or less and $I$ is a closed interval associated to $F(x)$, made by Monroy-Perez and Anzaldo-Meneses [2, 3, 4], also described in [8] (see pg. 4). In the present paper it will be a bijection between the geodesic
in $J^k(\mathbb{R}, \mathbb{R}^n)$ and the pairs $(F, I)$, module translation $F(x) \to F(x - x_0)$, where $F(x) = (F^1(x), \cdots, F^n(x))$ is a polynomial vector of degree $k$ or less and $I$ is a closed interval associated to $F(x)$, see Definition 3.1 for more detail of $I$.

In Section 3, it will be described how to build a geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$ given a pair $(F, I)$ and prove the following main result.

**Theorem B.** The prescription described in Section 3 yields a geodesic in $J(\mathbb{R}, \mathbb{R}^n)$ parameterized by arclength. Conversely, any arc-length parameterized geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$ can be achieved by this prescription applied to some polynomial vector $F(x)$ of degree $k$ or less.

$J^k(\mathbb{R}, \mathbb{R}^n)$ comes with a projection $\Pi: J^k(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}^{n+1}$ onto the Euclidean plane, which projects the frame $X_1, \cdots, X_{n+1}$ onto the standard coordinate frame $\{\partial/\partial x, \partial/\partial \theta_0, \cdots, \partial/\partial \theta^n\}$ of $\mathbb{R}^{n+1}$, see Section 2 for the meaning of the coordinates.

In sub-Riemannian geometry, a curve tangent to $D$ is called $C^1$-rigid or singular if it is a critical point of the endpoint map, see [14] chapter 3, or [1, 11], in other words, given an initial and end points, a curve is $C^1$-rigid if it is the only curve tangent to $D$ joining the given points. This property does not depend of the sub-Riemannian metric only on the distribution $D$, and it is said that the $C^1$-curve is minimizing regardless of how we measure length, that is, it is geodesic by virtue of its singular nature alone. Sometimes, a $C^1$-curve is not a solution to the geodesic equations and it is called abnormal geodesics, while, if it is also a solution to the geodesic equations it is called binormal geodesics.

It is well known that $J^1(\mathbb{R}, \mathbb{R})$ does not have singular curves. In the case of $J^k(\mathbb{R}, \mathbb{R})$, with $k > 1$, $C^1$ curves are tangent to $X_2$ (see [15]), that is, $C^1$-curves correspond the the polynomial $F(x) = 1$ and they are binormal geodesics. The third main result characterizes the $C^1$-curves in $J^k(\mathbb{R}, \mathbb{R}^n)$ and ensures that they can be achieved by Theorem B.

**Theorem C.** $C^1$-curves in $J^k(\mathbb{R}, \mathbb{R}^n)$ are binormal and they correspond to constant vector polynomials such that $||F(x)||^2 = 1$.

Using Theorem B, the geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$ will be classified into two main families: line-geodesics and non-line-geodesics: We say that a geodesic $\gamma(t)$ is a line-geodesics if $\gamma(t)$ corresponds to a constant polynomial vector and its projection to $\mathbb{R}^{n+1}$ is a line, in particular, binormal geodesics are line-geodesics. We say that a geodesic $\gamma(t)$ is a non-line-geodesics if $\gamma(t)$ corresponds to a non-constant polynomial and its Hill interval is compact. Moreover, if $I = [x_0, x_1]$, we say that a non-line-geodesic $\gamma(t)$ is $x$-periodic (or regular), if $x_0$ and $x_1$ are regular points of $||F(x)||^2$, that is, exist $L(F, I)$ such that $x(t + L(F, I)) = x(t)$. While, $\gamma(t)$ is critical if one point or both are critical points of $||F(x)||^2$; in this case the $x$-coordinate has an asymptotic behavior to the critical point and then the $x$-coordinate has an infinite period.
The fourth main result is the answer to a question by Enrico Le Donne: Does $J^k(\mathbb{R}, \mathbb{R}^n)$ have periodic geodesics?

**Theorem D.** $J^k(\mathbb{R}, \mathbb{R}^n)$ does not have periodic geodesics.

Following this classification, the only candidates to be periodic are $x$-periodic geodesics; so the focus is on non-constant vectors corresponding to $x$-periodic geodesics.

Remark 1: Viewing $J^k(\mathbb{R}, \mathbb{R}^n)$ as a Carnot group, Theorem D is a particular case of the conjecture made by Enrico Le Donne.

**Conjecture 1.** Carnot groups do not have periodic geodesics.

Remark 2: In control theory a “chained normal form” is a control system that is locally diffeomorphic to the canonical distribution for $J^k(\mathbb{R}, \mathbb{R}^n)$, see [16].

1.1. **Outline of paper.** The outline of the paper is as follows. In Section 2, the $k$-th jet space $J^k(\mathbb{R}, \mathbb{R}^n)$ is presented as a subRiemannian manifold, as well as, the notation that will be followed throughout the work. The subRiemannian geodesic flow is defined and the proof of Theorem A is given. Finally, the Carnot structure of $J^k(\mathbb{R}, \mathbb{R}^n)$ is presented. In Section 3, the prescription for constructing geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$ given the pair $(F, I)$ is described, the Hamilton equation are computed and Theorem B is proved. In Section 4, the abnormal equation is computed to show Theorem C. In Section 5, the proof of Theorem D is given.

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2. $J^k(\mathbb{R}, \mathbb{R}^n)$ as a subRiemannian manifold

The $k$-jet of a smooth function $f: \mathbb{R} \to \mathbb{R}^n$ at a point $x_0 \in \mathbb{R}$ is its $k$-th order Taylor expansion at $x_0$. We will encode this $k$-jet as a $(k+2)$-tuple of real numbers as follows:

$$(j^k f) = (x_0, f_k(x_0), \ldots, f_1(x_0), f(x_0)) \in \mathbb{R}^{n(k+1)+1}.$$

As $f$ varies over smooth functions and $x_0$ over $\mathbb{R}$, these $k$-jets sweep out the $k$-jet space. $J^k(\mathbb{R}, \mathbb{R}^n)$ is diffeomorphic to $\mathbb{R}^n(k+1)+1$ and we will use the global coordinates

$$(x, u_k, \ldots, u_1, u_0) \in \mathbb{R}^{n(k+1)+1}.$$

Where, $u_i = (u_i^1, \ldots, u_i^n)$ and, if $f = u_0$, then $u_1 = du_0/dx$, and more general, $u_{i+1} = du_i/dx$, $j \geq 1$. These equations are rewritten into $du_0 = u_1 dx$, and in general, $du_i = u_{i+1} dx$, we see that $J^k(\mathbb{R}, \mathbb{R}^n)$ is endowed with a
natural rank \((n + 1)\) distribution \(D \subset T^k J^k\) characterized by the \(nk\) Pfaffian equations

\[
\begin{align*}
0 &= du_0 - u_1 dx \\
0 &= du_1 - u_2 dx \\
\vdots &= \vdots \\
0 &= du_k - u_{k-1} dx.
\end{align*}
\]

\(J^k(\mathbb{R}, \mathbb{R}^n)\) has a natural definition using the coordinates \(u_i\), but they do not reflect the symmetries of the dynamics, see the proof of Theorem A in Section 3. We will introduce the alternate coordinates \(\theta_i\) for \(J^k(\mathbb{R}, \mathbb{R}^n)\) describes in [2, 3] and also introduced in [8, 9], they are exponential coordinates of the second type, see [6] Section 6.2.;

\[
\begin{align*}
\theta_0 &= u_k \\
\theta_1 &= xu_k - u_{k-1} \\
\vdots &= \vdots \\
\theta_k &= x^k u_k - x^k u_{k-1} dx + \cdots + (-1)^k u_0.
\end{align*}
\]

\(D\) is globally framed by \((n + 1)\) vector fields:

\[
(2.1) \quad X_0 = \frac{\partial}{\partial x}, \quad X^j_0 = \sum_{i=0}^k \frac{x^i}{i!} \frac{\partial}{\partial \theta^j_i} \text{ for } 1 \leq j \leq n.
\]

A subRiemannian structure on \(J^k(\mathbb{R}, \mathbb{R}^n)\) is defined by declaring these \((n+1)\) vector fields to be orthonormal. In these coordinates the subRiemannian metric is defined by restricting \(ds^2 = dx^2 + (d\theta^1_0)^2 + \cdots + (d\theta^n_0)^2\) to \(D\).

During this work we will use the convention \(\theta^j_i\), where \(i = 0, \cdots k\) and \(j = 1, \cdots, n\), that is, \(i\) is used to denotes the vector \(\theta_i\) and \(j\) denote the \(j\)-th entry of the vector \(\theta_i\).

2.1. Hamiltonian. Let \((p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)\) be the traditional coordinates for the cotangent bundle \(T^* J^k\), or abbreviated as \((p, q)\). Also, let \(P_{X_0}, P_{X^1_0}, \cdots, P_{X^n_0} : T^* J^k \to \mathbb{R}\) be the momentum functions of the vector fields \(X_0, X^1_0, \cdots, X^n_0\), in the coordinates \((p, q)\); the momentum functions are given by

\[
(2.2) \quad P_{X_0} = p_x, \quad P_{X^j_0} = \sum_{i=0}^k \frac{x^i}{i!} p_{\theta^j_i} \text{ for } 0 \leq j \leq k.
\]

Then the Hamiltonian governing the subRiemannian geodesic flow on \(J^k(\mathbb{R}, \mathbb{R}^n)\) is

\[
(2.3) \quad H = \frac{1}{2}(P_{X_0}^2 + P_{X^1_0}^2 + \cdots + P_{X^n_0}^2)
\]
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(see [14], pg 8). We will see in Section 3 that the condition $H = 1/2$ implies that the geodesics are parameterized by arc-length.

2.2. Proof of Theorem A.

Proof. The Hamiltonian $H$ does not depend on the coordinate $\theta^i_j$ because the Hamilton equations $p_{\theta^i_j}$ is a constant of motion. Then $\{H, p_{\theta^i_j}\}$ is a set of $n(k+1) + 1$ constants of motion that Poisson commute and they are linearly independent. □

2.3. Carnot Group structure. The frame $\{X_0, X^i_0, \cdots, X^n_0\}$ generates $(n(k+1) + 1)$-dimensional nilpotent Lie algebra, under the iterated bracket. That is,

$$X^i_0 = [X_0, X^j_0], \cdots, X^i_j = [X_0, X^{k-1}^j], \cdots 0 = [X_0, X^n_k],$$

all the other Lie brackets $[X^i_m, X^j_n]$ are zero. Then the frame $\{X_0, X^i_j\}$ with $0 \leq i \leq k$ and $1 \leq j \leq n$ forms a $n(k+1) + 1$-dimensional graded nilpotent Lie algebra:

$$\mathfrak{g}_k = V_1 \oplus \cdots \oplus V_{k+1}, V_1 = \{X_0, X^i_0\}, V_i = \{X^i_{i-1}\}, 1 \leq i \leq k, 1 \leq j \leq n.$$

Like any graded nilpotent Lie algebra, this algebra has an associated Lie group which is a Carnot group $G$ w.r.t the subRiemannian structure. We can identify $G$ with $J^k(\mathbb{R}, \mathbb{R}^n)$, using the flows of $\{X_0, X^i_j\}$. For more detail on the jets space as a Carnot group see [7].

3. Geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$

This Section describes how to build a geodesic on $J^k(\mathbb{R}, \mathbb{R}^n)$: Let us formalize the definition of the interval $I$.

Definition 3.1. We say that a closed interval $I$ is a Hill interval, associated to $F(x)$, if $F^2(x) < 1$ for all $x$ in the interior of $I$ and $G^2(x) = 1$ for $x$ in the boundary of $I$. Then, $i$ is compact if and only if $F(x)$ is not a constant polynomial, if $I$ is in the form $[x_0, x_1]$, $x_0$ and $x_1$ are called endpoints of the Hill interval.

Consider the Hamiltonian system of one degree of freedom defined on the plane phase space $(p_x, x)$ and with potential $1/2||F(x)||^2$, in other words, a Hamiltonian function given by

(3.1) $$H_F(p_x, x) = \frac{1}{2}p_x^2 + \frac{1}{2}||F(x)||^2;$$

then, the Hamilton equations are give by

(3.2) $$\dot{x} = p_x, \quad \dot{p}_x = (\frac{dF}{dx}, F(x)),$$

where $dF/dx$ is the derivative of the polynomial vector and $(\ ,\ )$ is the Euclidean dot product on $\mathbb{R}^n$. Since the Hamiltonian is autonomous, we choose $H_F = 1/2$; then the dynamic takes place in the point where $||F(x)||^2 \leq 1.$
If $F(x)$ is not the constant polynomial vector, and $I = [x_0, x_1]$ is the Hill interval, then $\dot{x} = 0$ if and only if $x = x_0, x_1$. Moreover, $x_0$ and $x_1$ are equilibrium points, if and only if, $x_0$ and $x_1$ are critical points of $||F(x)||^2$, in other words, $0 = (dF/dx, F(x))$.

Having found the solution $x(t)$, next we solve

$$\dot{\theta}_0^j(t) = F^j(x(t)),$$

for $\theta_0^j$. Then, $c(t) = (x(t), \theta_0(t))$ is a curve on $\mathbb{R}^{n+1}$ parameterized by arclength. Finally, we solve the horizontal lift equation associated to the curve $c(t)$

$$\dot{\theta}_1^j = x(t)F^j(x(t)),$$

$$\dot{\theta}_2^j = \frac{x^2(t)}{2!}F^j(x(t)),$$

$$\vdots = \vdots$$

$$\dot{\theta}_k^j = \frac{x^k(t)}{k!}F^j(x(t)).$$

### 3.1. Hamilton equations.

To prove Theorem B, we need to write down the Hamilton equations for the geodesic flow. Since the Hamiltonian function 2.3 is a left invariant function on the cotangent bundle of the Lie group $G$, the 'Lie-Poisson bracket' structure can be used for such Hamiltonian flows to find the equations, see Appendix [5] or chapter 4 [13]. That is, if $X$ and $Y$ are left invariant vector fields then

$$\{P_X, P_Y\} = -P_{[X,Y]}.$$

In this context, the Hamilton equations are read as $\dot{f} = \{f, H\}$. With the Hamiltonian of this system, they expand to

$$\dot{f} = \{f, P_0\}P_0 + \{f, P_{X_0}^1\}P_{X_0}^1 + \cdots + \{f, P_{X_0}^n\}P_{X_0}^n.$$

Using $\{P_0, P_{X_0}^j\} = -P_{X_0}^j$, we see that $P_0$ and $P_{X_0}^j$ evolves according to the equations

$$\dot{P}_0 = -P_{X_0}^1P_{X_0}^1 - \cdots - P_{X_0}^nP_{X_0}^n, \quad \dot{P}_{X_0}^j = P_0P_{X_0}^j \text{ for } 1 \leq j \leq n.$$

For $1 < i < k$, we have $\{P_{X_i}, P_0\} = P_{X_{i+1}}^j$ and $\{P_{X_i}, P_{X_{i+1}}^n\} = 0$, so

$$\dot{P}_{X_i}^j = P_0P_{X_{i+1}}^j,$$

$$\vdots = \vdots$$

$$\dot{P}_{X_k}^j = P_0P_{X_k}^j,$$

$$\dot{P}_{X_k}^j = 0,$$

for

$$\{P_{X_k}, P_{X_{k+1}}^n\} = 0.$$
for all $1 \leq j \leq n$. We also compute the Hamilton equations for the coordinates $(x, \theta_0, \cdots, \theta_n)$,

\begin{equation}
\dot{x} = P_0 \quad \dot{\theta}_j^i = \frac{x^i}{i!} P_{X_0} \text{ for } 0 \leq i \leq k \text{ and } 1 \leq j \leq n.
\end{equation}

### 3.2. Proof of Theorem B.

**Proof.** Let $\gamma(t)$ be a curve corresponding to the pair $(F, I)$, that is, the coordinates $x, \theta_0^i, \theta_j^i$ are solutions to the equations (3.1), (3.3) and (3.4), we will associate to $\gamma(t)$ some momentum functions and show that they hold equations (3.6) and (3.7), respectively.

Let $(p_x(t), x(t))$ be the solution to the equation (3.2) with $x(t)$ laying in the $I$, comparing with the geodesic equation from (3.8), we define $P_0 := p_x$. In the same way, comparing the equations (3.3) and (3.4) with the Hamilton equations (3.8) and (3.7) for $\theta_j^i$ and $\theta_i^j$, we define $P_{X_0}^j(t) := F^j(x(t))$ and $P_{X_1}^j(t) := \frac{dx}{dt} F^j(x(t))$. Then using the change rule we have

\begin{equation}
\dot{P}_{X_0}^j(t) = \frac{d}{dt} F^j(x(t)) = \frac{dF^j}{dx} \dot{x} = P_{X_1}^j P_0,
\end{equation}

which is the equation (3.6). In the same way

\begin{equation}
\dot{P}_{X_1}^j(t) = \frac{d}{dt} \frac{dF^j}{dx} (x(t)) = \frac{d^2 F^j}{dx dx} \dot{x} = P_{X_1}^j P_0.
\end{equation}

Since $F^j(x)$ is a polynomial of degree $k$ or less, we obtain $\dot{P}_{X_1}^j(t) = 0$ for all $j = 1, \cdots, n$, and the equation (3.9) is the same as equation (3.7).

Conversely, let $\gamma(t)$ be a geodesic parameterized by arc-length with the initial condition $\gamma(0)$, that is, $\gamma(t)$ is the projection to the solution $(p(t), \gamma(t))$ of the Hamiltonian function (2.3), we will show that the coordinates $x, \theta_0^i, \theta_j^i$ of the geodesic $\gamma(t)$ hold the equations (3.1), (3.3) and (3.4), respectively.

Being $\gamma(t)$ a solution to the Hamilton equations $p_{\theta_j^i}(t)$ is constant, if $a_j^i := i! p_{\theta_j^i}$ and $F^j(x) := a_j^i + a_j^i x + \cdots + a_j^i x^k$ for all $1 \leq j \leq n$, then, using these expressions and $x_p = P_{X_0}$, the Hamiltonian function (2.3) became

$$H = \frac{1}{2} (P_{X_0}^2 + P_{X_1}^2 + \cdots + P_{X_n}^2) = \frac{1}{2} (p_x^2 + ||F(x)||^2) = H_F.$$ 

Thus the $x$-coordinate of the geodesic $\gamma(t)$ is a solution to the Hamiltonian system of one degree of freedom with potential $1/2||F(x)||$, defined by equation (3.1), where the initial condition $x(0)$ lays in a Hill interval $I$ and, so does $x(t)$. In the same way, using the solution $x(t)$ and the Hamilton equation for $\theta_0^i$, that is, $\dot{\theta}_0^i = \partial H/\partial p_{\theta_0^i} = F^j(x(t))$, thus the $\theta_0^i$ coordinate of the geodesic $\gamma(t)$ is a solution to equation (3.3). Finally, the Hamilton equation for $\theta_j^i$, that is, $\dot{\theta}_j^i = \partial H/\partial p_{\theta_j^i} = \frac{x^i}{i!} F^j(x(t))$ is equivalent to the horizontal equation (3.4). Thus, $\gamma(t)$ is a geodesic corresponding to the pair $(F, I)$. \qed
3.3. Geodesics Classification in $J^k(\mathbb{R}, \mathbb{R}^n)$. Using the bijection between geodesics in $J^k(\mathbb{R}, \mathbb{R}^n)$ and the pair $(F, I)$, the geodesics are classified. Let $\gamma(t)$ be a geodesic corresponding to $(F, I)$, as said before the first dichotomy is if the projected curve $\pi(\gamma(t)) = c(t)$ is a line or not.

- We say that $\gamma(t)$ is a line-geodesic if $F(x)$ is the constant polynomial vector, since equation (3.3) implies that the curve $c(t) = (x(t), \theta_0(t))$ in $\mathbb{R}^{n+1}$ is a line.
- We say that $\gamma(t)$ is a non-line-geodesic if $F(x)$ is not the constant polynomial vector with Hill interval $I = [x_0, x_1]$, since equation (3.2) implies that the $x$-dynamics takes place in $I$ and curve $c(t) = (x(t), \theta_0(t))$ in $\mathbb{R}^{n+1}$ is not a line.

Let $\gamma(t)$ be a non-line-geodesic corresponding to $(F, I)$, where $I = [x_0, x_1]$, the second dichotomy refers to the qualitative behavior of the $x(t)$ dynamic.

- We say that $\gamma(t)$ is $x$-periodic or regular, that is, exist $L(F, I)$ such that $x(t + L(F, I)) = x(t)$, if $x_0$ and $x_1$ are regular points of the potential $1/2||F(x)||^2$, if and only if, $x_0$ and $x_1$ are simple roots of $1 - ||F(x)||^2$, if and only if, $1 - ||F(x)||^2 = (x - x_0)(x_1 - x)q(x)$, where $q(x)$ is not zero if $x$ is in $I$.
- We say that $\gamma(t)$ is critical, if one or both endpoints $x_0$ and $x_1$ are critical points of the potential $1/2||F(x)||^2$, if and only if, one or both endpoints $x_0$ and $x_1$ are not simple roots of $1 - ||F(x)||^2$. Then, by equation (3.1), the critical points are equilibrium points of a one degree of freedom system, and the solution $x(t)$ has an asymptotic behavior to the critical points.

3.3.1. Periods. $x$-periodic geodesics have the property that the change undergone by the coordinates $\theta_j^i$ after one $x$-period $L(F, I)$ is finite and does not depend on the initial point. This is summarized in the following proposition.

**Proposition 3.1.** Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in $J^k(\mathbb{R}, \mathbb{R}^n)$ be an $x$-periodic geodesic corresponding to the pair $(F, I)$. Then the $x$-period is

$$
L(F, I) = 2\frac{\int_I dx}{\sqrt{1 - ||F(x)||^2}},
$$

and is twice the time it takes for the $x$-curve to cross its Hill interval exactly once. After one period, the changes $\Delta\theta_j^i := \theta_j^i(t_0 + L) - \theta_j^i(t_0)$ for $i = 0, 1, \ldots, k$ and $j = 1, \cdots, n$ undergone by $\theta_j^i$ are given by

$$
\Delta\theta_j^i(F, I) = \frac{2}{i!} \int_I \frac{x^i F_j(x) dx}{\sqrt{1 - ||F(x)||^2}}.
$$

The proof of this Proposition is equivalent to the proofs of Proposition 4.1 from [8] (pg. 13) or Proposition 2.1 from [9] (pg. 2). In [8] an argument of classical mechanics was used, see [12] pg. 25 equation (11.5); while, in
a generating function to find action-angle coordinates for Hamiltonian systems was constructed, see [5], Section 50.

Then a \( x \)-periodic geodesic \( \gamma(t) \) corresponding to the pair \( (F, I) \) is periodic if and only if \( \Delta \theta_i^j(F, I) = 0 \) for all \( i = 0, 1, \ldots, k \) and \( j = 1, \ldots, n \).

4. Abnormal Geodesics and proof of Theorem C

Proof. To prove Theorem C, we will compute the abnormal geodesics using the Pontryagin’s maximum principle [1]. Let us look at the following optimal control problem:

\[ \dot{\gamma}(t) = u_0 X_0 + u_1 X_1^1 + \cdots + u_n X_n^n, \]

with the boundary conditions \( \gamma(0) \) and \( \gamma(T) \) and performance the functional

\[ \ell = \int_0^T \sqrt{u_0^2 + u_1^2 + \cdots + u_n^2} dt \to \min, \]

with the condition \( u_0^2 + u_1^2 + \cdots + u_n^2 = 1 \). Then the Hamiltonian associated to the optimal control problem (4.1) and (4.2) for the abnormal case is

\[ H(p, q, u) = u_0 p_x + \sum_{j=1}^{n} \sum_{i=0}^{k} p_{\theta_i^j} u_j x_{i-1}^i \]

From Pontryagin’s Maximum principle for this Hamiltonian we obtain a Hamiltonian system for the variables \( p \):

\[ \dot{p}_x = -\frac{\partial H}{\partial x} = -\sum_{j=1}^{n} \sum_{i=1}^{k} p_{\theta_i^j} u_j \frac{x_{i-1}^i}{(i-1)!}, \quad \dot{p}_{\theta_i^j} = -\frac{\partial H}{\partial \theta_i^j} = 0, \]

for all \( i = 0, \ldots, k \) and \( j = 1, \ldots, n \), the maximum condition is

\[ \max_{u \in \mathbb{R}^{n+1}} H(p(t), q(t), u), \]

where \( u(t) \) and \( q(t) \) is the optimal process, and the condition \( p \neq 0 \) of non-triviality. So, Theorem C is equivalent to showing that the optimal process corresponds to \( u_0 = 0 \) and \( u_j \) constant different than zero such that \( u_1^2 + \cdots + u_n^2 = 1 \).

From the maximum condition, we obtain

\[ 0 = \frac{\partial H}{\partial u_0} = p_x \quad 0 = \frac{\partial H}{\partial u_j} = \sum_{i=0}^{k} p_{\theta_i^j} \frac{x_i^i}{i!}; \]

differentiating these equations with respect to time, we get that

\[ 0 = \sum_{j=1}^{n} \sum_{i=1}^{k} p_{\theta_i^j} u_j \frac{x_{i-1}^i}{(i-1)!}, \quad 0 = u_0 \sum_{i=1}^{k} p_{\theta_i^j} \frac{x_{i-1}^i}{(i-1)!} \]

with \( j = 1, \ldots, n \). The second equation in (4.6) implies that \( u_0 = 0 \) or \( \sum_{i=1}^{k} p_{\theta_i^j} \frac{x_{i-1}^i}{(i-1)!} = 0 \). If \( u_0 = 0 \), then \( x \) is constant, since \( \dot{x} = u_0 \) according
to equation (4.1). If $\sum_{i=1}^{k} p_{\theta_{i}} x_{i}^{i-1} = 0$, we can differentiate again to get $u_{0} \sum_{i=2}^{k} p_{\theta_{i}} x_{i}^{i-2} = 0$. This equation also implies that $\sum_{i=2}^{k} p_{\theta_{i}} x_{i}^{i-2} = 0$ or $u_{0} = 0$. Following this process, it is concluded that $u_{0} = 0$, and therefore $x$ is constant.

First equation in (4.6) is rewritten as interior product:

$$0 = (u_{1}, \ldots, u_{n}) \cdot \left( \sum_{i=1}^{k} p_{\theta_{i}} x_{i}^{i-1}, \ldots, \sum_{i=1}^{k} p_{\theta_{i}} x_{i}^{i-1} \right).$$

This is a lineal equation and it has $(n - 1)$ lineal independent solutions, which can be written in terms of the constants $p_{\theta_{i}}$ and $x$ for $j = 1, \ldots, n$. Hence, $u_{j}$ is constant.

Then, to build the abnormal geodesic $\gamma(t)$ as in the prescription of Section 3, we define $F_{j}(x) = u_{j}$ and the condition $u_{0}^{2} + \cdots + u_{n}^{2} = 1$, which implies $||F(x)||^{2} = 1$. Therefore, the abnormal geodesics are line-geodesics and a solution to the Hamiltonian flow from (2.3).

5. PROOF OF THEOREM D

Because that period $L(F, I)$ in equation (3.10) is finite, we can define an inner product in the space of polynomials of degree $k$ or less as follows

$$(5.1) \quad < P_{1}(x), P_{2}(x) >_{F} := \int_{I} \frac{P_{1}(x)P_{2}(x)dx}{\sqrt{1 - F^{2}(x)}}.$$

This inner product is not degenerated and will be the key to the proof of Theorem D.

5.1. Proof of Theorem D.

Proof. It will be proceeded by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^{k}(\mathbb{R}, \mathbb{R}^{n})$ corresponding to the pair $(F, I)$, where $F(x)$ is not a constant polynomial vector; then $\Delta \theta_{i}^{j}(F, I) = 0$ for all $i = 0, \ldots, k$ and $j = 1, \ldots, n$.

In the context of the space of polynomials of degree $k$ or less with inner product $<, >_{F}$, the condition $\Delta \theta_{i}^{j}(F, I) = 0$ for all $i$ and $j$ is equivalent to each $F^{j}(x)$ being perpendicular to $x^{i}$ for all $i \in 0, 1, \ldots, k$ ($0 = \Delta \theta_{i}^{j}(F, I) = < x^{i}, F^{j}(x) >_{F}$). But $\{x^{i}\}$ is a basis for the space of polynomials of degree $k$ or less, then each $F^{j}(x)$ is perpendicular to any vector, so each $F^{j}(x)$ is zero since the inner product is not degenerated. This is a contradiction to the assumption that $F(x)$ is not a constant polynomial.

REFERENCES


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