NO PERIODIC GEODESICS IN JET SPACE

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Abstract. The $J^k$ space of $k$-jets of a real function of one real variable $x$ admits the structure of a subRiemannian manifold, which then has an associated Hamiltonian geodesic flow, and it is integrable. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does $J^k$ have periodic geodesics? This study will find the action-angle coordinates in $T^* J^k$ for the geodesic flow and demonstrate that geodesics in $J^k$ are never periodic.

1. Introduction

This paper is the first attempt to prove that Carnot groups do not have periodic subRiemannian geodesics; Enrico Le Donne made this conjecture. Here, we will establish the first case we found, which also has a simple and elegant proof.

This work is the continuation of [1, 2], in [1] $J^k$ was presented as subRiemannian manifold, the subRiemannian geodesic flow was defined, and its integrability was verified. In [2], the subRiemannian geodesics in $J^k$ were classified, and some of their minimizing properties were studied. The main goal of this paper is to prove:

Theorem A. $J^k$ does not have periodic geodesics.

Following the classification of geodesics from [2] (see pg. 5), the only candidates to be periodic are the ones called $x$-periodic (the other geodesics are not periodic on the $x$-coordinate); so we are focusing on the $x$-periodic geodesics.

An essential tool during this work is the bijection made by Monroy-Perez and Anzaldo-Meneses [3, 4, 5], also described in [2] (see pg. 4), between geodesics on $J^k$ and the pair $(F, I)$ (module translation $F(x) \rightarrow F(x - x_0)$), where $F(x)$ is a polynomial of degree bounded by $k$ and $I$ is a closed interval called Hill interval. Let us formalize its definition.

Key words and phrases. Carnot group, Jet space, integrable system, Goursat distribution, sub Riemannian geometry, Hamilton-Jacobi, periodic geodesics.
Definition 1. A closed interval $I$ is called Hill interval of $F(x)$, if for each $x$ inside $I$ then $F^2(x) < 1$ and $F^2(x) = 1$ if $x$ is in the boundary of $I$.

By definition, the Hill interval $I$ of a constant polynomial $F^2(x) = c^2 < 1$ is $\mathbb{R}$, while the Hill interval $I$ of the constant polynomial $F(x) = \pm 1$ is a single point. Also, $I$ is compact, if and only if, $F(x)$ is not a constant polynomial; in this case, if $I$ is in the form $[x_0, x_1]$, then $F^2(x_1) = F^2(x_0) = 1$. This terminology comes from celestial mechanics, and $I$ is the region where the dynamics governed by the fundamental equation (3.5) take place.

Geodesics corresponding to constant polynomials are called horizontal lines since their projection to $(x, \theta_0)$ planes are lines. In particular, geodesic corresponding to $F(x) = \pm 1$ are abnormal geodesics (see [6], [7] or [8]). Then this work will be restricted to geodesics associated with non-constant polynomials. $x$-periodic geodesics correspond to the pair $(F, [x_0, x_1])$, where $x_0$ and $x_1$ are regular points of $F(x)$, which implies they are simple roots of $1 - F^2(x)$.

Outline of the paper. In Section 2, Proposition 1 is introduced and Theorem A is proved. The main purpose of Section 3 is to prove Proposition 1. In sub-Section 3.1, the subRiemannian structure and the subRiemannian Hamiltonian geodesic function are introduced. In sub-Section 3.2, a generating function is presented and a canonical transformation from traditional coordinates in $T^*J^k$ to action-angle coordinates $(\mu, \phi)$ for the Hamiltonian systems are shown. In sub-Section 3.3, Proposition 1 is proved.

Acknowledgments. I want to express my gratitude to Enrico Le Donne for asking us about the existence of periodic geodesics and thus posing the problem. I want to thank my advisor Richard Montgomery for his invaluable help. This paper was developed with the support of the scholarship (CVU 619610) from “Consejo de Ciencia y Tecnologia” (CONACYT).

2. Proof of theorem A

Throughout the work the alternate coordinates $(x, \theta_0, \ldots, \theta_k)$ will be used, the meaning of which meaning is introduced in the Section 3 and described in more detail in [3, 4] or [2]. $x$-periodic geodesics have the property that the change undergone by the coordinates $\theta_i$ after one $x$-period is finite and does not depend on the initial point. We summarize the above discussion with the following proposition.
Proposition 1. Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in $J^k$ be an $x$-periodic geodesic corresponding to the pair $(F, I)$. Then the $x$-period is

$$L(F, I) = 2 \int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

Moreover, it is twice the time it takes for the $x$-curve to cross its Hill interval exactly once. After one period, the changes $\Delta \theta_i := \theta_i(t_0 + L) - \theta_i(t_0)$ for $i = 0, 1, \ldots, k$ undergone by $\theta_i$ are given by

$$\Delta \theta_i(F, I) = 2 \frac{i!}{i!} \int_I x^i F(x) dx \sqrt{1 - F^2(x)}.$$

In [2], a subRiemannian manifold $\mathbb{R}^3_F$, called magnetic space, was introduced and a similar statement like Proposition 1 was proved, see Proposition 4.1 from [2] (pg. 13), with an argument of classical mechanics, see [9] page 25 equation (11.5).

In the context of the space of polynomials of degree bounded by $k$ with inner product $\langle, \rangle_F$, the condition $\Delta \theta_i(F, I) = 0$ is equivalent to $F(x)$ being perpendicular to $x^i$ (0 = $\Delta \theta_i(F, I)$ = $\langle x^i, F(x) \rangle_F$), so $F(x)$ being perpendicular to $x^i$ for all $i$ in 0, 1, $\cdots$, $k$. However, the set $\{x^i\}$ with 0 $\leq$ $i$ $\leq$ $k$ is a base for the space of polynomials with degree bounded by $k$, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero since the inner product is non-degenerate. Being $F(x)$ equals 0 contradicts the assumption that $F(x)$ is not a constant polynomial.

2.1. Proof of Theorem A.

Proof. We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^k$ corresponding to the pair $(F, I)$, where $F(x)$ is not constant, then $\Delta \theta_i(F, I) = 0$ for all $i$ in 0, $\cdots$, $k$.

In the context of the space of polynomials of degree bounded by $k$ with inner product $\langle, \rangle_F$, the condition $\Delta \theta_i(F, I) = 0$ is equivalent to $F(x)$ being perpendicular to $x^i$ (0 = $\Delta \theta_i(F, I)$ = $\langle x^i, F(x) \rangle_F$), so $F(x)$ being perpendicular to $x^i$ for all $i$ in 0, 1, $\cdots$, $k$. However, the set $\{x^i\}$ with 0 $\leq$ $i$ $\leq$ $k$ is a base for the space of polynomials with degree bounded by $k$, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero since the inner product is non-degenerate. Being $F(x)$ equals 0 contradicts the assumption that $F(x)$ is not a constant polynomial.

Coming work: The proof of the conjecture in the meta-abelian group $\mathbb{G}$, that is, $\mathbb{G}$ is such that 0 = $[\mathbb{G}, \mathbb{G}, [\mathbb{G}, \mathbb{G}]].$
3. Proof of Proposition 1

3.1. $J^k$ as a subRiemannian manifold. The subRiemannian structure on $J^k$ will be here briefly described. For more details, see [1, 2]. We see $J^k$ as $\mathbb{R}^{k+2}$, using $(x, \theta_0, \cdots, \theta_k)$ as global coordinates, then $J^k$ is endowed with a natural rank 2 distribution $D \subset T J^k$ characterized by the $k$ Pfaffian equations

$$0 = d\theta_i - \frac{1}{i!} x^i d\theta_0, \quad i = 1, \cdots, k.$$  

$D$ is globally framed by two vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}.$$  

A subRiemannian structure on $J^k$ is defined by declaring these two vector fields to be orthonormal. In these coordinates the subRiemannian metric is given by restricting $ds^2 = dx^2 + d\theta_0^2$ to $D$.

3.1.1. Sub-Riemannian geodesic flow. Here it is emphasized that the projections of the solution curves for the Hamiltonian geodesic flow are geodesics, that is, if $(p(t), \gamma(t))$ is a solution for the Hamiltonian geodesic flow then $\gamma(t)$ is a geodesic on $J^k$.

Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates on $T^* J^k$, or in short way as $(p,q)$. Let $P_1, P_2 : T^* J^k \to \mathbb{R}$ be the momentum functions of the vector fields $X_1, X_2$, see [6] 8 pg or see [10], in terms of the coordinates $(p,q)$ are given by

$$P_1(p,q) := p_x, \quad P_2(p,q) := \sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}.$$  

Then the Hamiltonian governing the geodesic on $J^k$ is

$$H_{SR}(p,q) := \frac{1}{2} (P_1^2 + P_2^2) = \frac{1}{2} p_x^2 + \frac{1}{2} \left( \sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!} \right)^2.$$  

It is noteworthy that $h = 1/2$ implies that the geodesic is parameterized by arc-length. It can be noticed that $H$ does not depend on $\theta_i$ for all $i$, then $p_{\theta_i}$’s define a $k + 1$ constants of motion.

**Lemma 1.** The subRiemannian geodesic flow in $J^k$ is integrable, if $(p(t), \gamma(t))$ is a solution then

$$\dot{\gamma}(t) = P_1(t) X_1 + P_2(t) X_2, \quad (P_1(t), P_2(t)) = (p_x(t), F(x(t))),$$  

where $p_{\theta_i} = i! a_i$ and $F(x) = \sum_{i=0}^{k} a_i x^i$. 

Proof. $H$ does not depend on $t$ and $\theta_i$ for all $i$, so $h := H_{sR}$ and $p_{\theta_i}$ are constants of motion, thus the Hamiltonian system is integrable. First equation form the Lemma 1 is consequence that $P_1$ and $P_2$ are linear in $p_x$ and $p_y$'s. We denote by $(a_0, \ldots, a_k)$ the level set $i!a_i = p_{\theta_i}$, then by definition of $P_1$ and $P_2$ given by equation 3.3.

3.1.2. Fundamental equation. The level set $(a_0, \ldots, a_k)$ defines a fundamental equation

$$H_F(p_x, x) := \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x) = H|_{(a_0, \ldots, a_k)}(p, q) = \frac{1}{2}.$$ 

Here $H_F(p_x, x)$ is a Hamiltonian function in the phase plane $(p_x, x)$, where the dynamic of $x(s)$ takes place in the Hill region $I = [x_0, x_1]$ and its solution $(p_x(t), x(t))$ with energy $h = 1/2$ lies in an algebraic curve or loop given by

$$\alpha(F, I) := \{(p_x, x) : \frac{1}{2} = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x) \text{ and } x_0 \leq x \leq x_1\},$$

and $\alpha(F, I)$ is close and simple.

Lemma 2. $\alpha(F, I)$ is smooth if and only if $x_0$ and $x_1$ are regular points of $F(x)$, in other words, $\alpha(F, I)$ is smooth if and only if the corresponding geodesic $\gamma(t)$ is $x$-periodic.

Proof. A point $\alpha = (p_x, x)$ in $\alpha(F, I)$ is smooth if and only

$$0 \neq \nabla H_F(p_x, x)|_{\alpha(F, I)} = (p_x, F(x)F'(x)),$$

then $\alpha$ is smooth for all $p_x \neq 0$, the points $\alpha(F, I)$ such that $p_x = 0$ correspond to endpoints of the Hill interval $I$, since the condition $p_x = 0$ implies $F^2(x) = 1$, the point $\alpha = (0, x_0)$ is smooth if $F'(x_0) \neq 0$, as well as, the point $\alpha = (0, x_1)$ is smooth if $F'(x_1) \neq 0$. Then $\alpha(F, I)$ is smooth if and only $x_0$ and $x_1$ are regular points of $F(x)$. Also, $\alpha(F, I)$ is smooth if and only if $H_F(p_x, x)|_{\alpha(F, I)}$ is never zero, which is equivalent to the Hamiltonian vector field is never zero on $\alpha(F, I)$.  

3.1.3. Arnold-Liouville manifold. The Arnold-Liouville manifold $M|_F$ is given by

$$M_F := \{(p, q) \in T^*J^k : \frac{1}{2} = H_F(p_x, x), \ p_{\theta_i} = i!a_i\}.$$ 

In the case $\gamma(t)$ is $x$-periodic, $M_F$ is diffeomorphic to $S^1 \times \mathbb{R}^{k+1}$, where $S^1$ is the simple closed and smooth curve $\alpha(F, I)$.

$\alpha(F, I)$ has two natural charts using $x$ as coordinates and given by solve the equation $H_F = 1/2$ with respect of $p_x$, namely, $(p_x, x) = (\pm\sqrt{1 - F^2(x)}, x)$. Having this in mind,
Lemma 3. Let $d\phi_t$ be the close one-form on $M_F \subset T^*J^k$ give by

\begin{equation}
(3.7) \quad d\phi_h := \left. \frac{p_x}{\Pi(F,I)} dx \right|_{M_F} \approx \frac{\sqrt{1 - F^2(x)}}{\Pi(F,I)} dx,
\end{equation}

where $\Pi(F,I)$ is the area enclosed by $\alpha(F,I)$. Then,

\[ \int_{\alpha(F,I)} d\phi_h = 1 \quad \frac{\partial}{\partial h} \Pi(F,I) = L(F,I). \]

as a consequence exist the inverse function $h(\Pi)$.

Proof. Let $\Omega(F,I)$ be the closed region by $\alpha(F,I)$, then $d\phi_h$ can be extended to $\Omega(F,I)$ and Stokes’ Theorem implies

\begin{equation}
(3.8) \quad \Pi(F,I) := \int_{\alpha(F,I)} p_x dx = \int_{\Omega(F,I)} dp_x \wedge dx,
\end{equation}

\[ = 2 \int_{I} \sqrt{2h - F^2(x)} dx. \]

This tell that $\int_{\alpha(F,I)} d\phi_h = 1$, thus $d\phi_h$ is not exact.

$\Pi(F,I)$ is a function of $h$, so

\begin{equation}
(3.9) \quad \frac{\partial}{\partial h} \Pi(F,I) = \frac{\partial}{\partial h} \int_{I} d\phi_h = \int_{I} \frac{2dx}{\sqrt{1 - F^2(x)}}.
\end{equation}

$\Pi(F,I)$ is also called an adiabatic invariant see [11] pg 297. We will use $\Pi$ when we use it as a variable and $\Pi(F,I)$ for the adiabatic invariant.

3.2. Action-angle variables in $T^*J^k$. We will consider the actions $\mu = (\Pi, a_0, \cdots, a_k)$ and find its angle coordinates $\phi = (\phi_h, \phi_0, \cdots, \phi_k)$, such the set $(\mu, \phi)$ of coordinates are an action-angle coordinates in $T^*J^k$.

Lemma 4. There exist a canonical transformation $\Phi(p,q) = (\mu, \phi)$, where $\phi_h$ is the local function defined by the close form $d\phi_h$ from Lemma 3 and

\[ \phi_i = -\int_{x}^{\tilde{x}} \frac{x^i F(\tilde{x}) d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i! \theta_i \quad x \in I \quad i = 0, \cdots, k. \]

To construct the canonical transformation $\Phi(p,q)$, we will look for its generating function $S(\mu,q)$, of the second type that satisfies the three following conditions.

\begin{equation}
(3.10) \quad p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mu}, \quad H(\frac{\partial S}{\partial q}, q) = h(\Pi) = \frac{1}{2},
\end{equation}
where $h(\Pi)$ is the function defined in Lema 3. For more detail on the definition of $S(\mu, q)$, see [11] Section 50 or [9].

To find $S(\mu, q)$, we will solve the subRiemannian Hamilton-Jacobi equation associated with the subRiemannian geodesic flow. For more details about the definition of this equation in subRiemannian geometry and its relations with the Eikonal equation, see [6] 8 pg or [2].

Proof. The subRiemannian Hamilton-Jacobi equation is given by

$$h|_{1/2} = \frac{1}{2} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{1}{2} \sum_{i=0}^{k} x^i \frac{\partial S}{\partial \theta_i}.$$

(3.11)

Take the ansatz

$$S(\mu, q) := f(x) + \sum_{i=0}^{k} i! a_i \theta_i,$$ as a solution. The equation (3.11) becomes equation (3.5), then the generating function is given by

$$S(\mu, q) = \int_{x_0}^{x} \sqrt{2h(\Pi) - F^2(\tilde{x})} d\tilde{x} + \sum_{i=0}^{n} i! a_i \theta_i \quad (3.12)$$

Here, $h(\Pi) = 1/2$ and $S(\mu, q)$ is a local function, since $x$ must lay in the Hill region $I$, that is, $S(\mu, q)$ is defined in the sub-set $\mu \times I \times \mathbb{R}^{k+1}$.

We can see that conditions 1 and 3 of equation (3.10) are satisfied: $p(\mu, q) = \partial S/\partial q$ and $H(p(\mu, q), q) = h$. To find the new coordinates $\phi$, we use the condition 2:

$$\frac{\partial S}{\partial h} = \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} = \phi_h,$$

$$\frac{\partial S}{\partial a_i} = - \int_{x_0}^{x} \frac{\tilde{x}^i F(\tilde{x}) d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i! \theta_i = \phi_i.$$

□

Note: In [2] a projection $\pi_F : J^k \to \mathbb{R}^3_F$ was built and the solution to the subRiemannian Hamilton-Jacobi equation on the magnetic space $\mathbb{R}^3_F$ was found. The solution given by equation (3.12) is the pull-back by $\pi_F$ of the solution previously found in $\mathbb{R}_F$, where $\pi_F$ is in fact, a subRiemannian submersion.

Corollary 1. $(\mu, \phi)$ are action-angle coordinates.

Proof. Using the Hamilton equations for the new coordinates $(\mu, \phi)$, we have $\phi_t = t$ and $\phi_i = \text{const}.$ □
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Note: that \( h \) and \( \phi_t \) are action-angles coordinates for the Hamiltonian \( H_F \).

3.2.1. Horizontal derivative. A horizontal derivative \( \nabla_{\text{hor}} \) of a function \( S : J^k \to \mathbb{R} \) is the unique horizontal vector field that satisfies; for every \( q \) in \( J^k \),

\[
\langle \nabla_{\text{hor}} S, v \rangle = dS(v), \quad \text{for } v \in D_q,
\]

where \( \langle, \rangle_q \) is the subRiemannian metric in \( D_q \). For more detail see [6] pg 14-15 or [10].

**Lemma 5.** Let \( \gamma(t) \) be a geodesic parameterized by arc-length corresponding to the pair \((F, I)\) and \( S_F \) the solution given by equation (3.12), then

\[
dS_F(\dot{\gamma})(t) = 1.
\]

**Proof.** Let us prove that \( \dot{\gamma}(t) = (\nabla_{\text{hor}} S_F)_{\gamma(t)} \), which is just a consequence that \( S_F \) is solution to the Hamilton-Jacobi equation, that is,

\[
X_1(S_F)|_{\gamma(t)} = \frac{\partial S}{\partial x}|_{\gamma(t)} = p_x(t),
\]

but, Lemma 1 implies that \( P_1(t) = p_x(t) \), so \( P_1(t) = X_1(S_F)|_{\gamma(t)} \). As well,

\[
X_2(S_F)|_{\gamma(t)} = \sum_{i=0}^k \frac{x^i(t)}{i!} \frac{\partial S}{\partial \theta_i}|_{\gamma(t)} = \sum_{i=0}^k a_i x^i(t) = F(x(t)),
\]

also, Lemma 1 implies that \( P_2(t) = F(x(t)) \), so \( P_2(t) = X_2(S_F)|_{\gamma(t)} \). As a consequence;

\[
\nabla_{\text{hor}} S|_{\gamma(t)} := X_1(S_F)|_{\gamma(t)} X_1 + X_2(S_F)|_{\gamma(t)} X_2 = P_1(t) X_1 + P_2(t) X_2,
\]

Lemma 1 implies \( P_1(t) X_1 + P_2(t) X_2 = \dot{\gamma}(t) \). Thus, \( \nabla_{\text{hor}} S = \dot{\gamma}(t) \) and \( dS_F(v)|_q = \langle \nabla_{\text{hor}} S_F, v \rangle \) for all \( D_q \). In particular,

\[
dS_F(\dot{\gamma}) = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 1,
\]

since \( t \) is arc-length parameter. \( \square \)

3.3. Proof of Proposition 1.

**Proof.** It is well-known that the fundamental system system \( H_F \) with energy \( 1/2 \) has period \( L(F, I) \) given by equation (2.1) and the relation between \( \Pi(F, I) \) and \( L(F, I) \) is given by Lemma 3, see [11] pg 281. Let \( \gamma(t) \) be a \( x \)-periodic corresponding to \((F, I)\), we are interested in seeing the change suffered by the coordinates \( \theta_i \) after one \( L(I, F) \). For that,
we consider the change in $S(\mu, q)$ after $\gamma(t)$ travel form $t$ to $t + L(F, I)$, in other words,

$$L(F, I) = \int_t^{t+L(F,I)} dS(\gamma(t)) dt = \Pi(F, I) + \sum_{i=0}^{n} i!a_i \Delta \theta_i(F, I).$$  

(3.14)

On the left side of the equation is a consequence of Lemma 5, and the right side is the integration term by term. The derivative of equation (3.14) with respect to $a_i$ to find $-\frac{\partial}{\partial a_i} \Pi(F, I) = i! \Delta \theta_i$, which is equivalent to equation (2.2).

We differentiate $\Delta \theta_i := \theta_i(t + L) - \theta_i(t)$ respect to $t$, to see that $\Delta \theta_i(F, I)$ is independent of the initial point. The derivative is

$$\frac{x^i(t + L)F(x(t + L))}{\sqrt{1 - F^2(x(t + L))}} - \frac{x^i(t)F(x(t))}{\sqrt{1 - F^2(x(t))}},$$

but $x(t + L) = x(t)$.

$\Box$

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