NON-INTEGRABLE SUBRIEMANNIAN GEODESIC FLOW ON $J^2(\mathbb{R}^2,\mathbb{R})$

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Abstract. The $J^2(\mathbb{R}^2,\mathbb{R})$ space of 2-jets of a real function of two real variables x and y admits the structure of a Carnot group with step 3. As any subRiemannia manifold, $J^2(\mathbb{R}^2,\mathbb{R})$ has an associated Hamiltonian geodesic flow, which is non-integrable. To prove this, we used the reduced Hamiltonian H_μ on $T^*\mathcal{H}$, given by a symplectic reduction of the subRiemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$, using the fact that $J^2(\mathbb{R}^2,\mathbb{R})$ is a meta-abelian group.

1. Introduction

Let $J^2(\mathbb{R}^2,\mathbb{R})$ be the space of 2-jets of a real function of two variables, then $J^2(\mathbb{R}^2,\mathbb{R})$ is an 8-dimensional Carnot group with step 3 and growth vector $(5,7,8)$. Let $\mathfrak{j}$ be the graded Lie algebra of $J^2(\mathbb{R}^2,\mathbb{R})$, that is,

$$\mathfrak{j} = \mathfrak{j}_1 \oplus \mathfrak{j}_2 \oplus \mathfrak{j}_3, \text{ such that } [\mathfrak{j}_1, \mathfrak{j}_r] = \mathfrak{j}_{r+1}.$$

Let $\pi : J^2(\mathbb{R}^2,\mathbb{R}) \to \mathfrak{j}_1$ be the canonical projection and let $\mathfrak{j}_1$ be endowed with the Euclidean metric, let us consider the subRiemannian metric on $J^2(\mathbb{R}^2,\mathbb{R})$ such that π is a subRiemannian submersion, then the subRiemannian structure is left-invariant under the Carnot group multiplication. Like any subRiemannian structure, the cotangent bundle $T^*J^2(\mathbb{R}^2,\mathbb{R})$ is endowed with a Hamiltonian system whose underlying Hamiltonian H_{sR} is that whose solutions curves are subRiemannian geodesics on $J^2(\mathbb{R}^2,\mathbb{R})$. We call this Hamiltonian system the geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$.

Theorem A. *The subRiemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ is non-integrable.*

Another example of a Carnot group with a non-integrable geodesic flow: the group of all 4 by 4 lower triangular matrices with 1s on the diagonal proved by R. Montgomery, M. Saphirom and A. Stolin, see [2]. The Carnot group with growth vector $(3,6,14)$ showed by I. Bizyaev, A. Borisov, A. Kilin, and I. Mamaev, see [9]. The free Carnot group

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$(2,3,5,8)$ with step 4 verified by L. V. Lokutsievskiy and Y. L. Sachkov, see [10].

2. $J^2(\mathbb{R}^2,\mathbb{R})$ as a Carnot group

The 2-jet of a smooth function $f:\mathbb{R}^2\to\mathbb{R}$ at a point $(x_0,y_0)\in\mathbb{R}^2$ is its 2-th order Taylor expansion at x_0. We will encode this 2-jet as a 8-tuple of real numbers $(j^k f)|_{(x_0,y_0)}$ as follows:

$$(x_0,y_0,\frac{\partial^2 f}{\partial x^2},\frac{\partial^2 f}{\partial x\partial y},\frac{\partial^2 f}{\partial y^2},\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},f)|_{(x_0,y_0)}\in\mathbb{R}^8$$

As f varies over smooth functions and (x_0,y_0) varies over $\mathbb{R}^2$, these 2-jets sweep out the 2-jet space, denoted by $J^2(\mathbb{R}^2,\mathbb{R})$. One can see that $J^2(\mathbb{R}^2,\mathbb{R})$ is diffeomorphic to $\mathbb{R}^8$ and its points are coordinatized according to

$$(x,y,u_{2,0},u_{1,1},u_{0,2},u_{1,0},u_{0,1},u)\in\mathbb{R}^8:=J^2(\mathbb{R}^2,\mathbb{R}).$$

Recall that if $u=f(x,y)$, then $u_{1,0}=du/dx$, $u_{0,1}=du/dy$, $u_{2,0}=du_{1,0}/dx$, $u_{1,1}=du_{1,0}/dy=du_{0,1}/dx$ and $u_{0,2}=du_{0,1}/dy$. We see that $J^2(\mathbb{R}^2,\mathbb{R})$ is endowed with a natural rank 5 distribution $\mathfrak{j}_1\subset TJ^2(\mathbb{R}^2,\mathbb{R})\simeq\mathfrak{j}$ characterized by the following Pfaffian equations

$$u_{1,0}dx+u_{0,1}dy-du=u_{2,0}dx+u_{1,1}dy-du_{1,0}=u_{1,1}dx+u_{0,2}dy-du_{0,1}=0.$$

A subRiemannian structure on a manifold consists of a non-integrable distribution together with a smooth inner product on the distribution. We arrive at our subRiemannian structure by observing that $\mathfrak{j}_1$ is globally framed by

$$\begin{aligned}
X_1&=\frac{\partial}{\partial x}+u_{1,0}\frac{\partial}{\partial u}+u_{2,0}\frac{\partial}{\partial u_{1,0}}+u_{1,1}\frac{\partial}{\partial u_{0,1}},\\
X_2&=\frac{\partial}{\partial y}+u_{0,1}\frac{\partial}{\partial u}+u_{1,1}\frac{\partial}{\partial u_{1,0}}+u_{0,2}\frac{\partial}{\partial u_{0,1}},\\
Y_{2,0}&=\frac{\partial}{\partial u_{2,0}},Y_{1,1}=\frac{\partial}{\partial u_{1,1}},Y_{0,2}=\frac{\partial}{\partial u_{0,2}}.
\end{aligned}$$

An equivalent way to define the subRiemannian metric is to declare these vector fields to be orthonormal. Now the restrictions of the one-forms $dx,dy,du_{2,0},du_{1,1},du_{0,2}$ to $\mathfrak{j}_1$ form a global co-frame for $\mathfrak{j}_1^*$ which is dual to our frame. Therefore an equivalent way to describe our subRiemannian structure is to say that its metric is $dx^2+dy^2+du_{2,0}^2+du_{1,1}^2+du_{0,2}^2$ restricted to $\mathfrak{j}_1$.

For more detail about the jet space as Carnot group, see [4].

The left-invariant vector fields $\{X_1, X_2, Y_{2,0}, Y_{1,1}, Y_{0,2}\}$ generates the following Lie algebra:

$$Y_{1,0} := [X_1, Y_{2,0}] = [X_2, Y_{1,1}], \quad Y_{0,1} := [X_1, Y_{1,1}] = [X_2, Y_{0,2}], \tag{2.1}$$

Equations (2.1) defined the left-invariant vector fields corresponding to the second layer.

$$Y := [X_1, Y_{1,0}] = [X_2, Y_{0,1}], \tag{2.2}$$

Equations (2.1) defined the left-invariant vector field corresponding to the third layer. All the other brackets are zero.

We say that a group $\mathbb{G}$ is meta-abelian if $[\mathbb{G},\mathbb{G}] = 0$ is abelian. The Lie bracket relationship in equations (2.1) and (2.2) show that $J^2(\mathbb{R}^2,\mathbb{R})$ is a meta-abelian Carnot group, we will use the symplectic reduction performance on [1] to prove the main Theorem.

Following the notation used in [1]: let $\mathfrak{a}$ be the maximal abelian ideal containing $[\mathfrak{j},\mathfrak{j}]$; thus the Lie bracket relationship in equations (2.1) and (2.2) implies that $\mathfrak{a}$ is framed by $\{Y_{2,0}, Y_{1,1}, Y_{0,2}, Y_{1,0}, Y_{0,1}, Y\}$. Let $\mathbb{A}$ be the normal abelian sub-group whose Lie algebra is $\mathfrak{a}$ and consider its action on $J^2(\mathbb{R}^2,\mathbb{R})$ by left multiplication. Thus the action is free and proper, so $J^2(\mathbb{R}^2,\mathbb{R})/\mathbb{A}$ is well defined, and $\mathcal{H} := J^2(\mathbb{R}^2,\mathbb{R})/\mathbb{A}$ is 2-dimensional Euclidean space such that $J^2(\mathbb{R}^2,\mathbb{R}) \simeq \mathcal{H} \ltimes \mathbb{A}$.

We say that $J^2(\mathbb{R}^2,\mathbb{R})$ is a 2-abelian extension since $\mathcal{H}$ is 2-dimensional Euclidean space, latter we will see that 2 is the degree of freedom of reduced Hamiltonian H_μ, see sub-Section 3.1. Therefore Theorem A is part of the classification of 2-abelian extension Carnot Groups with the non-integrable geodesic flow.

2.1. **The exponential coordinates of the second kind.** The jet space $J^2(\mathbb{R}^2,\mathbb{R})$ has a natural definition using the coordinates x, y, and u's; however, these coordinates do not easily show the symmetries of the system. The canonical coordinates defined in [1] exhibit the symmetries,

We recall that the exponential map $\exp : \mathfrak{j} \to J^2(\mathbb{R}^2,\mathbb{R})$ is a global diffeomorphism, this allow us to endow $J^2(\mathbb{R}^2,\mathbb{R})$ with coordinates $(x, y, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ in the following way: a point g in $J^2(\mathbb{R}^2,\mathbb{R})$ is given by

$$g := \exp(\theta_1 Y_{2,0} + \theta_2 Y_{1,1} + \theta_3 Y_{0,2} + \theta_4 Y_{1,0} + \theta_5 Y_{0,1} + \theta_6 Y) * \exp(y X_2) * \exp(x X_1).$$

Then the horizontal left-invariant vector fields are given by

$$X_1 := \frac{\partial}{\partial x}, \qquad X_2 := \frac{\partial}{\partial y}, \tag{2.3}$$

the vector fields from equation (2.3) corresponding to the independent variable, while the following correspond to second derivatives

$$
\begin{aligned}
Y_{2,0} &:= \frac{\partial}{\partial\theta_1} + x\frac{\partial}{\partial\theta_4} + \frac{x^2}{2!}\frac{\partial}{\partial\theta_6},\\
Y_{1,1} &:= \frac{\partial}{\partial\theta_2} + y\frac{\partial}{\partial\theta_4} + x\frac{\partial}{\partial\theta_5} + xy\frac{\partial}{\partial\theta_6},\\
Y_{0,2} &:= \frac{\partial}{\partial\theta_3} + y\frac{\partial}{\partial\theta_5} + \frac{y^2}{2!}\frac{\partial}{\partial\theta_6}.
\end{aligned}
\tag{2.4}
$$

The left-invariant vector fields from equation (2.3) and (2.4) just depend on the independent variables x and y. All the meta-abelian Carnot groups have this property, which is the heart of the symplectic reduction. For more detail, see [1].

3. Geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$

Let us consider the traditional coordinates on $T^*J^2(\mathbb{R}^2, \mathbb{R})$, that is, $p := (p_x, p_y, p_1, p_2, p_3, p_4, p_5, p_6)$ are the momentums associated to canonical coordinates, see [5] and [6] for more details. Let λ be the tautological one-form; then the momentum functions associated to the left-invariant vector fields on the first layer $\mathfrak{j}_1$ are given by

$$P_1 := \lambda(X_1),\ P_2 := \lambda(X_2),\ P_{2,0} := \lambda(Y_{2,0}),\ P_{1,1} := \lambda(Y^2),\ P_{0,2} := \lambda(Y_{0,2}).$$

See [7] or [8] for more detail about the momentum functions. Then the Hamiltonian governing the subRiemannian geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$ is

$$H_{sR} := \frac{1}{2}(P_1^2 + P_2^2 + P_{2,0}^2 + P_{1,1}^2 + P_{0,2}^2). \tag{3.1}$$

See [7] or [8] for more detail about the definition of H_{sR}.

The Hamiltonian function H_{sR} does not depend on the coordinates θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6, so they are cycle coordinate, in other words, p_1, p_2, p_3, p_4, p_5 and p_6 are constants of motion. Moreover, H_{sR} is invariant under the action of $\mathbb{A}$, then these constants of motion correspond to the momentum map $J : T^*J^2(\mathbb{R}^2, \mathbb{R}) \to \mathfrak{a}^*$ defined by action, see [1] for more details.

3.1. **The reduced Hamiltonian.** By general theory, the symplectic reduced space is diffeomorphic to $T(G/A) \simeq T^*\mathcal{H}$ and the reduced Hamiltonian is a two-degree-of-freedom system with a polynomial potential of degree four on variables x and y, and depending on the parameters $\mu = (a_1, a_2, a_3, a_4, a_5, a_6)$ in $\mathfrak{a}^*$, given by

$$H_\mu(p_x, p_y, x, y) := \frac{1}{2}(p_x^2 + p_2^2 + \phi_\mu(x.y)), \tag{3.2}$$

where the potential $\phi_\mu(x,y)$ is given by

$$(a_1+a_4x+\frac{x^2}{2!}a_6)^2+(a_2+a_5x+a_4y+a_6xy)^2+(a_3+a_5y+a_6\frac{y^2}{2!})^2, \tag{3.3}$$

Setting $p^1=a_1$, $p^2=a_1$, $p^3=a_1$, $p^1_2=a^2_1$, $p^2_2=a^2_2$ and $p^1_3=a^3_1$, we obtain $H_{sR}=H_\mu$.

3.2. **Proof of Theorem A.** One of the main consequences of the symplectic reduction is that it is enough to verify the integrability of H_μ for all μ in $\mathfrak{a}^*$ to prove the integrability of geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$. Thus, to prove Theorem A, it is enough to exhibit a μ such that H_μ is not integrable.

Proof. If $\mu=(a_1,a_2,a_3,a_4,a_5,a_6)=(0,0,0,0,0,0,0,a)$ and $a\neq 0$, by the definition of the potential given by equation (3.3) is with the form $\phi_\mu(x,y)=a^2(\frac{1}{4}x^4+x^2y^2+\frac{1}{4}y^4)$. Let H_μ be given by equation (3.2), then H_μ is non-integrable. Indeed, this fact is a consequence of the classification of the two-degree-of-freedom Hamiltonian systems with a homogeneous potential of degree 4 made by J. Llibre, A. Mahdi, and C. Valls, in [3]. □

Appendix A. The $\mathfrak{a}^*$ value one-form $\alpha_{J^2(\mathbb{R}^2,\mathbb{R})}$

In [1], we showed that the mathematical object relating the subRiemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ and the reduced Hamiltonian on $T^*\mathcal{H}$ is $\mathfrak{a}^*$ value one-form $\alpha_{J^2(\mathbb{R}^2,\mathbb{R})}$ on $\mathfrak{j}_1\simeq\mathbb{R}^5$ given by

$$\begin{aligned}\alpha_{J^2(\mathbb{R}^2,\mathbb{R})} &= d\theta_1\otimes(e_1+xe_4+\frac{x^2}{2!}e_6)\\ &+d\theta_2\otimes(e_2+xe_5+ye_4+xye_6)\\ &+d\theta_3\otimes(e_3+ye_5+\frac{y^2}{2!}e_6).\end{aligned} \tag{A.1}$$

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