NON-INTEGRABLE SUBRIEMANNIAN GEODESIC FLOW ON $J^2(\mathbb{R}^2, \mathbb{R})$

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Abstract. The $J^2(\mathbb{R}^2, \mathbb{R})$ space of 2-jets of a real function of two real variables $x$ and $y$ admits the structure of a Carnot group with step 3. As any subRiemannian manifold, $J^2(\mathbb{R}^2, \mathbb{R})$ has an associated Hamiltonian geodesic flow, which is non-integrable. To prove this, we used the reduced Hamiltonian $H_\mu$ on $T^*H$, given by a symplectic reduction of the subRiemannian geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$, using the fact that $J^2(\mathbb{R}^2, \mathbb{R})$ is a meta-abelian group.

1. Introduction

Let $J^2(\mathbb{R}^2, \mathbb{R})$ be the space of 2-jets of a real function of two variables, then $J^2(\mathbb{R}^2, \mathbb{R})$ is an 8-dimensional Carnot group with step 3 and growth vector $(5, 7, 8)$. Let $j$ be the graded Lie algebra of $J^2(\mathbb{R}^2, \mathbb{R})$, that is,

$$j = j_1 \oplus j_2 \oplus j_3,$$

such that $[j_1, j_r] = j_{r+1}$.

Let $\pi : J^2(\mathbb{R}^2, \mathbb{R}) \to j_1$ be the canonical projection and let $j_1$ be endowed with the Euclidean metric, let us consider the subRiemannian metric on $J^2(\mathbb{R}^2, \mathbb{R})$ such that $\pi$ is a subRiemannian submersion, then the subRiemannian structure is left-invariant under the Carnot group multiplication. Like any subRiemannian structure, the cotangent bundle $T^*J^2(\mathbb{R}^2, \mathbb{R})$ is endowed with a Hamiltonian system whose underlying Hamiltonian $H_{sR}$ is that whose solutions curves are subRiemannian geodesics on $J^2(\mathbb{R}^2, \mathbb{R})$. We call this Hamiltonian system the geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$.

Theorem A. The subRiemannian geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$ is non-integrable.

Another example of a Carnot group with a non-integrable geodesic flow: the group of all 4 by 4 lower triangular matrices with 1s on the diagonal proved by R. Montgomery, M. Saphirom and A. Stolin, see [2]. The Carnot group with growth vector $(3, 6, 14)$ showed by I. Bizyaev, A. Borisov, A. Kilin, and I. Mamaev, see [9]. The free Carnot group...
(2, 3, 5, 8) with step 4 verified by L. V. Lokutsievskiy and Y. L. Sachkov, see [10].

2. $J^2(\mathbb{R}^2, \mathbb{R})$ as a Carnot group

The 2-jet of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $(x_0, y_0) \in \mathbb{R}^2$ is its 2-th order Taylor expansion at $x_0$. We will encode this 2-jet as a 8-tuple of real numbers $(j^2 f)\big|_{(x_0, y_0)}$ as follows:

\[(x_0, y_0, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f)\big|_{(x_0, y_0)} \in \mathbb{R}^8\]

As $f$ varies over smooth functions and $(x_0, y_0)$ varies over $\mathbb{R}^2$, these 2-jets sweep out the 2-jet space, denoted by $J^2(\mathbb{R}^2, \mathbb{R})$. One can see that $J^2(\mathbb{R}^2, \mathbb{R})$ is diffeomorphic to $\mathbb{R}^8$ and its points are coordinatized according to

\[(x, y, u_{2,0}, u_{1,1}, u_{1,0}, u_{0,2}, u_{0,1}, u) \in \mathbb{R}^8 := J^2(\mathbb{R}^2, \mathbb{R}).\]

Recall that if $u = f(x, y)$, then $u_{1,0} = du/dx, u_{0,1} = du/dy, u_{2,0} = du_{1,0}/dx$, $u_{1,1} = du_{1,0}/dy = du_{0,1}/dx$ and $u_{0,2} = du_{0,1}/dy$. We see that $J^2(\mathbb{R}^2, \mathbb{R})$ is endowed with a natural rank 5 distribution $J_1 \subset T J^2(\mathbb{R}^2, \mathbb{R}) \simeq j$ characterized by the following Pfaffian equations

\[u_{1,0}dx + u_{0,1}dy - du = u_{2,0}dx + u_{1,1}dy - du_{1,0} = u_{1,1}dx + u_{0,2}dy - du_{0,1} = 0.\]

A subRiemannian structure on a manifold consists of a non-integrable distribution together with a smooth inner product on the distribution. We arrive at our subRiemannian structure by observing that $J_1$ is globally framed by

\[X_1 = \frac{\partial}{\partial x} + u_{1,0} \frac{\partial}{\partial u} + u_{2,0} \frac{\partial}{\partial u_{1,0}} + u_{1,1} \frac{\partial}{\partial u_{0,1}},\]

\[X_2 = \frac{\partial}{\partial y} + u_{0,1} \frac{\partial}{\partial u} + u_{1,1} \frac{\partial}{\partial u_{1,0}} + u_{0,2} \frac{\partial}{\partial u_{0,1}},\]

\[Y_{2,0} = \frac{\partial}{\partial u_{2,0}}, Y_{1,1} = \frac{\partial}{\partial u_{1,1}}, Y_{0,2} = \frac{\partial}{\partial u_{0,2}}.\]

An equivalent way to define the subRiemannian metric is to declare these vector fields to be orthonormal. Now the restrictions of the one-forms $dx, dy, du_{2,0}, du_{1,1}, du_{0,2}$ to $J_1$ form a global co-frame for $J^*_1$ which is dual to our frame. Therefore an equivalent way to describe our subRiemannian structure is to say that its metric is $dx^2 + dy^2 + du_{2,0}^2 + du_{1,1}^2 + du_{0,2}^2$ restricted to $J_1$.

For more detail about the jet space as Carnot group, see [4].
The left-invariant vector fields \( \{ X_1, X_2, Y_{2,0}, Y_{1,1}, Y_{0,2} \} \) generates the following Lie algebra:

\[
Y_{1,0} := [X_1, Y_{2,0}] = [X_2, Y_{1,1}]
\]

Equations (2.1) defined the left-invariant vector fields corresponding to the second layer.

Equations (2.1) defined the left-invariant vector field corresponding to the third layer. All the other brackets are zero.

We say that a group \( G \) is meta-abelian if \([G,G] = 0\) is abelian. The Lie bracket relationship in equations (2.1) and (2.2) show that \( J^2(\mathbb{R}^2, \mathbb{R}) \) is a meta-abelian Carnot group, we will use the symplectic reduction performance on \([1]\) to prove the main Theorem.

Following the notation used in \([1]\): let \( a \) be the maximal abelian ideal containing \([j,j]\); thus the Lie bracket relationship in equations (2.1) and (2.2) implies that \( a \) is framed by \( \{ Y_{2,0}, Y_{1,1}, Y_{0,2}, Y_{1,0}, Y_{0,1}, Y \} \). Let \( A \) be the normal abelian sub-group whose Lie algebra is \( a \) and consider its action on \( J^2(\mathbb{R}^2, \mathbb{R}) \) by left multiplication. Thus the action is free and proper, so \( J^2(\mathbb{R}^2, \mathbb{R})/A \) is well defined, and \( \mathcal{H} := J^2(\mathbb{R}^2, \mathbb{R})/A \) is 2-dimensional Euclidean space such that \( J^2(\mathbb{R}^2, \mathbb{R}) \simeq \mathcal{H} \times A \).

We say that \( J^2(\mathbb{R}^2, \mathbb{R}) \) is a 2-abelian extension since \( \mathcal{H} \) is 2-dimensional Euclidean space, latter we will see that 2 is the degree of freedom of reduced Hamiltonian \( H_\mu \), see sub-Section 3.1. Therefore Theorem A is part of the classification of 2-abelian extension Carnot Groups with the non-integrable geodesic flow.

2.1. The exponential coordinates of the second kind. The jet space \( J^2(\mathbb{R}^2, \mathbb{R}) \) has a natural definition using the coordinates \( x, y, \) and \( u \)'s; however, these coordinates do not easily show the symmetries of the system. The canonical coordinates defined in \([1]\) exhibit the symmetries.

We recall that the exponential map \( \exp : j \to J^2(\mathbb{R}^2, \mathbb{R}) \) is a global diffeomorphism, this allow us to endow \( J^2(\mathbb{R}^2, \mathbb{R}) \) with coordinates \( (x, y, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \) in the following way: a point \( g \) in \( J^2(\mathbb{R}^2, \mathbb{R}) \) is given by

\[
g := \exp(\theta_1 Y_{2,0} + \theta_2 Y_{1,1} + \theta_3 Y_{0,2} + \theta_4 Y_{1,0} + \theta_5 Y_{0,1} + \theta_6 Y) * \exp(y X_2) * \exp(x X_1).
\]

Then the horizontal left-invariant vector fields are given by

\[
(2.3) \quad X_1 := \frac{\partial}{\partial x}, \quad X_2 := \frac{\partial}{\partial y},
\]
the vector fields from equation (2.3) corresponding to the independent variable, while the following correspond to second derivatives

\[ Y_{2,0} := \frac{\partial}{\partial \theta_1} + x \frac{\partial}{\partial \theta_4} + \frac{x^2}{2!} \frac{\partial}{\partial \theta_6}, \]

\[ Y_{1,1} := \frac{\partial}{\partial \theta_2} + y \frac{\partial}{\partial \theta_4} + x \frac{\partial}{\partial \theta_5} + xy \frac{\partial}{\partial \theta_6}, \]

\[ Y_{0,2} := \frac{\partial}{\partial \theta_3} + y \frac{\partial}{\partial \theta_5} + \frac{y^2}{2!} \frac{\partial}{\partial \theta_6}. \]

(2.4)

The left-invariant vector fields from equation (2.3) and (2.4) just depend on the independent variables \(x\) and \(y\). All the meta-abelian Carnot groups have this property, which is the heart of the symplectic reduction. For more detail, see [1].

3. Geodesic flow on \(J^2(\mathbb{R}^2, \mathbb{R})\)

Let us consider the traditional coordinates on \(T^*J^2(\mathbb{R}^2, \mathbb{R})\), that is, \(p := (p_x, p_y, p_1, p_2, p_3, p_4, p_5, p_6)\) are the momentums associated to canonical coordinates, see [5] and [6] for more details. Let \(\lambda\) be the tautological one-form; then the momentum functions associated to the left-invariant vector fields on the first layer \(j_1\) are given by

\[ P_{1} := \lambda(X_1), \quad P_{2} := \lambda(X_2), \quad P_{2,0} := \lambda(Y_{2,0}), \quad P_{1,1} := \lambda(Y_{1,1}), \quad P_{0,2} := \lambda(Y_{0,2}). \]

See [7] or [8] for more detail about the momentum functions. Then the Hamiltonian governing the subRiemannian geodesic flow on \(J^2(\mathbb{R}^2, \mathbb{R})\) is

\[ H_{sR} := \frac{1}{2}(P_1^2 + P_2^2 + P_{2,0}^2 + P_{1,1}^2 + P_{0,2}^2). \]

(3.1)

See [7] or [8] for more detail about the definition of \(H_{sR}\).

The Hamiltonian function \(H_{sR}\) does not depend on the coordinates \(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\) and \(\theta_6\), so they are cycle coordinate, in other words, \(p_1, p_2, p_3, p_4, p_5\) and \(p_6\) are constants of motion. Moreover, \(H_{sR}\) is invariant under the action of \(A\), then these constants of motion correspond to the momentum map \(J : T^*J^2(\mathbb{R}^2, \mathbb{R}) \rightarrow a^*\) defined by action, see [1] for more details.

3.1. The reduced Hamiltonian. By general theory, the symplectic reduced space is diffeomorphic to \(T^*(G/A) \simeq T^*\mathcal{H}\) and the reduced Hamiltonian is a two-degree-of-freedom system with a polynomial potential of degree four on variables \(x\) and \(y\), and depending on the parameters \(\mu = (a_1, a_2, a_3, a_4, a_5, a_6)\) in \(a^*\), given by

\[ H_{\mu}(p_x, p_y, x, y) := \frac{1}{2}(p_x^2 + p_y^2 + \phi_{\mu}(x, y)), \]

(3.2)
where the potential $\phi_\mu(x, y)$ is given by

\begin{equation}
(a_1 + a_4 x + \frac{x^2}{2!} a_6)^2 + (a_2 + a_5 x + a_6 y + a_6 x y)^2 + (a_3 + a_5 y + a_6 y^2)^2,
\end{equation}

Setting $p^1 = a_1$, $p^2 = a_1$, $p^3 = a_1$, $p^1_2 = a_2$, $p^2_2 = a_2$, and $p^1_3 = a_3$, we obtain $H_{\mu} = H_\mu$.

3.2. Proof of Theorem A. One of the main consequences of the symplectic reduction is that it is enough to verify the integrability of $H_\mu$ for all $\mu$ in $\mathfrak{a}^*$ to prove the integrability of geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$.

Thus, to prove Theorem A, it is enough to exhibit a $\mu$ such that $H_\mu$ is not integrable.

Proof. If $\mu = (a_1, a_2, a_3, a_4, a_5, a_6) = (0, 0, 0, 0, 0, a)$ and $a \neq 0$, by the definition of the potential given by equation (3.3) is with the form $\phi_\mu(x, y) = a^2(\frac{1}{4} x^4 + x^2 y^2 + \frac{1}{4} y^4)$. Let $H_\mu$ be given by equation (3.2), then $H_\mu$ is non-integrable. Indeed, this fact is a consequence of the classification of the two-degree-of-freedom Hamiltonian systems with a homogeneous potential of degree 4 made by J. Llibre, A. Mahdi, and C. Valls, in [3].

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Appendix A. The $\mathfrak{a}^*$ value one-form $\alpha_{J^2(\mathbb{R}^2, \mathbb{R})}$

In [1], we showed that the mathematical object relating the subRiemannian geodesic flow on $J^2(\mathbb{R}^2, \mathbb{R})$ and the reduced Hamiltonian on $T^*\mathcal{H}$ is $\mathfrak{a}^*$ value one-form $\alpha_{J^2(\mathbb{R}^2, \mathbb{R})}$ on $j_1 \simeq \mathbb{R}^5$ given by

\begin{equation}
\alpha_{J^2(\mathbb{R}^2, \mathbb{R})} = d\theta_1 \otimes (e_1 + x e_4 + \frac{x^2}{2!} e_6) + d\theta_2 \otimes (e_2 + x e_5 + y e_4 + x y e_6) + d\theta_3 \otimes (e_3 + y e_5 + \frac{y^2}{2!} e_6).
\end{equation}

\section*{References}


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