# SUB-RIEMANNIAN GEODESIC FLOW ON META-ABELIAN CARNOT GROUPS

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Abstract. The paper establishes a correspondence, making a symplectic reduction, between the regular geodesics in a meta-abelian Carnot group $\mathbb{G}$ and the space of solutions to a family of classical electro-mechanical systems on Euclidean space. The correspondence characterizes when these normal subRiemannian geodesic flows are integrable or admit no closed geodesics. Moreover, we can classify the integrable subRiemannian geodesic flow on the Carnot group with a rank higher than 2, which is a more general perspective than the previously done.

## 1. Introduction

This paper is devoted to establishing a correspondence between the normal geodesics in a meta-abelian Carnot group $\mathbb{G}$ and the space of solutions to a family of classical electro-mechanical systems on a Euclidean space $\mathcal{H}$, sub-space of the group's first level. Let $n$ be the dimension of $\mathcal{H}$ and $m$ its codimension within $\mathbb{G}$. Then the systems of our family have $n$-degrees of freedom and are defined by polynomial static electromagnetic fields depending linearly on $m$ parameters. See equation (1.1). This correspondence allows us to quickly characterize when these normal subRiemannian geodesic flows are integrable or admit no closed geodesics. This perspective also allows us to understand the cut loci better. See Theorem B. Moreover, we can classify the integrable subRiemannian geodesic flow on the Carnot group with a rank higher than 2, see Section 8, which is a more general perspective than the previously done by B. Kruglikov, A. Vollmer, G Lukes-Gerakopulos, in [1].

A Carnot group $\mathbb{G}$ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ of the left-invariant vector fields is graded stratified nilpotent Lie algebra of step $s$. Let $\mathfrak{g}_1$ be the first layer of $\mathfrak{g}$, with the Euclidean inner product and dimension $d_1$, then the canonical projection $\pi : \mathbb{G} \to \mathbb{R}^{d_1} \simeq \mathfrak{g}_1$ induces a left-invariant subRiemannian inner product

*Key words and phrases.* Carnot group, sub-Riemannian geometry, symplectic reduction, integrable-systems, non-integrable system,

in $\mathbb{G}$. Since $\mathfrak{g}$ is nilpotent, the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism. In general the dimension of the layer $\mathfrak{g}_r$ is $d_r$.

We say that $\mathbb{G}$ is meta-abelian if $[\mathbb{G}, \mathbb{G}]$ is abelian. Let $\mathbb{G}$ be a meta-abelian Carnot group and let $\mathfrak{a}$ be a maximal abelian ideal of $\mathfrak{g}$ among all those ideals containing $[\mathfrak{g}, \mathfrak{g}]$, thus $\mathfrak{a}$ defines a sub-group $\mathbb{A}$ of $\mathbb{G}$, and we can see $\mathbb{G}$ as $\mathcal{H}$ extension of $\mathbb{A}$, that is, exists a short exact sequence

$$1 \to \mathbb{A} \hookrightarrow^{i_{\mathbb{A}}} \mathbb{G} \to^{\pi_{\mathbb{A}}} \mathcal{H} \to 1,$$

which is equivalent to $\mathbb{G} \simeq \mathcal{H} \ltimes \mathbb{A}$. $\mathfrak{a}$ might not be unique, so $\mathbb{A}$ might not. Thus, the projection $\pi_{\mathbb{A}}$ is canonical up to the choice of $\mathfrak{a}$. If $\mathfrak{a} \cap \mathfrak{g} = 0$, then $\pi_{\mathbb{A}}$ is just the canonical projection $\pi$. If $\mathcal{V} := \mathfrak{a} \cap \mathfrak{g} \neq 0$, then $\mathbb{A}$ acts by translation on $\mathbb{R}^{d_1}$, so quotient the $\mathbb{R}^{d_1}/\mathbb{A}$ is well defined and exist a orthonormal projection $\pi_{\mathcal{H}} : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}/\mathbb{A} \simeq \mathcal{H}$, such that $\pi_{\mathbb{A}} = \pi_{\mathcal{H}} \circ \pi$. Let $n_1$ be the dimension of $\mathcal{V}$.

| Group | Dimension | Isomorphic |
|---|---|---|
| $\mathbb{G}$ | $n + m = d_1 + \cdots + d_s$ | $\mathbb{R}^{n+m}$ |
| $\mathbb{A} \simeq \mathcal{V} \times [G, G]$ | $m = n_1 + d_2 + \cdots + d_s$ | $\mathbb{R}^m$ |
| $[\mathbb{G}, \mathbb{G}]$ | $d_2 + \cdots + d_s$ | $\mathbb{R}^{d_2+\cdots+d_s}$ |
| $\mathfrak{g}_1 \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}] \simeq \mathcal{H} \oplus \mathcal{V}$ | $d_1 = n + n_1$ | $\mathbb{R}^{d_1}$ |
| $\mathcal{H} := \mathbb{G}/\mathbb{A}$ | $n$ | $\mathbb{R}^n$ |
| $\mathcal{V} := \mathcal{H}^{\perp} \subset \mathfrak{g}_1$ | $n_1$ | $\mathbb{R}^{n_1}$ |

Let $T^*\mathbb{G}$ be the cotangent bundle of $\mathbb{G}$ with traditional coordinates $(p, g)$, and let $H_{sR}$ be subRiemannian kinetic energy, then $H_{sR}$ is $\mathbb{A}$-invariant, and the geodesic flow has $m$ constant of motion in involution. Let $T^*\mathcal{H}$ be the cotangent bundle of $\mathcal{H}$ with the exponential coordinates of the second kind $(p_x, x)$, the Hamiltonian structure for a charged particle under the influence of a static electromagnetic field is given by a magnetic potential $\mathcal{A}$ and effective potential or electrostatic potential $\phi$, see [2], [3] or [4] for more details.

The mathematical object relating the Hamiltonian structures is a $\mathfrak{a}^*$ value one-form $\alpha_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}} + \beta_{\mathbb{G}}$ in $\Omega^1(\mathbb{R}^{d_1}, \mathfrak{a})$, where $\mathcal{A}_{\mathbb{G}}$ is in $\Omega^1(\mathcal{H}, \mathfrak{a}^*)$ and $\beta_{\mathbb{G}}$ is in $\Omega^1(\mathcal{V}, \mathfrak{a}^*)$. Let $\mu$ be in $\mathfrak{a}^*$ and $\alpha_\mu$ the paring of $\alpha_{\mathbb{G}}$ with $\mu$, then

$$\alpha_\mu(x) :=< \mu, \alpha_{\mathbb{G}} >= \sum_{i=1}^{n=dim(\mathcal{H})} \mathcal{A}_{i,\mu}(x) dx_i + \sum_{k=1}^{n_1=dim(\mathcal{V})} \beta_{k,\mu}(x) d\theta_k \tag{1.1}$$

is a one-form on $\mathbb{R}^{d_1}$ with local coordinates $x_i$'s and $\theta_k$'s. $\alpha_\mu$ just depends on the coordinates $x$'s in a polynomial way with degree $s - 1$

and defines Hamiltonian $H_\mu$ in $T^*\mathcal{H}$, given by

$$\begin{aligned}
H_\mu &:= \frac{1}{2}||p_x + \alpha_\mu(x)||^2_{\mathbb{R}^{d_1}} \\
&= \frac{1}{2}||p_x + \mathcal{A}_\mu(x)||^2_{\mathcal{H}} + \frac{1}{2}\phi(x) \qquad (1.2)\\
&= \frac{1}{2}\sum_{i=1}^{n}(p_i + \sum_{j,r}^{d_r,s} A^r_{i,j_r}(x)p^r_{j_r})^2 + \frac{1}{2}\sum_{k=1}^{n_1}(p_k + \sum_{j,r}^{d_r,s} \beta^r_{b,j}(x)p^r_j)^2
\end{aligned}$$

In the second line $\phi(x) := ||\beta_\mu(x)||^2_{\mathcal{V}}$ is a polynomial. In the third line The $p_i$ are momenta for $\mathcal{H}$, the $p_k$ are momenta for $\mathcal{V}$ and the $p^r_i$ are momenta for $[\mathbb{G}, \mathbb{G}]$. Hence $p_k$'s and $p^r_j$'s are the $\mu$'s - they are linear coordinates on $\mathfrak{g}^*$, and are constants of motion. Understanding and explaining this expression for $H_{sR}$, understood as a rewriting of the subRiemannian kinetic energy , is at the heart of work.

Let $J : T^*\mathbb{G} \to \mathfrak{a}^*$ be the momentum map associated to the action of $\mathbb{A}$. We say that $\gamma(t)$ is a subRiemannian geodesic in $\mathbb{G}$ parameterized by arc-length and has momentum $\mu$, if $\gamma(t)$ is the projection of the geodesic flow for $H_{sR}$ with energy $1/2$ and $J(p(t), \gamma(t)) = \mu$. We say that $c(t)$ is $\alpha_{\mathbb{G}}$-curve for $\mu$ in $\mathcal{H}$, if $c(t)$ is the projection of the Hamiltonian flow for $H_\mu$ with energy $1/2$, we call the Hamiltonian system given by $H_\mu$ an $\alpha_{\mathbb{G}}$-system. We shall find a horizontal lift of $\alpha_{\mathbb{G}}$-curve $c(t)$ in $\mathcal{H}$ to $\mathbb{G}$, see sub-Section 4.1, and prove the main Theorem of this work.

**Theorem A.** *Let $\mathbb{G}$ be a meta-abelian Carnot group and $\mathfrak{a}$ a choice of maximal abelian ideal ($[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$). Then there exists an $\mathfrak{a}^*$ value polynomial one-form $\alpha_{\mathbb{G}}$ on $\mathcal{H} = \mathbb{G}/\mathbb{A}$ of the form $\alpha_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}} + \beta_{\mathbb{G}}$, $x \in \mathcal{H}$ with the following significance. If $c(t)$ is a $\alpha_{\mathbb{G}}$-curve for $\mu$, then its horizontal-lift is a normal subRiemannian geodesic in $\mathbb{G}$ with momentum $\mu$. Conversely, if $\gamma(t)$ is a normal subRiemannian geodesic in $\mathbb{G}$ with momentum $\mu$, then the curve $c(t) = \pi_{\mathbb{A}}(\gamma(t))$ is an $\alpha_{\mathbb{G}}$-curve for $\mu$.*

The Theorem is proved by showing that the symplectic reduction of the subRiemannian flow on $T^*\mathbb{G}$ yields the $\alpha_{\mathbb{G}}$-systems. Then, we reduce the study of subRiemannian geodesic in meta-abelian Carnot groups to the study of the $\alpha_{\mathbb{G}}$-systems. An example of this is the second main Theorem, but first let us introduce the following definitions.

When $\mathcal{A}_\mu(x) = 0$ the electromagnetic field is just a static electric field, as well as when $\beta_\mu(x) = 0$ the electromagnetic field is just magnetic. Then, we recall the $\alpha_{\mathbb{G}}$-system as $\beta_{\mathbb{G}}$-system if $\mathcal{A}_\mu(x) = 0$, and as $\mathcal{A}_{\mathbb{G}}$-system if $\beta_\mu(x) = 0$.

The map exp endows $\mathbb{G}$ with the exponential coordinates $g = \Phi(x, \theta)$ of the second kind; during this work, we will call them exponential coordinates of the second kind since the $x$'s and $\theta$'s are the coordinates for $\mathcal{H}$ and $\mathbb{A}$ inside $\mathbb{G}$, respectively, see sub-section 2.3.

**Theorem B.** *Let $\mathbb{G}$ be a meta-abelian Carnot group,*

1. *Let $\mathbb{G}$ be such that $\mathbb{G}$ has a $\beta_{\mathbb{G}}$-system, then $\mathbb{G}$ does not have periodic normal geodesics.*
2. *If $c(t)$ is $L$-periodic in $\mathcal{H}$ and $\gamma(t)$ in $\mathbb{G}$ is its horizontal lift, then the upper bound of the cut time for $\gamma(t)$ is $L$.*
3. *The geodesic flow in $\mathbb{G}$ is integrable if and only if $H_\mu$ is integrable for all $\mu$ in $\mathfrak{a}^*$.*
4. *Let $(x, \theta)$ be the coordinates for $G$ and let $\Psi(x)$ be an eigenvector for the quantum Hamiltonian $\hat{H}_\mu := \hat{H}(\frac{\partial}{\partial x}, x; \mu)$, that is, $\hat{H}_\mu \Psi(x) = \lambda_\mu \Psi(x)$ and let $\Psi(\theta, t)$ be the following function*

$$\Psi(\theta, t) := \exp\left(\frac{i}{2\hbar}\left(-\lambda_\mu t + \sum_{\ell=1}^{m} a^\ell \theta_\ell\right)\right). \tag{1.3}$$

*Then $\Psi(x, \theta, t) := \Psi(x)\Psi(\theta, t)$ is a solution for the sub-Riemannian Schrodinger equation on $G$, that is,*

$$\hat{H}_{sR}\Psi(x, \theta, t) = \lambda i\hbar \frac{\partial}{\partial t}\Psi(x, \theta, t).$$

To remark the degree of freedom $n$ of the reduce Hamiltonian $H_\mu$, we will introduce the following definition

**Definition 1.1.** *We say that a meta-abelian Carnot group is $n$-abelian extension if $n$ is the co-dimension of the maximal abelian ideal $\mathfrak{a}$ in $\mathbb{G}$.*

The fourth main contribution of the paper is the approach to meta-abelian Carnot groups as $n$-abelian extensions. In the last two sections, we provide

**Theorem C.** *The meta-abelian Carnot groups which integrability of subRiemannian geodesic flow has been decide are the following:*

1. *Let $\mathbb{G}$ be 1-abelian extension Carnot group, then the geodesic flow in $\mathbb{G}$ is integrable.*
2. *The known 2-abelian extension with integrable geodesic flow are the following:*

| *Group* | $n+m$ | $d_1$ | *Step* | *System* |
|---|---|---|---|---|
| $F_{23}$ | 5 | 2 | 3 | $\mathcal{A}$-*system* |
| $N_{6,2,5a^*}$ | 6 | 2 | 3 | $\mathcal{A}$-*system* |
| $En(2)$ | 6 | 3 | 3 | $\beta$-*system* |

(3) *The known 2-abelian extension with non-integrable geodesic flow are the following:*

| *Group* | $n+m$ | $d_1$ | *Step* | *System* |
|---|---|---|---|---|
| $F_{24}$ | 8 | 2 | 4 | $\mathcal{A}$-*system* |
| $N_{6,2,5}$ | 6 | 3 | 3 | $\beta$-*system* |
| $J^2(\mathbb{R}^2,\mathbb{R})$ | 6 | 5 | 3 | $\beta$-*system* |

(4) *The $n$-abelian extension $Eng(n)$ is non-commutative integrable*

**1-abelian extension.** This sub-Section introduces the example of the 1-Abelard extension. The first consequence of Theorem B is the proof of part one.

*Proof.* The reduced system is one-degree of freedom, then $H_\mu$ is always integrable, and so is $H_{sR}$. □

The quantum version of the integrability of the subRiemannian geodesic flow is the following.

**Proposition 1.1.** *Let $\mathbb{G}$ be 1-abelian extension Carnot group and let $\hat{H}_\mu$ be the quantum Hamiltonian, then $\hat{H}_\mu$ is a self-adjoint operator.*

The proof is the same that the one given by Y. C. de Verdire and C. Letrouit in [35], see Section 3.

Any 1-abelian extension $\mathbb{G} \simeq \mathbb{R} \ltimes \mathbb{A}$ is a sub-group of $J^k(\mathbb{R},\mathbb{R}^n)$. In this context, the Heisenberg group is the easiest example and is diffeomorphic to $J^2(\mathbb{R},\mathbb{R})$. We will start with the example of Engel's group denoted by $Eng \simeq J^2(\mathbb{R},\mathbb{R})$.

*Eng as 1-abelian extension.* Let $Eng$ be the 4-dimension Carnot group with growth vector $(2,3,4)$. The first layer $\mathfrak{g}_1$ is framed by $\{X,Y\}$, and the following relationships give its Lie algebra:

$$Y^2 := [X^1,Y^1], \qquad Y^3 := [X^1,Y^2].$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is given by $Y^1$, $Y^2$ and $Y^3$. So in this case $\mathbb{A} \simeq \mathbb{R}^4$, $\mathcal{V} = \mathbb{R}$, $[Eng, Eng] \simeq \mathbb{R}^2$ and $Eng \simeq \mathbb{R}^1 \times \mathbb{A}$.

The base of left-invariant vector fields for $\mathfrak{g}_1$ in the coordinates $(x,\theta)$ is given by

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial \theta_1} + x\frac{\partial}{\partial \theta_2} + \frac{x^2}{2}\frac{\partial}{\partial \theta_3}.$$

Then the Hamiltonian function in $T^*Eng$ with the traditional coordinates $(p,g) = (p_x, p_\theta, x, \theta)$ is given by

$$H_{sR}(p,g) = \frac{p_x^2}{2} + \frac{1}{2}(p_1 + p_2 x + p_3\frac{x^2}{2})^2.$$

Since $H_{sR}$ does not depend on the variables $\theta^1$, $\theta^2$ and $\theta^3$, then they are cycle coordinates and $p_1$, $p_2$ and $p_3$ (see [5] or [6] for the definition of cycle coordinate). These momentum functions correspond to the frame of killing vector fields $\frac{\partial}{\partial\theta_1}$, $\frac{\partial}{\partial\theta_2}$, and $\frac{\partial}{\partial\theta_3}$.

The $\mathfrak{a}^*$ value one-form $\alpha_{Eng}$ is given by

$$\alpha_{Eng} = d\theta_0 \otimes (e^1 + xe^2 + \frac{x^2}{2}e^3).$$

Notice the polynomial is degree two since the step $s = 3$. If $\mu = (a_1, a_2, a_3)$ is in $\mathfrak{a}^*$ and $F(x) := \beta_\mu(\frac{\partial}{\partial\theta_0})$. Then the reduced Hamiltonian is given by

$$H_\mu(p_x, x) = \frac{p_x^2}{2} + \frac{1}{2}(a_0 + a_1x + a_2\frac{x^2}{2})^2 = \frac{1}{2}(p_x^2 + F^2(x)).$$

Let $c(t) = x(t)$ be a $\mu$-curve, then the horizontal lift equation is the following

$$\dot\gamma = \dot x(t)X^1(x(t)) + F(x(t))Y^1(x(t)). \tag{1.4}$$

When $a_2 = 0$, the above system is the harmonic oscillator, and the geodesic $\gamma(t)$ in $Eng$ is the lift of a geodesic in the Heisenberg group.

After a translation the system is written as $\frac{1}{2}(p_x^2 + (a_1 + a_2\frac{x^2}{2})^2)$, which is the an-harmonic oscillator. Richard Montgomery used the an-harmonic oscillator to quantize Engel's group $Eng$, see [7].

**Outline.** In Section 2, the subRiemannian structure in $\mathbb{G}$ is described, and the notation for the left-invariant vector fields followed during the paper is introduced. The formal definition of the action and the exponential coordinates of the second kind $(x, \theta)$ are introduced in the sub-sections 2.1 and 2.3, respectively. In Section 3, the cotangent bundle $T^*G$ is endowed with the traditional coordinates $(p, g)$ and subRiemannian geodesic flow. The properties of the traditional coordinates $(p_x, p_\theta, x, \theta)$ are shown in sub-section 3.2. In Section 4, the cotangent bundle $T^*\mathcal{H}$ is provided with the traditional coordinates $(p_x, x)$, the horizontal lift of a $\alpha_{\mathbb{G}}$-curve $c(t)$ to $\mathbb{G}$ is described. In Section 5, Theorem A is proved. In Section 6, some essential properties to prove Theorem B of the $\alpha_{\mathbb{G}}$-system is exposed by the cases: electric, magnetic and electromagnetic.

In sub-Section 6.4, we consider the case $\mathfrak{a} \neq [\mathfrak{g}, \mathfrak{g}]$ and explain some details about the intermediate Hamiltonian $H_\tau : T^*\mathbb{R}^{d_1} \to \mathbb{R}$ in the reduction: the action of $[\mathbb{G}, \mathbb{G}]$ on $\mathbb{G}$ induce a lift action on $T^*\mathbb{G}$, then $H_\tau$ is a $\mathbb{A}$-invariant Hamiltonian function on the cotangent bundle $T^*\mathbb{R}^{d_1}$ for a charged particle under the influence of the static magnetic field, the mathematical object relating the Hamiltonian structures $H_{sR}$ and

$H_\tau$ is a $[\mathfrak{g},\mathfrak{g}]^*$ value one-form $\beta_{\mathbb{G}}$ in $\Omega^1(\mathbb{R}^{d_1},\mathfrak{a})$. We say that $\tilde{c}(t)$ is $\eta_{\mathbb{G}}$-curve for $\tau$ in $\mathbb{R}^{d_1}$, if $\tilde{c}(t)$ is the projection of the Hamiltonian flow for $H_\tau$ with energy $1/2$. Theorem D is the fourth main result of this work and establishes a correspondence between a geodesic $\gamma(t)$ in $\mathbb{G}$ and $\eta_{\mathbb{G}}$-curve $\tilde{c}(t)$.

In Section 7, Theorem 7 is shown. In Section 8, the method to classify the integrable meta-abelian Carnot group is proposed In Section 9, a $n$-abelian extension $Eng(n)$ is presented, whose subRiemannian geodesic flow is invariant under the action of $SO(n)$ and the reduced Hamiltonian $H_\mu$, after a translation, is the radial-an-harmonic oscillator, which is an example of non-commutative integrability.

**Acknowledgments.** I want to express my gratitude to Richard Montgomery and Enrico Le Donne: To Richard for being my advisor and for his invaluable help. To Enrico for my time at Fribourg University, where he introduced the definition of the meta-abelian Carnot group and the exponential coordinates of the second kind, also for showed me the proof of Proposition 2.1. I want to thank Cyril LeTrouit for explained to me the subRiemannian Schrodinger equation and the quantum consequences of the symplectic reduction, see sub-Section 7.4.

I want to thank Nicola Paddeu, Andrei Ardentov, Yuri Sachkov, Felipe Monroy-Perez , Luis Garcia-Naranjo Angel Carrillo-Hoyo and Hector Sanchez-Morgado, for e-mail conversations and talks regarding the course of this work. This paper was developed with the support of the scholarship (CVU 619610) from "Consejo de Ciencia y Tecnologia" (CONACYT).

## 2. $\mathbb{G}$ as subRiemannian manifold

A Carnot group $\mathbb{G}$ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ of the left-invariant vector fields is graded stratified nilpotent Lie algebra of step $s$, that is, the Lie algebra $\mathfrak{g}$ of step $s$ satisfies:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \quad [\mathfrak{g}_1,\mathfrak{g}_r] = \mathfrak{g}_{r+1}, \quad \mathfrak{g}_{s+1} = 0, \quad dim\ \mathfrak{g}_r := d_r.$$

We remark that the left-invariant subRiemannian structure in $\mathbb{G}$ is given by declaring the left-invariant vector fields in $\mathfrak{g}_1$ be orthonormal.

Let $g$ be in $\mathbb{G}$, then the formal definition of the canonical projection $\pi$ is given by

$$\pi(g) := g \;\; mod \;\; [G,G]. \tag{2.1}$$

The canonical injection $i : [G,G] \hookrightarrow \mathbb{G}$ and the projection $\pi$ define a short sequence

$$0 \to \mathbb{A} \hookrightarrow^{i_{\mathbb{A}}} \mathbb{G} \to^{\pi_{\mathbb{A}}} \mathcal{H} \to 0,$$

which is equivalent to $\mathbb{G} \simeq \mathbb{R}^{n+m}$.

Let $\mathbb{G}$ be meta-abelian abelian Carnot and let $\mathfrak{a}$ be its maximal abelian ideal contannig $[\mathbb{G}\mathbb{G}]$, then projection $\pi_{\mathbb{A}}$ is given by

$$\pi_{\mathbb{A}}(g) := g \ \ mod \ \ \mathbb{A}. \tag{2.2}$$

Note: During this work, all the natural maps corresponding to $\mathbb{G}$, as $\pi$ and $i$ are not labeled. The maps depending on the selection of $\mathfrak{a}$, as $\pi_{\mathbb{A}}$, are labeled with $\mathbb{A}$, and the intermediate maps are labeled with the target, as $\pi_{\mathcal{H}}$.

Given the maximal ableian Lie algebra containing $[\mathfrak{g}, \mathfrak{g}]$, the first layer $\mathfrak{g}_1$ has a two natural left-invariant sub-space: $\mathfrak{v} := \mathfrak{a} \cap \mathfrak{g}_1$, and $\mathfrak{h}$ the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{g}_1$, thus $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{v}$. Moreover, the map $\pi$ is compatible with the splitting of $\mathfrak{g}_1$ and $\mathbb{R}^{d_1}$, that is, $d\pi(\mathfrak{h}) = \mathcal{H}$ and $d\pi(\mathfrak{v}) = \mathcal{V}$, where we identity $T\mathcal{H} \simeq \mathcal{H}$ and $T\mathcal{V} \simeq \mathcal{V}$.

We will introduce the notation used during this work: Let $X^i$ be the base for $\mathfrak{h}$ with $1 \leq i \leq n$, let $\{Y^\ell\}$ be the base for $\mathfrak{a}$ with $1 \leq \ell \leq m$. An alternative notation is; $\{Y^k\}$ be the base for $\mathfrak{v}$ with $1 \leq k \leq n_1$ and let $\{Y^j_r\}$ be the base for $\mathfrak{g}_r$ whit $1 < r \leq s$ and $1 \leq j \leq d_r$. Then, we will use the index $i$'s for vector fields in $\mathfrak{h}$, $\ell$'s for vector fields in $\mathfrak{a}$, and when we want to distinguish between the different layers, we will use $k$'s for vector fields in $\mathfrak{v}$ and $r$'s and $j$'s for vector fields in $\mathfrak{g}_s$ .

**Corollary 2.1.** *Let $\mathbb{G}$ be a meta-abelian Carnot group, then $\mathfrak{h}$ is orthonormal to $\mathfrak{v}$ with respect to the subRiemannian structure.*

$\mathbb{G}$ has a canonical projection $\pi : \mathbb{G} \to \mathfrak{g}_1$, such that $d\pi$ has a canonical inverse map $hor : T\mathbb{R}^{d_1} \to T\mathbb{G}$ called the horizontal lift, that is, if $v = (v_1, \cdots, v_n, u_1, \cdots, u_{n_1})$ is in $T\mathbb{R}^{d_1}$, then $hor(v) := \sum_{i=1}^n v_i X^i + \sum_{k=1}^{n_1} u_k Y^j$ and $d\pi \circ hor = Id_{\mathfrak{g}_1}$, we say that $hor$ is a horizontal lift with respect to $d\pi$.

*hor* defines an inverse map in the space of curves $\tilde{c}(t)$ in $\mathbb{R}^{d_1}$ with coordinates $(x, \theta)$, which is unique up to a constant of integration, also called the horizontal lift. The Pfaffian system gives another construction of *hor* in the following way. Let us consider the frame $\{Y^j_r\}$, with $1 < r \leq s$ and $1 \leq j \leq d_r$. Then $\{Y^j_r\}$ defines a co-frame of left-invariant one-form $\{\Theta^r_j\}$, with $1 < r \leq s$ and $1 \leq j \leq d_r$ such that $\Theta^{r_1}_{j_1}(Y^j_r) = \delta^{r_1}_r \delta^j_{j_1}$ and $\Theta^r_j(\mathfrak{g}_1) = 0$, where $\delta$ is the delta of Kronecker. Let $\tilde{c}(t) = (x(t), \theta(t))$ be in $\mathfrak{g}_1$, then the $\tilde{c}(t)$ defines a curve $\gamma(t)$ in $\mathbb{G}$ and tangent to $\mathfrak{g}_1$ by solving the equations

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \dot{x}_i(t) X^i(\tilde{c}(t)) + \sum_{k=1}^{n_1} \dot{\theta}_k(t) Y^k(\tilde{c}(t)). \tag{2.3}$$

Let us formalize the definition.

**Definition 2.1.** *Let $\tilde{c}(t)$ be in $\mathcal{H}$, we say that $\gamma(t)$ is the horizontal lift of $\tilde{c}$ if $\gamma(t)$ is the solution to the equation* (2.3).

2.0.1. *Horizontal lift $hor_{\mathbb{G}}$.* The horizontal lift $hor_{\mathbb{R}^{d_1}}$ with respect the projection $\pi_{\mathcal{H}}$ is defined as follows if $(v_1, \cdots, v_n)$ is in $\mathcal{H}$ then

$$hor_{\mathbb{R}^{d_1}}(v_1, \cdots, v_n) := (v_1, \cdots, v_n, 0, \cdots, 0) \in \mathbb{R}^{d_1},$$

by construction $d\pi_{\mathcal{H}} \circ hor_{\mathbb{R}^{d_1}} = Id_{\mathcal{H}}$.

Then the horizontal lift $hor_{\mathbb{G}}$ with respect the projection $\pi_{\mathbb{A}}$ is given by $hor \circ hor_{\mathbb{R}^{d_1}}$, let us formalize the definition.

**Definition 2.2.** *The horizontal lift $hor_{\mathbb{G}}$ with respect the projection $\pi_{\mathbb{A}}$ is given by $hor_{\mathbb{A}} := hor \circ hor_{\mathbb{R}^{d_1}}$, that is,*

$$hor_{\mathbb{A}}(v_1, \cdots, v_n) = \sum_{i=1}^{n} v_i X_i.$$

2.1. **Action of $\mathbb{A}$.** The action of the sub-group $\mathbb{A}$ on $\mathbb{G}$ is given by the multiplication by the left. Let us be more clear, let $a$ be in $\mathbb{A}$ then $a$ defines the translation in the following way

$$\varphi(a, g) := a * g. \tag{2.4}$$

Then by definition $\varphi(a_1 * a_2, g) = \varphi(a_2 * a_1, g)$.

If $\xi$ is in $\mathfrak{g}$ then the action of $\mathbb{A}$ on $\mathbb{G}$ defines the infinitesimal generator map $\sigma : \mathfrak{a} \to \mathfrak{g}$ in the following way

$$\sigma(\xi)(g) = \frac{d}{dt}\varphi(t\xi, g)|_{t=0} = \frac{d}{dt}\exp(t\xi) * g|_{t=0}. \tag{2.5}$$

For clarity, we shall distinguish between the group $\mathbb{A}$ as a sub-group of $\mathbb{G}$ and as a group acting in $\mathbb{G}$. We will make the convention to use $\mathbb{A}_{\mathbb{G}}$ as a sub-group and $\mathfrak{a}_{\mathbb{G}}$ for its sub-Lie algebra. In addition, we distinguish between the Euclidean space as $n$-dimensional vector space sub-manifold and as a sub-manifold of $\mathbb{G}$, then we use $\mathcal{H}_{\mathbb{G}}$ as a sub-manifold. Then we write $\mathbb{G} \simeq \mathcal{H}_{\mathbb{G}} \ltimes \mathbb{A}_{\mathbb{G}}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a}_{\mathbb{G}}$ also, we use $\mathcal{H}$ as the configuration space of the $\alpha_{\mathbb{G}}$-systems and $\mathbb{A}$ as a group acting in $\mathbb{G}$, as well as, $\mathfrak{a}$ for its Lie algebra.

We say that the vector field $X$ and the map is $\sigma$ are equivariant if $X(a * g) = (L_a)_* X(g)$ and $\sigma(a * g) = (L_a)_* \sigma(g)$. For more details of this definition, see [8] pg 108 or [9] pg 161. The general property of infinitesimal generators and the fact that $\mathbb{A}$ is abelian imply that the left-invariant vector fields and $\sigma(g)$ are $\mathbb{A}$-equivariant.

$\sigma(g)$ send the canonical base $e^{\ell}$ for $\mathbb{A}$, with $1 \leq \ell \leq m$, into the frame of Killing vector fields $\sigma^{\ell}(g)$. Thus, the frame $\sigma^{\ell}(g)$ defines a co-frame $\omega_{\ell}(g)$ with $1 \leq \ell \leq m$, such that, $\omega_{\ell_1}(\sigma^{\ell_2})(g) = \delta_{\ell_1}^{\ell_2}$. so, the co-frame $\sigma_{\ell}$

is $\mathbb{A}$-equivariant. We also split the base $e^\ell$, $\sigma^\ell(g)$ and $\omega_\ell(g)$ according to the grading of $\mathbb{G}$ as we did with the left-invariant vector field; that is, we use $e^k$, $\sigma^k(g)$ and $\omega_k(g)$ for $1 \leq k \leq n_1$ and $e^j_r$, $\sigma^j_r(g)$ and $\omega^r_j(g)$ for $1 \leq j \leq d_r$ and $1 < r \leq s$ such that, $\omega^{r_1}_{\ell_1}(\sigma^{\ell_2}_{r_2})(g) = \delta^{\ell_2}_{\ell_1}\delta^{r_1}_{r_2}$.

We remark that $\mathfrak{a}_{\mathbb{G}}$ and $\sigma(\mathfrak{a})$ as abstract Lie algebras and a sub-vector space inside $\mathfrak{g}$ are the same, however, inside $\mathfrak{g}$, they are different Lie algebras, in general only the left-invariant vector fields in $\sigma(\mathfrak{a})$ and $\mathfrak{g}_G$ are the one in the last layer $\mathfrak{g}_s$.

2.2. $\mathbb{G}$ **as** $\mathbb{A}$**-principle bundle.** We can think of $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ as a principle $\mathbb{A}$-bundle. In our case, we have identified the base $\mathcal{H}$ with a sub-vector space $\mathcal{H} \subset \mathfrak{g}_1$, one which is complementary to $\mathfrak{a} \subset \mathfrak{g}$ so that

$$\mathcal{H} \oplus \mathfrak{a} = \mathfrak{g}.$$

This way, $\mathcal{H}$ also defines a connection on our principal bundle $\pi_{\mathbb{A}}$. Note: $\mathfrak{a}$ represents the vertical space for $\pi_{\mathbb{A}}$. And by left-translation about $\mathbb{G}$, $\mathcal{H} \subset \mathfrak{g}$ defines an $\mathbb{A}$-invariant choice of horizontal, as required for connections on principal $\mathbb{A}$-bundles.

For more bundles with connections, see [10] Chapter 8, [9] Chapter 12, or [8] sub-Chapter 2.9.

2.2.1. *Connection form.* The connection one-form $\omega_g$ on $\mathbb{G}$ is a $\mathfrak{a}$ value one-form given by

$$\omega(g) = \sum_{\ell=1}^{m} \omega_\ell \otimes e^\ell(g) = \sum_{k=1}^{n_1} \omega_k \otimes e^k|_g + \sum_{r=2}^{s}\sum_{\ell=1}^{d_s} \omega^r_j \otimes e^j_r(g). \tag{2.6}$$

$\mathbb{A}$-equivariant; $(L_a)_*\omega(g) = \omega(a * g)$. By definition $ker\ \omega(g) = \mathcal{H}_{\mathbb{G}}(g)$ and $\omega \circ \sigma(g) = Id_{\mathfrak{a}}$. The first condition tells $d\pi_{\mathbb{G}}$ is the canonical identification between $\mathcal{H}$ and $\mathcal{H}_{\mathbb{G}}$ , while, the second asserts that $\omega$ is the canonical identification between $\mathfrak{a}$ and $\mathfrak{a}_{\mathbb{G}}$

$$0 \longrightarrow \mathfrak{a} \longrightarrow^{\sigma(g)} \mathfrak{g}|_g \longrightarrow^{d\pi_{\mathbb{A}}} T_x\mathcal{H} \longrightarrow 0$$

A connection is a splitting of the exact sequence

$$0 \longleftarrow \mathfrak{g} \longleftarrow^{\omega(g)} \mathfrak{g}|_g \longleftarrow^{h_x} T_x\mathcal{H} \longleftarrow 0$$

where $h_x : T_x\mathcal{H} \to \mathfrak{g}|_g$ is the lineal map given by

$$h_x(e^i) := X^i(g) - \sum_{\ell=1}^{m} \omega_\ell(X^i)|_g \sigma^\ell(g),$$

where $e^i$ is the canonical base for $\mathcal{H}$. If we define $\tilde{X}^i := h(e^i)$, then the vector fields $X^i$'s are a base for $\mathcal{H}_{\mathbb{G}}$, so $h_x(\mathcal{H}) = \mathcal{H}_{\mathbb{G}}$ and $d\pi_{\mathbb{A}} \circ h_x = Id_{T_x\mathbb{R}^m}$. We summarize saying that, $X^i$'s is a base for the sub-space $\mathcal{H}_{\mathbb{G}}|_g$, while, $\sigma_\ell$'s is a base for the space $\mathfrak{a}_{\mathbb{G}}|_g$.

2.2.2. *The $\mathfrak{a}^*$ value one-form $\alpha_{\mathbb{G}}$.* Let us introduce the formal definition of the $\mathfrak{a}$ value one-form $\alpha_{\mathbb{G}}$ on $\mathbb{R}^{n_1}$. The horizontal map *hor* allows us to define a linear projection $\Pi_{\mathbb{R}^{d_1}}$ from $T^*G$ to $T^*\mathcal{H}$, let $\lambda$ in $T^*G$ then $\Pi_{\mathbb{R}^{d_1}}(\lambda) := \lambda \circ hor$.

**Definition 2.3.** *The $\mathfrak{a}^*$ value one-form $\alpha_{\mathbb{G}}$ on $\mathbb{R}^{d_1}$ is given by*

$$\alpha_{\mathbb{G}} := \Pi_{\mathbb{R}^{d_1}}(\omega)(g).$$

$\alpha_{\mathbb{G}}$ and $\omega(g)$ are $\mathbb{A}$ equivariant one-form on $T^*R^{d_1}$ and $T^*\mathbb{G}$, to give a formal definition of the derivative with respect a vector field $v$ in $\mathcal{H}$, we use the covariant derivative.

2.2.3. *Covariant derivative.* The connection on $\mathbb{G}$ induce a covariant derivative $\nabla$ on the space of $\mathbb{A}$-equivariant zero-form $\Omega^0(\mathbb{G},\mathcal{H})$, that is, $\nabla : \Omega^0(\mathbb{G},\mathcal{H}) \to \Omega^1(\mathbb{G},\mathcal{H})$ such that for any $\mathbb{A}$-equivariant vector field $Z$ on $\mathbb{G}$ the contraction of the covariant derivative with a vector field $v(x)$ on $\mathcal{H}$ is

$$\nabla_v Z := (\sum_{i=1}^{n} v_i(x)\tilde{X}^i)Z \text{ and satisfies } \nabla_v fZ = df(v)Z + f(x)\nabla_v Z.$$

See [9], and [10] for more details.

2.3. **Exponential coordinates of the second kind** $(x,\theta)$**.** We use this frame to give coordinates to the Carnot group in the following way a point $g$. We define a map from the coordinates $x$'s and $\theta$'s

$$\begin{aligned} &\Phi(x) := \exp(x_n X^n) * \cdots * \exp(x_1 X^1) \\ &\Phi(\theta) := \exp(\sum_{j=1}^{m} \theta_\ell Y^\ell) = \exp(\sum_{k=1}^{n_1} \theta_k Y^k + \sum_{r=2}^{s}\sum_{j=1}^{d_r} \theta_j^r Y_r^j) \end{aligned} \tag{2.7}$$

**Definition 2.4.** *The exponential coordinates $(x,\theta)$ are given by unique chart $(\mathbb{R}^{n+m},\Phi)$ where a point $g$ of is given by $g := \Phi(x,\theta) := \Phi(\theta) * \Phi(x)$.*

**Proposition 2.1.** *Let $\mathbb{G}$ be a meta-abelian Carnot group, then the horizontal left-invariant vector fields $\{X^i\}$ and $\{Y^k\}$ only depend on the coordinates $x$'s in a polynomial way of degree $s-1$, so does $\{Y_r^j\}$. Let $g = (x,\theta)$ be in $\mathbb{G}$, then : the left-invariant vector fields and the* *left-invariant one-forms on $\mathbb{G}$ are given by*

$$X^1(g) = \frac{\partial}{\partial x_1}, \quad X^i = \frac{\partial}{\partial x_i} + \sum_{r=2}^{s}\sum_{j=1}^{d_r} A^r_{i,j}(x)\frac{\partial}{\partial \theta^r_j} \qquad 2 \leq i \leq n,$$

$$Y^k(g) = \frac{\partial}{\partial \theta_k} + \sum_{r=2}^{s}\sum_{j=1}^{d_r} \beta^r_{k,j}(x)\frac{\partial}{\partial \theta^r_j} \qquad 1 \leq k \leq n_1,$$

$$\Theta_k(g) = d\theta_k, \quad \Theta^r_j(g) = d\theta^r_j - \sum_{i=1}^{n} \mathcal{A}^r_{ij}(x)dx_i,$$

*where $A^{i,j}_r(x)$ and $\beta^{k,j_1}_r(x)$ are homogeneous polynomial functions on the horizontal coordinates of degree $r-1$ or zero.*

*Proof.* We will use that $\frac{\partial}{\partial x_i} := \frac{d}{dt}\Phi(x+te_i,\theta)$, as well as, $\frac{\partial}{\partial \theta^r_j} := \frac{d}{dt}\Phi(x,\theta+te^r_j)$ to write the base $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \theta_k}, \frac{\partial}{\partial \theta^r_j}\}$ in terms of the base $\{X^i.Y^k, Y^r_j\}$, then we will see that change of base matrix is an upper diagonal matrix with the following form,

$$\begin{pmatrix} \frac{\partial}{\partial x_i} \\ \frac{\partial}{\partial \theta_k} \\ \frac{\partial}{\partial \theta^1_j} \\ \vdots \\ \frac{\partial}{\partial \theta^s_j} \end{pmatrix} = \begin{pmatrix} Id_{\mathfrak{g}_1} & Q_{1,2}(x) & \cdots & \cdots & Q_{1,s}(x) \\ 0 & Id_{\mathfrak{g}_2} & Q_{2,3}(x) & \ddots & Q_{2,s}(x) \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & Id_{\mathfrak{g}_{s-1}} & Q_{s-1,s}(x) \\ 0 & \cdots & \cdots & 0 & Id_{\mathfrak{g}_s} \end{pmatrix} \begin{pmatrix} X^i \\ Y^k \\ Y^j_2 \\ \vdots \\ Y^j_s \end{pmatrix}.$$

To calculate these expressions we will introduce the following notation $g_i := \exp(x_1X_1) * \cdots * \exp(x_{i-1}X_{i-1})$, then we notice that

$$\Phi(x+te_i,\theta) = \Phi(x,\theta) * Ad_{g_i}\exp(tX_i) = g * Ad_{g_i}\exp(tX_i)$$

Let us start with the case $i = 1$,

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{d}{dt}\Phi(x+te_1,\theta)|_{t=0} = \frac{d}{dt} g * \exp(tX^1)|_{t=0} \\ &= (L_g)_* X^1 = X^1(g). \end{aligned}$$

Let us calculate for $1 < i$,

$$\begin{aligned} \frac{\partial}{\partial x_i} &= \frac{d}{dt}\Phi(x+te^j,\theta)|_{t=0} = \frac{d}{dt} g Ad_{g_j}\exp(tX^j)|_{t=0} \\ &= (L_g)_* \exp(-x_1 ad_{X^1})\cdots\exp(-x_m ad_{X^m})(X^j) \\ &= X^j(g) + \sum_{r=2}^{s}\sum_{j_1=1}^{d_r} Q^{j,r}_{j_1}(x) Y^{j_1}_r. \end{aligned}$$

We finish with the case $1 \leq j \leq d_1$ and $1 < r \leq s$,

$$\begin{aligned}\frac{\partial}{\partial \theta_j^r} &= \frac{d}{dt}\Phi(x,\theta + te_j^r)|_{t=0} = \frac{d}{dt} gAd_{g_m}\exp(tY_j^r)|_{t=0} \\ &= (L_g)_* \exp(-x_1 ad_{X_1})\cdots\exp(-x_m ad_{X_m})(Y_r^j) \\ &= Y_r^j(g) + \sum_{r_1=r+1}^{s}\sum_{j_1=1}^{n_{r_1}} \tilde{Q}_{j_1}^{j,r_1}(x) Y_{r_1}^{j_1}.\end{aligned}$$

□

**Lemma 2.1.** *Let $\mathbb{G}$ be a meta-abelian Carnot group and let $g = (x, \theta)$ be in $\mathbb{G}$, then :*

*(1) The frame of Killing vector field associated $\sigma_\ell(g)$ and its co-frame are $\sigma_\ell(g) := \frac{\partial}{\partial\theta_\ell}$ and $\omega_\ell(g) = d\theta_\ell$.*

*(2) $\omega(g) = \sum_{\ell=1}^m d\theta_\ell \otimes e^\ell$ and $\tilde{X}_i(g) = \frac{\partial}{\partial x_i}$.*

*(3) The relationships between the covariant derivative $\nabla$ and the Lie bracket [ , ] are given by*

$$\nabla_{e^{i_1}} X^i - \nabla_{e^i} X^{i_1} = [X^{i_1}, X^i], \;\; \nabla_{e^i} Y^k = [X^i, Y^k], \;\; \nabla_{e^i} Y_r^j = [X^i, Y_r^j].$$

*(4) The coefficient $\mathcal{A}_{i,\mu}$ and $\beta_{j,\mu}$ are given by*

$$\mathcal{A}_{i,\mu}(x) := \sum_{j=1,r=2}^{d_r,s} a_j^r \mathcal{A}_{i,j}^r(x) \quad \beta_{k,\mu}(x) = \sum_{j=1,r=2}^{d_r,s} a_j + a_j^r \beta_{k,j}^r(x).$$

*Proof.* (1) By definition

$$\begin{aligned}\varphi(\exp(tY_r^j), g) &= \exp(tY_r^j) * g = \exp(tY^\ell) * \Phi(\theta) * \Phi(x) \\ &= \Phi(\theta + te_r^j) * \Phi(x) = \Phi(\theta + te_r^j, x).\end{aligned}$$

Then $\sigma_r^j = \frac{d}{dt}\Phi(\theta + te_r^j, x) = \frac{\partial}{\partial\theta_j^r}$, so $\sigma_r^j = d\theta_j^r$.

(2) By part (2) and the definition $\omega_g$ and $\tilde{X}_i(g)$.

(3) By part (1) and the definition of $\nabla_{e^i}$.

(4) Let $\mu = \sum_{k=1}^{n_1} a_k e^k + \sum_{j=2,r=2}^{d_r,s} a_j^r e_r^j$, by definition

$$\mathcal{A}_{i,\mu}(x) =< \mu, \omega(X^i) >= \sum_{\ell=1}^{m} \omega_\ell \otimes e^\ell(X^i,\mu) = \sum_{\ell=1,r=2}^{s,d_s} a_j^r A_{i,j}^r(x),$$

$$\begin{aligned}\beta_{k,\mu}(x) &=< \mu, \omega(Y^k) >= \sum_{k_1=1}^{n_1} \omega_{k_1}(Y^k)\mu(e_{k_1}) + \sum_{\ell=1,r=2}^{s,n_s} \omega_j^r(Y^k)\mu(e_j^r) \\ &= a_k + \sum_{j=1,r=2}^{s,d_s} a_j^r \beta_{k,j}^r(x).\end{aligned}$$

□

We define functions $\beta^r_{j,\mu} :=< \mu, \omega(Y^j_r) >$ with $1 \leq j \leq d_r$ and $1 < r \leq s$.

**Lemma 2.2.** *The relation between the partial derivatives of the family of functions $A_{i,\mu}$, $\beta_{k,\mu}(x)$ and $\beta^r_{j,\mu}$ and the Lie bracket $[\ ,\ ]$ of the left-invariant vector fields $X^i$, $Y^j$ and $Y^j_r$ holds the following relationships*

$$\frac{\partial}{\partial x_{i_1}} A_{i,\mu} - \frac{\partial}{\partial x_i} A_{i_1,\mu} =< \mu, \omega([X^{i_1}, X^i]) >$$

$$\frac{\partial}{\partial x_i} \beta_{j,\mu} =< \mu, \omega([X^i, Y^j]) > \quad \frac{\partial}{\partial x_i} \beta^r_{j,\mu} =< \mu, \omega([X^i, Y^j_r]) > .$$

*Proof.* Then,

$$\frac{\partial}{\partial x_{i_1}} \mathcal{A}_{i,\mu} := \nabla_{e^{i_1}} \omega(X^i, \mu) = \omega(\nabla_{e^{i_1}} X^i, \mu).$$

So

$$\frac{\partial}{\partial x_{i_1}} \mathcal{A}_{i,\mu} - \frac{\partial}{\partial x_i} \mathcal{A}_{i_i,\mu} = \omega(\nabla_{e^{i_1}} X^i - \nabla_{e^i} X^{i_1}, \mu) =< \mu, \omega([X^{i_1}, X^i]) >,$$

same proof for $\beta_{k,\mu}$ and $\beta^r_{j,\mu}$. □

## 3. The contangent bundle of $T^*\mathbb{G}$

Let $T^*\mathbb{G}$ be the cotangent bundle of $\mathbb{G}$, with the traditional coordinates $(p, q)$. A vector field $X$ in $\mathbb{G}$ define a function $P_X : T^*\mathbb{G} \to \mathbb{R}$, by $p(X)$. $T^*\mathbb{G}$ is endowed with the Poison bracket $\{\ ,\ \}_{T^*\mathbb{G}}$, the Lie algebra of vector fields and the Lie algebra of momentum functions are related by the equation

$$\{P_X, P_Y\}_{T^*\mathbb{G}} = p([X, Y]). \tag{3.1}$$

The frames $\{X^i\}$ and $\{Y^k\}$ and $\{Y^j_r\}$ define the frame $\{P_i\}$, $\{P_k\}$ and $\{P^r_j\}$ of momentum functions form $T^*\mathbb{G} \to \mathbb{R}$. Thus, the Hamiltonian function governing the geodesic flow is given by

$$H_{sR} = \frac{1}{2}(\sum_{i=1}^{n} P_i^2 + \sum_{k=1}^{n_1} P_k^2). \tag{3.2}$$

Where the condition $H_{sR} = 1/2$ implies that the geodesics are parameterized by arc-length.

**Lemma 3.1.** *Let $T^*\mathbb{G}$ be the cotangent bundle of a meta-abelian Carnot group and $H_{sR}$ the Hamiltonian function given by* (3.2)*, then*

*(1) The tangent vector of the geodesic $\gamma(t)$ is*

$$\dot{\gamma}(t) = \sum_{i=1}^{n} P_i(t) X^i(t) + \sum_{k=1}^{n_1} P_k(t) Y^k(t). \tag{3.3}$$

*(2) The evolution of $P_i$, $P_k$ and $P_j^r$ with respect of the subRiemannian geodesic flow is given by*

$$\begin{aligned}
\dot{P}_i &= \sum_{i_1=1}^{n} P_{[X^i, X^{i_1}]} P_{i_1} + \sum_{k=1}^{n_1} P_{[X^i, Y^k]} P_k \\
\dot{P}_k &= \sum_{i=1}^{n} P_{[Y^k, X^i]} P_i + \sum_{k_1=1}^{n_1} P_{[Y^k, Y^{k_1}]} P_{k_1} \\
\dot{P}_j^r &= \sum_{i=1}^{n} P_{[Y_r^j, X^i]} P_i + \sum_{k=1}^{n_1} P_{[Y_r^j, Y^k]} P_k.
\end{aligned} \tag{3.4}$$

*Proof.* (1) The momentum function $\{P_i\}$'s and $\{P_k\}$'s are lineal in $p$.

(2) We use the Poisson bracket $\{\ ,\ \}_{T^*\mathbb{G}}$ to fiend equations (3.4), that is, $\dot{f} = \{f, H_{sR}\}_{T^*\mathbb{G}}$.

□

3.1. **The momentum map** $J(p,g)$**.** We will denote by $P_\xi := p(\sigma_\xi)$ the $m$-dimensional algebra of momentum functions defined by the algebra of Killing vector field $\sigma(\mathfrak{a})$. The algebra of momentum functions $P_\xi$ defines a momentum map $J : T^*\mathbb{G} \to \mathfrak{a}^*$ given by the $\mu$ in $\mathfrak{a}^*$ such that $< \mu, \xi >= P_\xi$.

The Hamiltonian action of $\mathbb{A}$ on $T^*\mathbb{G}$ is given by the Hamiltonian flow associated to the function $P_\xi$, under this action $J$ is $\mathbb{A}$-equivariant and the relation between $J$ and $\omega_g$ is given by the following Lemma

**Lemma 3.2.** *Let $J : T^*\mathbb{G} \to \mathfrak{a}^*$ be the momentum map in the cotangent bundle $T^*\mathbb{G}$ of a meta-abelian Carnot group $\mathbb{G}$ then*

*(1) The momentum functions $P_\xi$ and momentum map $J$ are $\mathbb{A}$-equivariant*
*(2) $J$ defines $m$ constant of motion in involution with $H_{sR}$.*
*(3) If $\xi$ is in $\mathfrak{a}$ then $< J(p,g), \xi >=< J(p,g), \omega(\sigma_\xi) >$.*

*Proof.* (1) This is a well-known statement, see [11] or [8] pg 149.

(2) Because $H_{sR}$ is $\mathbb{A}$-invariant, the functions $P_\xi$'s are in involution since $\{P_{\xi_1}, P_{\xi_2}\}_{T^*\mathbb{G}} = -p([\xi_1, \xi_2]) = 0$ and Poisson commute with $H$.

(3) Let $\xi = \sum_{\ell=1}^m \xi_\ell e^\ell$ be in $\mathfrak{a}$, then $\sigma(\xi) = \sum_{\ell=1}^m \xi_\ell \sigma^\ell$ and $P_\xi = \sum_{\ell=1}^m \xi_\ell P_\ell$ where $P_\ell$'s are a constant of motions. Let us fix the level set

$P_\ell = a^\ell$, then $J(p,g) = \mu = \sum_{\ell=1}^m a^\ell e_\ell$ and

$$< J(p,g), \xi >=< \mu, \xi >= \sum_{\ell=\ell_1=1}^{m} a^\ell \xi_{\ell_1} e_\ell(e^{\ell_1}) = \sum_{\ell=1}^{m} a^\ell \xi_\ell.$$

On the other side, $\omega_g(\sigma_\xi, \mu) = \sum_{\ell=1}^n a_\ell \xi^\ell$.

So $< J(p,g), \xi >= \omega_g(\sigma_\xi, \mu)$. □

3.2. **The traditional coordinates** $(p_x, p_\theta, x, \theta)$ **on** $T^*\mathbb{G}$**.** Given the exponential coordinates of the second kind $(x,\theta)$, we denote by $p_i$'s, $p_k$'s and $p_j^r$'s the momentum associated these coordinates $x_i$'s $\theta_k$ and $\theta_j^r$.

**Proposition 3.1.** *Let $\mathbb{G}$ be a meta-abelian Carnot group. Then the horizontal momentum function $\{P_i\}$ and $\{P_k\}$ are given by*

$$P_i = p_i + A_{i,j}^r(x) p_j^r, \quad P_k = p_k + \beta_{j,j_1}^r(x) p_{j_1}^r.$$

*Where $A_{i,j}^r(x)$ and $\beta_{k,j}^r$ are given by proposition 2.1. As a consequence, the proof of the third line of equation 1.2. Thus $p_k$'s and $p_j^r$'s are constant of motions.*

*Proof.* Part one of Proposition 2.1 implies that $P_i$ and $P_j$ have the desired expression by the definition of momentum function. So does $H_{sR}$, by equation (3.2). $p_k$'s and $p_j^r$'s are constant of motion since $\theta_k$'s and $\theta_j^r$'s are cyclic coordinates. □

## 4. The contagent bundle $T^*\mathcal{H}$ and the $\alpha_{\mathbb{G}}$-system

Let $T^*\mathcal{H}$ be the cotangent bundle of $\mathbb{R}^n$ with the canonical symplectic structure and the traditional coordinates $(p_x, x)$, here we will use the Poisson bracket $\{\ ,\ \}_{T^*\mathcal{H}}$ which in coordinates is given by

$$\{f,g\}_{T^*\mathcal{H}} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i}.$$

See [5] or [6] for more details.

An $\alpha_{\mathbb{G}}$-system is a pair given by $(T^*\mathbb{R}^n, \alpha_{\mathbb{G}})$, then the pair $(T^*\mathbb{R}^n, \mu)$ defines the Hamiltonian function given by equation (1.2).

**Lemma 4.1.** *The Hamilton equations for the $\alpha_{\mathbb{G}}$-system are given by*

$$\dot{c}(t) := (p_1(t) + A_{1,\mu}(c(t)), \cdots, p_n(t) + A_{n,\mu}(c(t))). \tag{4.1}$$

*and*

$$\frac{d}{dt}(p_i + A_{i,\mu}) = \sum_{i_1=1}^{n}(p_i + A_{i,\mu})\left(\frac{\partial \mathcal{A}_{i_1,\mu}}{\partial x_i} - \frac{\partial \mathcal{A}_{i,\mu}}{\partial x_{i_1}}\right) + \sum_{j=1}^{n_1} \beta_{j,\mu}\frac{\partial \beta_{j,\mu}}{\partial x_i} \tag{4.2}$$

*Proof.* Hamilton equation $\dot{x} = \frac{\partial H_\mu}{\partial x}$ implies equation 4.1. Equation 4.6 is a consequence of the Hamilton equation using the Poisson structure; in other words, $\dot{f}$ read as $\{f, H_{sR}\}_{T^*\mathcal{H}}$. $\square$

4.1. **Symplectic reconstruction: Horizontal lift of $c(t)$ in $\mathcal{H}$.** A $\alpha_{\mathbb{G}}$-curve $c(t)$ for $\mu$ defines a curve $\gamma(t)$ in $\mathbb{G}$, this process is called the horizontal lift of $c(t)$ and is unique up to a constant of integration. For the sake of clarity, we will introduce the horizontal lift of $c(t)$ as the composition of two horizontal lifts, one from $c(t)$ to $\tilde{c}(t)$ a curve in $\mathbb{R}^{d_1}$ and the horizontal lift defined by the Carnot structure and defined in 4.1, that is, from the curve $\tilde{c}(t)$ to $\gamma(t)$.

Let us set the following function on $T^*\mathcal{H}$:

$$F_{i,\mu}(p_x, x) := p_i + A_{i,\mu}(x), \quad F_{k,\mu}(p_x, x) = \beta_{k,\mu}(x), \tag{4.3}$$

where $1 \leq i \leq n$ and $1 \leq k \leq n_1$. Let $c(t)$ be a $\alpha_{\mathbb{G}}$-curve and $(p(t), c(t))$ the Hamiltonian flow associated to $c(t)$, by evaluating the $F_{i,\mu}$ and $F_{k,\mu}$ along the flow $(p(t), c(t))$ we get the control functions associated to $c(t)$. Thus $c(t)$ defines $\tilde{c}(t)$ as the solution to the control problem

$$\frac{d}{dt}\tilde{c}(t) = \sum_{i=1}^{n} \dot{x}_i(t)e^i + \sum_{k=1}^{n_1} F_{k,\mu}(t)e^k = \sum_{i=1}^{n} F_{i,\mu}(t)e^i + \sum_{k=1}^{n_1} F_{k,\mu}(t)e^k. \tag{4.4}$$

(By abuse of notation we have written $F_{i,\mu}(t)$ for $F_{i,\mu}(c(t), p(t))$ etc.)

We formalize the above discussion with the following definition.

**Definition 4.1.** *Let $c(t)$ be in $\mathcal{H}$, we say that $\tilde{c}(t)$ is the horizontal lift of $c(t)$ to $\mathbb{R}^{d_1}$ if $\tilde{c}(t)$ is the solution to the equation* (4.4).

*Moreover, we say that $\gamma(t)$ is the horizontal lift of $c(t)$ if $\gamma(t)$ is the solution to the following equation*

$$\begin{aligned} \dot{\gamma}(t) &= \sum_{i=1}^{n} \dot{x}_i(t)X^i(c(t)) + \sum_{k=1}^{n_1} F_{k,\mu}(t)Y^k(c(t)) \\ &= \sum_{i=1}^{n} F_{i,\mu}(t)X^i(c(t)) + \sum_{k=1}^{n_1} F_{k,\mu}(t)Y^k(c(t)). \end{aligned} \tag{4.5}$$

Compare (4.5) with the equation (1.4) from Engel's example.

4.2. **The algebra of functions.** We define $F^r_{k,\mu}(p_x, x) := \beta^r_{k,\mu}(x)$ with $1 \leq j \leq d_r$ and $1 < r \leq s$.

**Proposition 4.1.** *The Lie algebra of functions $\{F_{i,\mu}, F_{k,\mu}, F^r_{i,\mu}\}$ with the Poisson bracket the Poisson bracket $\{\ ,\ \}_{T^*\mathcal{H}}$ is equivalent to $\mathfrak{g}$.*

*Proof.* By the definition of the Poisson bracket $\{\ ,\ \}_{T^*\mathcal{H}}$, it follows that

$$\begin{aligned}\{F_{i_1,\mu}, F_{i,\mu}\}_{T^*\mathcal{H}} &= \{p_{i_1}, A_{i,\mu}\} + \{A_{i_1,\mu}, p_i\} \\ &= \frac{\partial}{\partial x_{i_1}} A_{i,\mu} - \frac{\partial}{\partial x_i} A_{i_1,\mu} =< \mu, \omega([X^{i_1}, X^i]) > .\end{aligned}$$

Same proof to find

$$\begin{aligned}\{F_{i,\mu}, F_{k,\mu}\}_{T^*\mathcal{H}} &=< \mu, \omega([X^i, Y^k]) >, \\ \{F_{i,\mu}, F^r_{j,\mu}\}_{T^*\mathcal{H}} &=< \mu, \omega([X^i, Y^j_r]) > .\end{aligned}$$

□

The equivalence between the algebra of left-invariant vector fields, left-invariant momentum functions, and control functions is summarized by the following equations:

$$\begin{aligned}\{P_i, P_{i_1}\}_{T^*\mathbb{G}} &=< \mu, \omega([X_i, X_{i_1}] >= \{F_{i,\mu}, F_{i,\mu}\}_{T^*\mathcal{H}} \\ \{P_i, P_k\}_{T^*\mathbb{G}} &=< \mu, \omega([X_i, Y_k] >= \{F_{i,\mu}, F_{k,\mu}\}_{T^*\mathcal{H}} \\ \{P_i, P^r_j\}_{T^*\mathbb{G}} &=< \mu, \omega([Y_i, Y^r_j] >= \{F_{i,\mu}, F^r_{j,\mu}\}_{T^*\mathcal{H}}.\end{aligned}$$

**Corollary 4.1.** *The Hamilton equations for the $\alpha_{\mathbb{G}}$-system in terms of the algebra of functions $\{F_{i,\mu}, F_{k,\mu}, F^r_{i,\mu}\}$ are given by equation and the following equations*

$$\begin{aligned}\dot{F}_{i,\mu} &= \sum_{i_1=1}^{n} F_{i_1,\mu}\{F_{i,\mu}, F_{i_1,\mu}\}_{T^*\mathcal{H}} + \sum_{k=1}^{n_1} F_{k,\mu}\{F_{i,\mu}, F_{k,\mu}\}_{T^*\mathcal{H}} \\ \dot{F}_{k,\mu} &= \sum_{i_1=1}^{n} F_{i,\mu}\{F_{k,\mu}, F_{i,\mu}\}_{T^*\mathcal{H}} + \sum_{k_1=1}^{n_1} F_{k_1,\mu}\{F_{k,\mu}, F_{k_1,\mu}\}_{T^*\mathcal{H}} \\ \dot{F}^r_{j,\mu} &= \sum_{i_1=1}^{n} F_{i,\mu}\{F^r_{j,\mu}, F_{i,\mu}\}_{T^*\mathcal{H}} + \sum_{j=1}^{n_1} F_{j,\mu}\{F^r_{j,\mu}, F_{j,\mu}\}_{T^*\mathcal{H}}\end{aligned} \tag{4.6}$$

*Proof.* By the definition $\dot{f} = \{f, H_{sR}\}$. □

## 5. Proof of Theorem A

*Proof.* Let $c(t)$ be a $\alpha_{\mathbb{G}}$-curve, that is, $(p_x(t), c(t))$ is solution to Hamiltonian system given by (4.1) and (4.2). If we define the momentum function in $T^*\mathbb{G}$ by $P_i(t) := F_{i,\mu}(t)$, $P_k(t) := F_{k,\mu}(t)$ and $P^r_{j_r}(t) := F^r_{j,\mu}(t)$. If $\gamma(t)$ is the lift of $c(t)$, $\gamma(t)$ satisfies the horizontal lift equation (4.5), then $\gamma(t)$ holds equation (3.3). Thus, it is enough to prove that $F_i(t)$, $F_k(t)$ and $F^r_j(t)$ satisfy the O.D.E. given by (3.4).

Indeed, Proposition 4.1 shows that the algebra of functions $(F_\mu)_i(t)$ and $(F_\mu)_j(t)$ generate the Lie algebra $\mathfrak{g}$ with Poisson bracket $\{\ ,\ \}_{T^*\mathcal{H}}$, moreover, $F_{i,\mu}(t)$, $F_{k,\mu}(t)$ and $F^r_{j,\mu}(t)$ satisfies equations (4.6), comparing equations from with equations (3.4); we conclude that $P_i(t) := F_{i,\mu}(t)$, $P_k(t) := F_{k,\mu}(t)$ and $P^r_{j,\mu}(t) := F^r_{j_r}(t)$ hold Hamilton equations, and the solution $(p(t), \gamma(t))$ have the level $\mu$ by construction.

Conversely, let $\gamma(t)$ be a normal geodesic in $\mathbb{G}$ with level set $\mu$ and $c(t) := \pi_{\mathbb{A}}(\gamma(t))$, then $\gamma(t)$ satisfies the equation (3.3) and it is enough to prove that $c(t)$ satisfies Hamilton equations for $H_\mu$. Proposition 3.1 implies that we can defined the level set $\mu = (a_k, a^r_j) = (p^k, p^j_r)$, then $H_{sR} = H_\mu = 1/2$, which tell us that $c(t)$ is a solution to the $n$-degree of freedom for the Hamiltonian system $H_\mu$. □

If we think $\mathbb{G}$ as $\mathbb{A}$-principle bundle, see sub-Section 2.2, then this Theorem is equivalent to the one proved by Richard Montgomery in [9] pg 164, in the context of abelian charge. In our problem, we have the extra condition that any close form in $\mathbb{R}^{d_1}$ is exact, then we can give a Hamiltonian structure to a charged particle under the electromagnetic field.

## 6. The $\alpha_{\mathbb{G}}$-system

Here we will describe some dynamical properties of the $\alpha_{\mathbb{G}}$-system that will be essential for the proof of Theorem B

6.1. *$L$*-**periodic curves** $c(t)$. Let us consider the case when $c(t)$ is a $L$-periodic curve. Let us study the horizontal lift of $c(t)$. We want to compute the change that suffers the coordinates $\theta$'s after a period $L$. This computation has been done before in a similar context. In [12] in the context of $J^k(\mathbb{R}, \mathbb{R})$, and in [13], outside the Carnot group world in the magnetic space $\mathbb{R}^3_F$.

Let us calculate the projection of $\dot\gamma(t)$ to the component $\theta_j$, that is,

$$\dot\theta_k = d\theta_k(\dot\gamma)(t) = F_{j,\mu}(t), \qquad \dot\theta^r_j = d\theta^r_j(\dot\gamma)(t) = \sum_{j=1}^{n_1} F_{j,\mu} d\theta(Y_j). \tag{6.1}$$

**Proposition 6.1.** *Let $\gamma(t) = (c(t), \theta(t))$ be a geodesic in $\mathbb{G}$ for $\mu$, such that $c(t)$ is $L$-periodic in $\mathcal{H}$. After one period, $c(t)$ travels one times around the curve $C := c([0, L])$, the changes $\Delta\theta_k(\mu) = \theta_k(t_0+L)-\theta_k(t_0)$ and $\Delta\theta^r_j(\mu) = \theta^r_j(t_0 + L) - \theta^r_j(t_0)$ undergone by $\theta_k$ and $\theta$ are given by*

$$\Delta\theta_k(\mu) = \int_C F_{k,\mu}, \quad \Delta\theta^r_j(\mu) = \sum_{j=1}^{n_1} \int_C F_{j,\mu} d\theta(Y^r_j) \tag{6.2}$$

*Moreover, $\Delta\theta_k(\mu)$ and $\Delta\theta^r_j(\mu)$ do not depend on the initial point, and it is invariant time reflection $\gamma(t) \to \gamma(-t)$.*

*Proof.* Let $c(t)$ be a $L$-periodic $\alpha_{\mathbb{G}}$-curve in $\mathcal{H}$, we will estimate the change in $\theta_j$ and $\theta_j^r$ after the curve $(t)$ travel one period $L$, that is we will integrate equation (6.1) from a time $t$ to time $t+L$,

$$\Delta\theta_k(\mu) := \int_t^{t+L} F_{k,\mu}(c(t))dt = \int_C F_{j,\mu}.$$

To prove that $\Delta\theta_j(\mu)$ and $\Delta\theta_j^r(\mu)$ do not depend on the initial point, we derivative with respect to the initial time $t$,

$$\frac{d}{dt}\Delta\theta_k(\mu) = F_{k,\mu}(c(t+L)) - F_{k,\mu}(c(t)) = 0.$$

Let us consider the curve $c_1(t) := c(-t)$, $c_1(t)$ is a $L$-period $\alpha_{\mathbb{G}}$-curve since the Hamiltonian system is reflexible with respect the time and $c_1(L) = c(-L) = c_1(0)$, then.

$$\begin{aligned}\int_t^{t+L} F_{k,\mu}(c_1(t))dt = \int_t^{t+L} F_{k,\mu}(c(-t))dt = \int_{-t}^{-t-L} F_{k,\mu}(c(t))dt \\ = \int_C F_{k,\mu}.\end{aligned}$$

□

We define $\mathcal{P}(\mathbb{G})$ the $n$-dimensional sub-vector space of the polynomials one-form on $\mathfrak{g}_1$ with degree bounded by $s-1$ given by spam of $\{\alpha^k, \alpha_r^j\}$, where $\alpha^k :=< e_k, \alpha_{\mathbb{G}} >$ and $\alpha_r^j :=< e_j^r, \alpha_{\mathbb{G}} >$. The Euclidean product $( \, , \, )_{\mathfrak{g}_1}$ in $\mathfrak{g}_1$ induce a inner product in $\mathcal{P}(\mathbb{G})$ in the following way; let $\alpha$ and $\alpha'$ be in $\mathcal{P}(\mathbb{G})$, then the inner product is given by

$$(\alpha, \alpha')_{\mathcal{P}(\mathbb{G})} = \int_C (\alpha, \alpha')_{\mathfrak{g}_1}. \tag{6.3}$$

The inner product is no-degenerate

$$p_x + \alpha_\mu|_{c(t)} := \sum_{i=1}^{n} F_{i,\mu} d\theta_i + \sum_{k=1}^{n_1} F_{k,\mu} d\theta_k. \tag{6.4}$$

**Corollary 6.1.**

$$\Delta\theta_j =< p_x + \alpha_\mu, \alpha^j >_{\mathcal{P}(\mathbb{G})}, \quad \Delta\theta_j^r =< p_x + \alpha_\mu, \alpha_r^j >_{\mathcal{P}(\mathbb{G})} \tag{6.5}$$

6.2. $\beta_{\mathbb{G}}$**-systems.**

$$H_\mu = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\phi(x). \tag{6.6}$$

The geodesics associated with the constant polynomial $\phi(x)$ are called line geodesics since their projection to $\mathfrak{g}_1 \simeq \mathbb{R}^{d_1}$ is a line. Let us assume $\phi(x)$ is not constant. So, exist a compact set $K \subset \mathcal{H}$ where the dynamics takes place, that is, if $x$ is in *int* $K$, then $0 \leq \phi(x) < 1$ and if $x$ is in $\partial K$, then $\phi(x) = 1$, the second condition implies that $p_x = 0$, so we say that $c(t)$ bounces at boundary of $K$.

6.3. **$\mathcal{A}_{\mathbb{G}}$-systems.**

$$H_\mu = \frac{1}{2}\sum_{i=1}^{n}(p_i + \mathcal{A}_{i,\mu})^2. \tag{6.7}$$

If $c(t)$ is $\mathcal{A}$-system, then $H_\mu = H_{sR} = \frac{1}{2}$, so $||\dot{c}(t)|| = 1$.

6.3.1. 2-*abelian extension.* The reduced Hamiltonian $H_\mu$ is given by

$$H_\mu = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y + A_\mu(x,y))^2.$$

Let $c(t)$ be a $\alpha_{\mathbb{G}}$-curve for $\mu$, since $H_\mu = H_{sR} = 1/2$, a simple computation shows that the curvature $\kappa(t)$ of $c(t)$ is given by

$$\kappa(t) = -\frac{\partial A_\mu}{\partial x}.$$

6.4. $\mathbb{G}$ **as** $[\mathbb{G},\mathbb{G}]$**-principle bunlde and the intermediate Hamiltonian** $H_\mu$**.** We will introduce $\mathbb{G}$ as $[\mathbb{G},\mathbb{G}]$-principle bundle and describe the

Let $\tilde{\Phi}$ be the action of $[\mathbb{G},\mathbb{G}]$ on $\mathbb{G}$ given by the left multiplication.

6.4.1. $\mathbb{G}$ *as* $[\mathfrak{g},\mathfrak{g}]$*-principle bundle.* We can thing of $\pi : \mathbb{G} \to \mathbb{R}^{d_1}$ as a principle $[\mathbb{G},\mathbb{G}]$-bundle, where $\tilde{\varphi}$ is the action of $[\mathbb{G},\mathbb{G}]$ on $\mathbb{G}$.

The action $\tilde{\varphi}$ defines the map $\tilde{\sigma} := [\mathfrak{g},\mathfrak{g}] \to \mathfrak{g}$, which also we can see $\tilde{\varphi}$ as the restriction of $\sigma$ to $[\mathfrak{g},\mathfrak{g}]$, then $\tilde{\sigma}$ provides a frame of Killing vector field and its co-frame. The connection form is given by

$$\tilde{\omega}(g) = \omega|_{[\mathfrak{g},\mathfrak{g}]}(g) = \sum_{j=1,r=2}^{n_r,s} \omega_j^r \otimes e_r^j.$$

**Definition 6.1.** *The* $[\mathfrak{g},\mathfrak{g}]$ *value one-form* $\eta_{\mathbb{G}}$ *on* $\mathbb{R}^{d_1}$ *is given by*

$$\eta_{\mathbb{G}} := \Pi_{\mathbb{R}^{d_1}}(\tilde{\omega})(g).$$

Let $\eta_\mu$ be We write $\eta_\mu := \alpha( \ ,\mu)$ in terms of the base $dx$'s in $T^*\mathbb{R}^{d_1}$, in the following way

$$\alpha_\mu = \sum_{i=1}^{n} \mathcal{A}_{i,\mu} dx_i + \sum_{k=1}^{n_1} \beta_{k,\mu} d\theta_k. \tag{6.8}$$

6.4.2. *The intermediate Hamiltonian $H_\mu$.* Let $T^*\mathbb{R}^{d_1}$ be the cotangent bundle of $\mathbb{R}^{d_1} \simeq \mathfrak{g}$, with the canonical symplectic structure and the traditional coordinates $(p_x, p_\theta, x, \theta)$. Let $\eta_{\mathbb{G}}$ be the $[\mathfrak{g}, \mathfrak{g}]$ value one-form on $\mathbb{R}^{d_1}$ given by $\mathcal{A}_{\mathbb{G}} + \tilde{\beta}_{\mathbb{G}}$, if $\tau$ is in $[\mathfrak{g}, \mathfrak{g}]^*$ then $\eta_\tau$

(6.9)

$$H_\tau := \frac{1}{2}||p_x dx + p_\theta d\theta + \eta_\tau||^2_{\mathbb{R}^{d_1}} = \frac{1}{2}\sum_{i=1}^{n}(p_i + \mathcal{A}_{i,\mu}(x))^2 + \sum_{k=1}^{n_1}(p_k + \beta_{k,\mu}(x))^2.$$

The Hamiltonian $H_\tau$ does not depend on the coordinates $\theta_j$'s, so the coordinates $\theta_j$ are cycle, and the $p_j$'s are constant of motion. This consequence of the action of $\mathbb{A}$ in $\mathbb{R}^{d_1}$ by translations, where if we grade $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_s$ and $\mathfrak{a}_1 = \mathfrak{v}$. Then $\mathfrak{g}$ acts on $\mathbb{R}^{d_1}$ by translations and the action of $\mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_s$ is trivial.

Let $\tilde{J}$ be the momentum map $\tilde{J}(p_x, p_\theta, x, \theta) = \mu_1$. We say that that $\tilde{c}(t)$ is $\eta_{\mathbb{G}}$-curve for $\tau$ and with momentum $\mu_1$, if $\tilde{c}(t)$ is the projection of the Hamiltonian flow for $H_\tau$ with energy $1/2$ and with momentum $\mu_1 := \tilde{J}(p_x(t), p_\theta(t), x(t), \theta(t))$, we call the Hamiltonian system given by $H_\tau$ an $\eta_{\mathbb{G}}$-system. The horizontal lift of $\eta_{\mathbb{G}}$-curve $\tilde{c}(t)$ in $\mathbb{R}^{d_1}$ to $\mathbb{G}$ is given by equation (2.3). The last main Theorem of this work.

**Theorem D.** *Let $\mathbb{G}$ be a meta-abelian Carnot group and $\tau$ be in $[\mathfrak{g}, \mathfrak{g}]^*$, then exist $[\mathfrak{g}, \mathfrak{g}]$ value polynomial one-form $\eta_{\mathbb{G}}$ such that if $\tilde{c}(t)$ is a $\eta_{\mathbb{G}}$-curve for $\tau$ and with level $\mu_1$, then its horizontal-lift is a normal subRiemannian geodesic in $\mathbb{G}$ with momentum $\mu = \mu_1 \times \tau$. Conversely, if $\gamma(t)$ is a normal geodesic in $\mathbb{G}$ with momentum $\mu = \mu_1 \times \tau$, then the curve $\tilde{c}(t) = \pi(\gamma(t))$ is $\eta_{\mathbb{G}}$-curve for $\tau$ and with level $\mu_1$.*

6.4.3. *Eng as 2-abelian extension.* The $[\mathfrak{g}, \mathfrak{g}]^*$ value one-form $\eta_{Eng}$ is given by

$$\eta_{Eng} = d\theta_0 \otimes (xe^2 + \frac{x^2}{2}e^3).$$

If $\tau = (a_2, a_3)$ is in $[\mathfrak{g}, \mathfrak{g}]^*$, then the reduced Hamiltonian is given by

$$H_\tau(p_x, p_y, x, y) = \frac{p_x^2}{2} + \frac{1}{2}(p_y + a_1 x + a_2\frac{x^2}{2})^2.$$

Let $\tilde{c}(t)$ be a $\beta_{Eng}$-curve for $\tau$, since $H_\tau = H_{sR} = 1/2$, a simple computation shows that the curvature $\kappa(t)$ of $\tilde{c}(t)$ is given by

$$\kappa(x(t)) = a_1 + a_2 x(t).$$

The $\eta_{Eng}$-curves are the solution to the Euler-Elastica problem, which one way to state the problem is like the curves in $\mathbb{R}^2$ such that is curvature is proportional to the distance to a given line called "directrix." In this case, the directrix is the line given by the solution $\kappa = 0$. see [14] or See for the relation of Euler-Elastica and Elliptic functions [15].

## 7. Proof Theorem B

7.1. **Non-periodic geodesics.** One necessary condition for a periodic geodesic in a meta-abelian Carnot group to be periodic is that $c(t)$ is periodic. Let $\gamma(t)$ be a geodesic in $\mathbb{G}$ such that $c(t)$ is $L$-periodic. We compute the changes that suffer the coordinates $\theta$'s after a period $L$.

*Proof.* We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^k$ corresponding to the pair $\mu$, where $\beta(x)$ is not constant one form, then $\Delta\theta_\ell(\mu) = 0$ with $1 \leq \ell \leq m$.

In the context of the space of polynomial one-forms $\mathcal{P}(\mathbb{G})$ with inner product $<,>_{\mathcal{P}}$, the conditions $\Delta\theta_\ell(\mu) = 0$ is equivalent to $\beta(x)$ being perpendicular to $\alpha_\ell$ ($0 = \Delta\theta_\ell(\mu) =< \beta, \alpha^\ell >_{\mathcal{P}}$), so $\beta(x)$ being perpendicular to $\alpha_\ell$ for $1 \leq \ell \leq n$. However, the set $\{\alpha_\ell\}$ with $1 \leq \ell \leq n$ is a base for $\mathcal{P}(\mathbb{G})$. Then $\beta$ is perpendicular to any one form on $\mathcal{P}(\mathbb{G})$, so $\beta$ is zero since the inner product is non-degenerate. Being $\beta$ equals zero contradicts the assumption that $\beta$ is not a constant one-form. □

The above result is a particular case of the conjecture "Carnot groups do not have periodic geodesic" by Enrico Le Donne. In [12], we proved the particular case $\mathbb{G} = J^k(\mathbb{R}, \mathbb{R})$.

7.2. **Cut time.** The upper bound $L$ does not depend on the initial point. The proof given here is the same as the one from [13] in the context of jet-space $J^k(\mathbb{R}, \mathbb{R})$; we will repeat the proof in this new language. The difference between the techniques using in the proof arises from the following dichotomy, $\dot{c}(t_0)$ is zero or not.

**Definition 7.1.** *Let $\gamma : \mathbb{R} \to \mathbb{G}$ be a subRiemannian geodesic parameterized by arc-length. The cut time of $\gamma$ is*

$$t_{cut}(\gamma) := \sup\{t > 0 : \ \gamma|_{[0,t]} \ \textit{ is length-minimizing}\}.$$

The cut time might depend on the initial point $\gamma(0)$, and some authors use the notation $t_{cut}(\gamma, \gamma(0))$ to specify the initial point. We will prove that the upper bound $L$ for cut time does not depend on the initial point, then we do not need to specify it. We will consider two cases; the first case is $\dot{c}(0) = 0$, while, in the second case is $\dot{c}(0) = 0$.

**Proposition 7.1.** *Let $\gamma(t)$ be a geodesic in $\mathbb{G}$ corresponding to $L$-periodic curve $c(t)$ such that $\dot{c}(0) = 0$, then $\gamma(L)$ is conjugate to $\gamma(0)$ along $\gamma$, so $\gamma(t)$ fails to fails to minimize beyond $L$.*

*Proof.* After a translation, we can assume that the periodic geodesic bounce on the origin, that is, $c(kL) = 0$, $\dot{x}(kL) = p_x(kL) = L$ and $W(kL) = 1$ for all $k = 0, 1, 2, 3, \ldots$. Let $y_j = F_j(0)$, where $F_j(x) =$

$\beta_\mu(e^j)$, then $W(kL) = ||(y_1, \cdots, y_n)|| = 1$ for all $k = 0, 1, 2, 3, \ldots$. Let us consider the Jacobi vector fields

$$W_1(t) = \sum_{i=1}^{n} F_{i,\mu}(t) X^i|_{c(t)} + \sum_{k=1}^{n_1} F_{k,\mu}(t) Y^k|_{c(t)}, \quad W_2(t) = \sum_{k=1}^{n_1} y_k \sigma^k|_{c(t)} \in \mathfrak{v}.$$

Therefore, the periodicity of $c(t)$ implies $W_1(jL) = \sum_{k=1}^{n_1} y_k Y^k|_0$ and $W_2(jL) = \sum_{k=1}^{n_1} y_k \sigma^k|_0$. By construction $Y_k|_0 = \sigma_k|_0$, so $W_1(jL) = W_2(jL)$. The space of Jacobi vector fields is a vector space, then $J := W_1(t) - W_2(t)$ is a Jacobi vector field such that $J(kL) = 0$ for all $k = 0, 1, 2, 3$.

To see that the field $J$ is not zero in the interior of the interval $(0, L)$, we consider the case again when $x(t)$, with $t$ in $(0, L)$, is inside $K$ or in $\partial K$. On one side, if $x(t)$ is inside $K$, then $\dot{c}(t) \neq 0$ and $J$ has non-zero component in $\mathcal{H}$. On the other side, if $x(t)$ is in $\partial K$ then $\dot{c}(t) = 0$, but $Y_j|_{c(t)}$ is tangent to $\mathfrak{v}$ if and only if $c(t) = 0$.

$J$ contributes 1 to the nullity of the Hessian of the action, thus establishing that $kL$ is a conjugate time to $s = 0$ along $\gamma$. The classic theory of calculus of variation tells that the geodesic $\gamma(t)$ fails to minimize beyond $L$. □

**Proposition 7.2.** *Let $\gamma(t)$ be a geodesic in $\mathbb{G}$ corresponding to $L$-periodic curve $c(t)$ such that $\dot{c}(0) \neq 0$, then exist $\gamma_1(t) \neq \gamma(t)$ such that $\gamma(0) = \gamma_1(0)$ and $\gamma(L) = \gamma_1(L)$, so $\gamma(t)$ fails to minimize beyond $L$.*

It is well-known that a geodesic $\gamma_1(t)$ fail to minimize beyond a time $L$ if exist another geodesic $\gamma_2(t) \neq \gamma_1(t)$ with $\gamma_1(0) = \gamma_2(0)$ such that $\gamma_1(L) = \gamma_2(L)$.

*Proof.* Let $c(t)$ be a $\alpha_{\mathbb{G}}$-curve for $\mu$ corresponding to $\gamma(t)$, such that is $L$-periodic and $\dot{c}(0) \neq 0$. Then, there are exactly two geodesic passing through $\gamma(0)$ and with value $\mu$, namely, the given one $\gamma_1(t)$ and $\gamma_2(t)$ characterized by $\dot{c}_2(0) = -\dot{c}_1(0)$. Then $c_1(t) = c_2(-t)$ for all $t$. By $L$-periodic $c_1(L) = c_1(0)$. Moreover, $c_1(t)$ defines the same curve $C := c_1[0, L] = c[0, L]$, thus Proposition 6.1 tell us that $\gamma_1(t)$ and $\gamma_2(t)$ have same period $\Delta\theta_\ell$. Thus,

$$\gamma_1(L) = \gamma_1(0) + (0, \Delta\theta_\ell(\mu)) = \gamma_2(L).$$

So $\gamma(t)$ fail to minimize beyond a time $L$ □

### 7.3. **Integrability.**

*Proof.* Let $H_\mu$ be integrable for all $\mu$ in $\mathfrak{a}^*$, that is, exist $n-1$ lineal independent constant of motion $I_k(p_x, x) : T^*\mathcal{H} \to \mathbb{R}$ in involution with $H_{sR}$. Let $\tilde{I}_k(x)$ be the lift of $I_k(p_x, x)$, then set of constant of motion

$\{\tilde{I}_k(p_x,x), p_\ell.H_{sR} = H_\mu\}$ are $(n+m)$-lineal independent functions in involution, so $H_{sR}$ is integrable. $\square$

7.4. **The sub-Riemannian Schrodinger equation on $G$.** Let us review the formal definition of both operator.

7.4.1. *The quantum Hamiltonian $\hat{H}_\mu$.* Let us use the second line of equation (1.2) as a definition of $H_\mu$, that is,

(7.1)
$$H_\mu = \frac{1}{2}||p_x - \mathcal{A}_\mu(x)||^2_{\mathcal{H}} + \frac{1}{2}\phi_\mu(x) = \frac{1}{2}\sum_{i=1}^{n}(p_x - \mathcal{A}_\mu(x))^2 + \frac{1}{2}\phi_\mu(x).$$

Therefore the quantum Hamiltonian $\hat{H}_\mu$ is given by

$$\hat{H}_\mu = -\frac{1}{2}\sum_{i=1}^{n}(\hbar\frac{\partial}{\partial x_i} - \mathcal{A}_{i,\mu}(x))^2 - \frac{1}{2}\phi_\mu(x). \tag{7.2}$$

Let us consider a $\Psi(x)$ such that $\hat{H}_\mu\Psi(x) = \lambda_\mu\Psi(x)$, then since the Hamiltonian $H_\mu$ is autonomous system, then $\Psi(x,t) = \Psi(x)\exp(i\lambda_\mu t)$ is a solution the Schrodinger equation $\hat{H}_\mu\Psi(x,t) = \lambda\frac{\partial}{\partial t}\Psi(x,t)$.

7.4.2. *The sub-Riemannian Schrodinger equation on $G$.* Since the Hamiltonian is purely kinetic the sub-Riemannian Schrodinger $\hat{H}_{sR}$ is minus the subRiemannian Laplacian, for more deatils about the subRiemannian Laplacian see [9] or [16]. The subRiemannian Laplacian $\mathbb{G}$ in a meta-abelian Carnot group is given by

$$\Delta_{sR} = (\sum_{i=1}^{n} X_i^2 + \sum_{k=1}^{n_1} Y_k^2). \tag{7.3}$$

Let us consider a $\Psi(x)$ such that $\hat{H}_\mu\Psi(x) = \lambda_\mu\Psi(x)$, then we need to prove that $\Psi(x,\theta,t) = \Psi(x)\Psi(\theta,t)$ where

$$\begin{aligned} \Psi(\theta,t) &= \exp(i(\frac{-\lambda_\mu t}{2\hbar} + \sum_{\ell=1}^{m}\frac{a^\ell\theta_\ell}{\hbar})) \\ &= \exp(i(\frac{-\lambda_\mu t}{2\hbar} + \sum_{k=1}^{n_1} a^k\theta_k + \sum_{r=2}^{s}\sum_{j=1}^{d_r}\frac{a_r^\ell\theta_\ell^r}{\hbar})). \end{aligned} \tag{7.4}$$

is a solution to the sub-Riemannian Schrodinger equation on $G$.

$$\bar{H}_{sR}\Psi(x,\theta,t) = -\frac{\hbar^2}{2}\Delta_{sR}\Psi(x,\theta,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,\theta,t). \tag{7.5}$$

*Proof.* Let us calculate $\Delta_{sR}\Psi(x,\theta)$ by parts, first let us calculate $X_i\Psi(x,\theta,t)$

$$\begin{aligned}
X_i^2\Psi(x,\theta,t) &= (\frac{\partial}{\partial x_i} + \sum_{r=2}^{s}\sum_{j=1}^{d_r} A_{i,j}^r(x)\frac{\partial}{\partial \theta_j^r})^2\Psi(x,\theta,t) \\
&= -\Psi(\theta,t)(\frac{\partial}{\partial x_i} - \frac{1}{\hbar^2}A_{i,\mu}(x))^2\Psi(x), \\
Y_k^2\Psi(x,\theta,t) &= \frac{1}{\hbar^2}\Psi(\theta,t)\beta_{k,\mu}(x)\Psi(x).
\end{aligned}$$

So

$$\begin{aligned}
\hat{H}_{sR}\Psi(x,\theta,t) &= \frac{1}{2}\Psi(\theta,t)(\sum_{i=1}^{n}(\hbar\frac{\partial}{\partial x_i} - A_{i,\mu}(x))^2 + \sum_{k=1}^{n_1}\beta_{k,\mu}(x))\Psi(x) \\
&= \frac{1}{2}\Psi(\theta,t)\hat{H}_\mu\Psi(x) = \frac{\lambda_\mu}{2}\Psi(x,\theta,t)
\end{aligned}$$

On the other side

$$i\hbar\frac{\partial}{\partial t}\Psi(x,\theta,t) = \frac{\lambda_\mu}{2}\Psi(x,\theta,t).$$

□

In case of the $\mathfrak{h}$ is abelian, $(\sum_{i=1}^{n} X_i^2)$ is just the Laplace operator $\Delta_{\mathcal{H}}$ on $\mathcal{H}$.

## 8. 2-abelian extensions

In this section, we propose the method to make the complete classification of integrable subRiemannian geodesic flow on meta-abelian Carnot group. We recall the history of the classification of the Hamiltonian system has 100 years of history. In particular, we will use the theory of integrable systems in magnetic space; see [17] or [4]. Also, we will use the theory of a conservative system with polynomial potential, see [18].

In the case of 2-abelian extensions, we need to find one new constant of motion depending on the variables $(p_x, p_y, x, y)$ in involution with $H_{sR}$ or prove that the constant of motion does not exist.

### 8.1. $\mathcal{A}_{\mathbb{G}}$-systems.

We remark that the general form of the reduced Hamiltonian $H_\mu$ is given by

$$H_\mu = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y + A_\mu(x,y))^2.$$

8.1.1. *$F_{23}$ or Cartan group as 2-abelian extension.* Let $F_{23}$ be the free-nilpotent Lie algebra with 2 generators of step 3 and growth vector $(2,3,5)$, also called Cartan group. The first layer $\mathfrak{g}_1$ is framed by $\{X^1, X^2\}$, and the following relationships give its Lie algebra.

$$Abelian Y_2^1 := [X^1, X^2], \qquad Y_3^1 := [X^1, Y_2^1], \qquad Y_3^2 := [X^2, Y_2^1]$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is given by $Y_1^1$, $Y_1^2$ and $Y_2^2$. So in this case $Car \simeq \mathbb{R}^2 \times [\mathbb{G}, \mathbb{G}]$, also

$$\alpha_{F_{23}} = dy \otimes (xe^1 + \frac{x^2}{2}e^2 + xye^3),$$

Notice that: the step $s = 3$ and the polynomial are degree two. If $\mu = (a_1, a_2, a_3)$ is in $\mathfrak{a}^*$ then the reduce Hamiltonian is

$$H_\mu(p_x, x) = \frac{1}{2}(p_x^2 + (p_y + a_1 x + a_2\frac{x^2}{2} + a_3 xy)^2).$$

The system is Arnold-Liouville integrable; indeed, the Poisson structure has a Casimir functions $C$ given by

$$C := P_5P_1 - P_2P_4 - \frac{1}{2}P_3^2 = a_3p_x - a_2p_y + a_1a_3y + \frac{a_3^2}{2}y^2.$$

Another way to make this integration is using the Noether theorem. The constants of motion given by the left action are $p_1$, $p_2$, $p_2$ and

$$I_1 = p_x + yp_1 + \theta_1 p_2 + \frac{y^2}{2}p_3, \qquad I_2 = p_y + \theta_1 p_3.$$

We can see that $I_1$ and $I_2$ are not in involution with $p_1$, $p_2$ and $p_3$, however, the lineal combination $I := p_5I_1 - p_4I_2 = C$, so $p_1$, $p_2$, $p_3$ and $I$ are in involution with $H_{sR}$.

The $\alpha_{F_{23}}$-curves are the solution to the Euler-Elastica problem; that is, the curvature of $c(t)$ is

$$\kappa(x(t), y(t)) = a_2x(t) + a_3y(t) + a_1, \tag{8.1}$$

and the directrix is given by the line $\kappa(x, y) = 0$. After a rotation in the plane $\mathbb{R}^2$ the Hamiltonian form equation became in (1.4). See Agrachev [16] Exercise 7.80.

8.1.2. $N_{6,2,5a^*}$. Let $N_{6,2,5a^*}$ be the 6 dimensional Carnot group with growth-vector $(2,3,5,6)$. Its first layer $\mathfrak{g}_1$ is framed by $\{X^1, X^2\}$, and the following relationships give its Lie algebra.

$$\begin{aligned} &Y_2^1 := [X^1, X^2], \;\; Y_3^1 := [X^1, Y_2^1], \\ &Y_3^2 := [X^2, Y_1^2], \;\; Y_1^4 := [X^1, Y_3^1] = [X^2, Y_3^2]. \end{aligned}$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is $[G,G]$, so in this case $F_{24} \simeq \mathbb{R}^2 \times [G,G]$, also

$$\alpha_{F_{24}} = dy \otimes (xe_1^2 + \frac{x^2}{2}e_1^3 + xye_2^3 + (\frac{x^3}{3!} + \frac{xy^2}{2})e_1^4),$$

$$I(p_x,p_y,x,y) := P_1P_5 - P_2P_4 + \frac{1}{2}P_3^2. \tag{8.2}$$

E. Le Donne and F. Tripaldi used the notation $N_{6,2,5a^*}$ in [19].

8.1.3. *$F_{24}$ as 2-abelian extension.* Let $F_{2,3}$ be the free-nilpotent Lie algebra with two generators of step 3 and growth vector $(2,3,5,8)$. Its first layer $\mathfrak{g}_1$ is framed by $\{X^1, X^2\}$, and the following relationships give its Lie algebra.

$$Y_1^2 := [X^1, X^2], \qquad Y_1^3 := [X_1, Y_1^2], \qquad Y_2^3 := [X_2, Y_1^2]$$
$$Y_1^4 := [X^1, Y_1^3], \qquad Y_2^4 := [X^1, Y_2^3] = [X^2, Y_1^3], \qquad Y_1^3 := [X^2, Y_2^3].$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is $[G,G]$, so in this case $F_{24} \simeq \mathbb{R}^2 \times [G,G]$, also

$$\alpha_{F_{24}} = dy \otimes (xe_1^2 + \frac{x^2}{2}e_1^3 + xye_2^3 + \frac{x^3}{3!}e_1^4 + \frac{yx^2}{2}e_2^4 + \frac{y^2x}{2}e_3^4),$$

8.2. **$\beta_{\mathbb{G}}$-systems.** We remark that the work done by J. Llibre, A. Mahdi, and C. Valls, in [18], it is enough to classify $\beta_{\mathbb{G}}$-system, where $\mathbb{G}$ is meta-abelian Carnot group with step 3.

8.2.1. *$Eng(2)$ as 2-abelian extension.* Let $Eng(2)$ be the 6-dimensional Carnot group and with growth vector $(3,5,6)$. The first layer $\mathfrak{g}_1$ is framed by $\{X^1, X^2, Y\}$, and its Lie algebra is given by the following relationships.

$$Y_2^1 := [X^1, Y], \qquad Y_2^2 := [X^2, Y], \qquad Y_3^1 := [X^1, Y_2^1] = [X^2, Y_2^2]$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is given by $Y$ $Y_2^1$, $Y_2^2$ and $Y_3^1$: So in this case $Eng(2) = \mathbb{R}^2 \ltimes \mathbb{A}$. $\alpha_{Eng(2)}$ associated with $Eng(2)$ is given by

$$\alpha_{Eng(2)} = d\theta_1 \otimes (e_1 + xe_2 + ye_3 + \frac{x^2+y^2}{2}e_4),$$

Notice that: the step $s=3$ and the polynomial are degree two. Then if $\mu = (a_1, a_2, a_3)$ in $\mathfrak{a}^*$,

$$H_\mu(p_x, x) = \frac{1}{2}(p_x^2 + p_y^2 + (a_0 + a_1x + a_2y + a_3\frac{x^2+y^2}{2})^2).$$

**Theorem 1.** *The subRiemannian geodesic flow on $Eng(2)$ is integrable.*

*Proof.* After a translation in the $(x, y)$ plane, the system became in

$$\frac{1}{2}(p_x^2 + p_y^2 + (b + a(x^2 + y^2))^2), \tag{8.3}$$

which is the radial an-harmonic oscillator, equation (8.3) is invariant under the action of the group $SO(2)$ and the reduced Hamiltonian $H_\mu$ is integrable, so does $Eng(2)$. □

E. Le Donne and F. Tripaldi used the notation $N_{3,6,1a^*}$ for this group in [19].

8.2.2. *The unit-lower-triangular as* 2*-abelian extension.* Let $A_{ult}$ be the 6-dimensional Carnot group and with growth vector $(3, 5, 6)$. Its first layer $\mathfrak{g}_1$ is framed by $\{X^1, X^2, Y\}$, and its Lie algebra is given by the following relationships.

$$Y_2^1 := [X^1, Y] \;\; Y_2^2 := [X^2, Y], \;\; Y_1^3 := [X^2, Y_2^1] = [X^1, Y_2^2],$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is given by $Y$ $Y_2^1$, $Y_2^2$ and $Y_3^1$: So in this case $A_{ult} = \mathbb{R}^2 \ltimes \mathbb{A}$. $\alpha$ associated with $A_{ult}$ is given by

$$\alpha_{A_{ult}} = d\theta_1 \otimes (e_1 + xe_2 + ye_3 + xye_4),$$

Then if $\mu = (a_1, a_2, a_3)$ in $\mathfrak{a}^*$,

$$H_\mu(p_x, x) = \frac{1}{2}(p_x^2 + p_y^2 + (a_0 + a_1 x + a_2 y + a_3 xy)^2).$$

**Theorem 2.** *The subRiemannian geodesic flow on $N_{3,6,1}$ is not integrable by meromorphic functions.*

This Proposition is a consequence of the classification of integrable systems by meromorphic functions of the Hamiltonian with the degree of freedom two and homogeneous potential of degree 4 in [18].

*Proof.* If $\mu = (0, 0, 0, a)$, then $H_\mu(p_x, x)$ is non-integrable by meromorphic functions. □

E. Le Donne and F. Tripaldi used the notation $N_{3,6,1a^*}$ for this group in [19].

8.2.3. $J^k(\mathbb{R}^2, \mathbb{R})$ *as* 2*-abelian extension.* The jet space has a natural Carnot structure, see [20, 9, 16] for the general construction, see [21] in the context of Goursat distributions. In [22] , we prove that the subRiemannian geodesic flow in $J^k(\mathbb{R}^2, \mathbb{R})$ is not integrable if $1 < k$, using the classification from [18].

## 9. General Engel's group $Eng(n)$ and the Radial An-harmonic Oscillator

Let $Eng(n)$ be the Carnot group of dimension $2n+2$ and with growth vector $(n+1, 2n+1, 2n+2)$. Its first layer $\mathfrak{g}_1$ is framed by $(n+1)$ left-invariant vector fields $\{X_1, \cdots, X_n, Y\}$, and its Lie algebra is given by the following relationships.

$$Y_2^i := [X^i, Y], \qquad Y_3^1 := [X^i, Y_2^i] \tag{9.1}$$

Otherwise, zero. The biggest algebra $\mathfrak{a}$ is given by $Y$, $Y_2^1, \cdots$, $Y_2^n$ and $Y_3^1$: So in this case $G = \mathbb{R}^n \ltimes \mathbb{A}$.

$$\alpha_{Eng(n)} = d\theta \otimes (e_1^1 + \sum_{i=1}^{n} x_i e_2^i + \frac{1}{2}||x||^2_{\mathbb{R}^n} e_3^1)\} \tag{9.2}$$

here the step $s = 3$, and the polynomial has degree two. $\mu = (a_0, \cdots, a_{n+1})$ in $\mathfrak{a}^*$, then

$$H_\mu(p_x, x) = \frac{1}{2}||p_x||^2_{\mathbb{R}^n} + \frac{1}{2}(a_0 + \sum_{i=1}^{m} a_i x_i + a_{m+1}\frac{1}{2}||x||^2_{\mathbb{R}^n})^2.$$

**Theorem 3.** *The geodesic flow on $Eng(n)$ is non-commutative integrable if $2 < n$.*

*Proof.* After a translation $H_\mu$ became in $\frac{1}{2}||p_x||^2_{\mathbb{R}^n} + \frac{1}{2}(a + b\frac{1}{2}||x||^2_{\mathbb{R}^n})^2$, Which is the an-harmonic oscillator, the group $SO(n)$ acts in $\mathbb{R}^n$ and the Hamiltonian is invariant under its action, having as a consequence. The action of $SO(n)$ on $Eng(n)$ provides with $n(n-1)/2$ constant of motion. □

The subRiemannian Laplacian is given by

$$\Delta_{sR}(g) := \frac{1}{2}(\Delta_{\mathbb{R}^n} + (\frac{\partial}{\partial\theta_0} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial\theta_2^i} + \frac{x^2}{2}\frac{\partial}{\partial\theta_3^1})^2). \tag{9.3}$$

Let $P(\theta)$ be the quadratic polynomial such $Y^2 P(\theta) = (a + b||x||^2_{\mathbb{R}^n})^2$, then the antz $\Psi(x,\theta) = \Psi(x) + P(\theta)$ yields us to

$$\Delta_{sR}\Psi(x,\theta) = \frac{1}{2}(\Delta_{\mathbb{R}^n} + (a_1 + a_2\frac{x^2}{2})^2),$$

which is the radial an-harmonic oscillator. In [23], Del Valle and Turbiner, solve the eingen-value problem.

## Appendix A. Open problems

A.1. **Non-periodic geodesic on Carnot groups.** Enrico Le Donne made the following conjecture.

**Conjecture 1.** *The Carnot groups do not have subRiemannian periodic geodesic.*

Nicola Paddeu has a proof for the normal geodesic, coming work, and part 1 of Theorem B it is the particular case for meta-abelian Carnot groups with $\beta_{\mathbb{G}}$-system. Then, the open problem concerns the abnormal geodesic. In addition, it will be nice to give alternative proof using the symplectic reduction, now in the context of meta-abelian Carnot groups with $\beta_{\mathbb{G}}$-systems.

A.2. **Metric lines.** Classify the relative equilibrium points in $H_\mu$ and metric lines.

**Conjecture 2.** *Let $\mathbb{G}$ be a meta-abelian abelian Carnot group with $\beta_{\mathbb{G}}$-system, then the relative equilibrium points in $T^*\mathcal{H}$ for $H_\mu$ are isolated.*

In [13], Richard Montgomery and I classify the singular dynamics in the context of the jet space $J^k(\mathbb{R}, \mathbb{R})$. Let us generalize the classification thinking that Conjecture 2 is valid. Let $c(t)$ be a $\beta$-curve for $\mu$; the classical dynamical system theory gives the following classification of the singular dynamics:

- $c(t)$ is homoclinic if $c(t) \to p$ when $t \to \infty$ and $c(t) \to p$ when $t \to -\infty$.
- $c(t)$ is heteroclinic if $c(t) \to p_1$ when $t \to \infty$ and $c(t) \to p_2$ when $t \to -\infty$. We add more detail to the dichotomy.
- $c(t)$ is heteroclinic of the direct type if $\beta_k(p_1)\beta_k(p_2) = 1$ for all $1 \leq k \leq n_1$.
- $c(t)$ is heteroclinic of the turn back type if $\beta_k(p_1)\beta_k(p_2) = -1$ for at least on $k$ such that $1 \leq k \leq n_1$.

**Definition A.1.** *We say that a geodesic $\gamma(t)$ is homoclinic, heteroclinic of the direct type or heteroclinic of the turn-back type, according to whether its $c(t)$ $\alpha$-curve is periodic, homoclinic, heteroclinic of the direct type or heteroclinic of the turn-back type.*

**Definition A.2.** *We say that a geodesic $\gamma : \mathbb{R} \to \mathbb{G}$ is a metric line if $|a-b| = dist_{\mathbb{G}}(\gamma(a), \gamma(b))$ for all compact set $[a, b] \subset \mathbb{R}$, where $|a-b|$ is the absolute value and $dist_{\mathbb{G}}(\gamma(a), \gamma(b))$ is the subRiemannian distance in $\mathbb{G}$.*

**Open Problem 1.** *Besides lines which are the metric lines in meta-abelian groups?*

Guess: The metric lines in meta-abelian Carnot group with a $\beta_{\mathbb{G}}$-system are the homoclinic geodesics and heteroclinic geodesics of the direct type.

In [24, 25, 26], A. Ardentov and G. Shackov showed the conjecture in the context of Engel's group $Eng$, and they proved that lift of the Euler-soliton is a metric line in $Eng$.

### A.3. **Equi-optimal.**

**Definition A.3.** *We say that the arc-length parameterized geodesic $\gamma : \mathbb{R} \to \mathbb{G}$ is equi-optimal if its cut lengths are independent of where we start on the geodesic. In other words, for any real $s$, let $\gamma_s(t) = \gamma(t-s)$ be the time translated version of $\gamma$, having new starting point $\gamma_s(0) = \gamma(s)$. Then $\gamma$ is equi-optimal if $t_{cut}(\gamma_s)$ is independent of $s$.*

*We say that a length space is equi-optimal if all the geodesics are equi-optimal.*

**Open Problem 2.** *Are the meta-abelian Carnot groups equi-optimal?*

A.4. **An equivalent definition of $\mathfrak{a}$.** In all the above examples, $\mathfrak{a}$ the maximal abelian ideal containing $[\mathfrak{g}, \mathfrak{g}]$ is equal to the maximal abelian ideal containing $\mathfrak{g}_s$.

**Open Problem 3.** *Let $\mathbb{G}$ be a meta-abelian Carnot group, under which condition is valid that $\mathfrak{a}$ is equal to the maximal abelian ideal containing $\mathfrak{g}_s$.*

Guess: $\mathbb{G} \neq \mathbb{G}_1 \times \mathbb{G}_2$.

## Appendix B. Proof

## Appendix C. Integrability

C.1. **Arnold-Liouville integrability.** We say that a $n$-degree of freedom Hamiltonian system in $T^*M$ is Arnold-Liouville integrable if exist $n$ constant of motion $F := (F_1, , F_n) : T^*M \to \mathbb{R}^n$ such that:

1. (independence) the rank of the Jacobian matrix of $F$ is rank $n$.
2. (first integral) $\{F_i, H\} = 0$ for all $i$.
3. (involution) $\{F_i, F_j\} = 0$ for all $i$ and $j$.

C.2. **Non-commutative integrability.** We say that a $n$-degree of freedom Hamiltonian system in $T^*M$ is non-commutative integrable if exist $\ell \geqslant n$ constant of motion $F := (F_1, , F_n) : T^*M \to \mathbb{R}^n$ such that:

1. (independence) the rank of the Jacobian matrix of $F$ is rank $\ell$.
2. (first integral) $\{F_i, H\} = 0$ for all $i$.
3. (isotropy) the matrix $\{F_i, F_j\}$ has a kernel of dimension $2n\ell$.

(4) (closure) for each $i, j$ there exist a function $P_{ij} : ImF \subset^{\ell} \to \mathbb{R}$ such that $\{F_i, F_j\} = P_{ij} \circ (F_1, \cdots, F_\ell)$.

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