1. In the circuit diagram below, switches $s_1$, $s_2$ and $s_3$ are randomly and independently set in the open or closed state. Let $A_i$ be the event that $s_i$ is open (for $i = 1, 2, 3$), and let $A$ be the event that there is a closed path from terminal 1 to terminal 2.

Suppose that $P(A_i) = p_i$, for $i = 1, 2, 3$. Determine $P(A)$ in terms of $p_1$, $p_2$ and $p_3$.

**Solution:**
There is a closed path from terminal 1 to terminal 2 if and only if either $s_1$ and $s_2$ are closed, or $s_3$ is closed. Therefore by independence

$$P(A) = P((A_1^c \cap A_2^c) \cup A_3^c)$$
$$= P(A_1^c \cap A_2^c) + P(A_3^c) - P(A_1^c \cap A_2^c \cap A_3^c)$$
$$= (1 - p_1)(1 - p_2) + (1 - p_3) - (1 - p_1)(1 - p_2)(1 - p_3).$$

**Alternate Solution:**
There is no closed path from terminal 1 to terminal 2 if and only if both $s_3$ is open, and either $s_1$ or $s_2$ are open. Thus, again by independence,

$$P(A) = 1 - P(A^c) = 1 - P((A_1 \cup A_2) \cap A_3)$$
$$= 1 - P(A_1 \cup A_2)P(A_3)$$
$$= 1 - (P(A_1) + P(A_2) - P(A_1 \cap A_2))P(A_3)$$
$$= 1 - (p_1 + p_2 - p_1p_2)p_3$$
$$= 1 - p_1p_3 - p_2p_3 + p_1p_2p_3.$$
2. A system consists of \( n \) identical components, each of which is operational with probability \( p \), independent of other components. The system is operational if at least \( m \) out of the \( n \) components are operational. What is the probability that the system is operational?

**Solution:**
Let \( X \) be the number of components that are operational. Then \( X \) is a Binomial random variable with parameters \( n \) and \( p \), and its PMF is

\[
p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k = 0, 1, 2, ..., n.
\]

The probability that \( m \) or more of the \( n \) components are operational is therefore

\[
P(X \geq m) = \sum_{k=m}^{n} p_X(k) = \sum_{k=m}^{n} \binom{n}{k} p^k (1 - p)^{n-k}.
\]

3. Alice and Bob have a chess match in which the first player to win a game wins the match. Each game has one of 3 possible outcomes: Bob wins, Alice wins, or the game is a draw. One game is played each day until someone wins, so the match is of potentially unlimited duration. The prize money starts at $100 on the first day, and goes up by $100 every subsequent day a match is played. Alice wins with probability 0.4, Bob wins with probability 0.3, and a draw occurs with probability 0.3.

a. What is the probability that Alice wins the match?

**Solution:**
Let \( A_k \) be the event that Alice wins the match on the \( k \)th game, and let \( A \) be the event that Alice wins the match. Then \( P(A_k) = (0.3)^{k-1} \cdot 0.4 \), and hence

\[
P(A) = \sum_{k=1}^{\infty} (0.3)^{k-1} \cdot 0.4 = 0.4 \cdot \sum_{k=1}^{\infty} (0.3)^{k-1} = (0.4) \cdot \sum_{k=0}^{\infty} (0.3)^k = \frac{0.4}{1 - 0.3} = \frac{4}{7}
\]

b. Determine the mean and standard deviation of the total prize money.

**Solution:**
Let \( X \) be the duration of the match, in days. The probability that a particular game is won by somebody is 0.3 + 0.4 = 0.7, so that \( X \) is a Geometric random variable with parameter 0.7. The total prize money is 100\( X \), which has mean, variance and standard deviation

\[
E[100X] = 100 \cdot E[X] = \frac{100}{0.7} = 142.86,
\]

\[
\text{Var}(100X) = 100^2 \cdot \text{Var}(X) = 100^2 \cdot \frac{1 - (0.7)}{(0.7)^2} = 100^2 \cdot (0.612244)
\]

\[
\sigma_{100X} = \sqrt{\text{Var}(100X)} = 100 \cdot (0.78246) = 78.25
\]
4. A 3-sided die and a coin, which are neither fair nor independent, are rolled and tossed, respectively. The die has faces \{1, 2, 3\} and the coin has sides labeled \{1, 2\}. Let \(X\) be the outcome of the die, and \(Y\) the outcome of the coin. The \textit{conditional} PMF \(p_{X|Y}(x|y)\) is given by the following table.

\[
\begin{array}{c|ccc}
 y & 1 & 2/8 & 5/8 \\
 & 2 & 1/8 & 3/8 \\
 & 1 & 2 & 3 \\
\end{array}
\]

Also, the \textit{marginal} PMF \(p_Y(y)\) is given by the following table.

\[
\begin{array}{c|c}
 y & 1/3 \\
 & 2/3 \\
\end{array}
\]

a. Fill in the following table giving the \textit{joint} PMF \(p_{X,Y}(x,y) = p_Y(y) \cdot p_{X|Y}(x|y)\)

\[
\begin{array}{c|ccc}
 y & 1 & 2/24 & 5/24 \\
 & 2 & 2/24 & 6/24 \\
 & 1 & 2 & 3 \\
\end{array}
\]

b. Fill in the following table giving the \textit{marginal} PMF \(p_X(x) = \sum_y p_{X,Y}(x,y)\)

\[
\begin{array}{c|ccc}
 x & 4/24 & 11/24 & 9/24 \\
 & 1 & 2 & 3 \\
\end{array}
\]

c. Fill in the following table giving the \textit{conditional} PMF \(p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}\)

\[
\begin{array}{c|ccc}
 y & 1/2 & 5/11 & 1/9 \\
 & 1/2 & 6/11 & 8/9 \\
 & 1 & 2 & 3 \\
\end{array}
\]

d. Given that the coin flip is 2, what is the probability that the die roll is 3?

\[
p_{X|Y}(3|2) = \frac{4/8}{1/2} = 1/2
\]
5. The number \( X \) of phone calls received by a call center within a certain time period is a Poisson random variable with parameter \( \lambda \). Determine the smallest positive number \( \lambda \) such that the probability of receiving at least one call is at least \( 1/2 \).

**Solution:**
We require that \( P(X \geq 1) \geq 1/2 \). Hence

\[
1 - P(X = 0) \geq \frac{1}{2} \quad \Rightarrow \quad 1 - \frac{1}{2} \geq P(X = 0) \quad \Rightarrow \quad p_X(0) \leq \frac{1}{2},
\]

and therefore

\[
e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-\lambda} \leq \frac{1}{2}
\]

\[
\therefore \quad -\lambda \leq \ln(1/2) = -\ln(2)
\]

\[
\therefore \quad \lambda \geq \ln(2)
\]

The smallest such \( \lambda \) is \( \lambda = \ln(2) = 0.6931 \). \( \blacksquare \)