Lecture 7 - Open mapping and inverse function theorems. Local analytic logarithms and roots. Behavior of analytic functions near critical points

---

Material: [G, §VIII.4, VIII.5]. [SS, Ch 3., Sec 6]

Recall Rouché's theorem.

Theorem 1 (Rouché's theorem)**.** *Let $f, g : U \to \mathbb{C}$ analytic on $U \subset \mathbb{C}$ open, $\gamma : I \to U$ a Jordan curve with Int $(\gamma) \subset U$. Assume f has no zero on $\gamma(I)$ and*

$$|f(z) - g(z)| \le |g(z)|, \qquad \forall\ z \in \gamma(I). \tag{1}$$

Then f and g have the same number of zeros inside γ, counting multiplicities.

A direct consequence of this is the Open Mapping theorem.

Theorem 2 (Open Mapping Theorem)**.** *If f is holomorphic and non-constant in a region Ω, then it is open (i.e., f maps open sets to open sets).*

Proof. Suppose f non-constant. It is enough to show that if $w_0 \in f(z_0)$ for some $z_0 \in \Omega$ then there exists $\rho > 0$ such that $D_\rho(w_0) \subset f(\Omega)$. Indeed, consider the function $g(z) = f(z) - w_0$. g is holomorphic, non-constant and has a zero at z_0. By the local representation, there exists $r > 0$ such that g does not vanish on $\dot{D}_r(z_0)$. In particular $\inf_{z \in C_{r/2}(z_0)} |g(z)| = C > 0$. For w such that $|w - w_0| < C$, the function $h(z) = f(z) - w$ satisfies

$$|h(z) - g(z)| = |w - w_0| < C \le |g(z)|, \qquad z \in C_{r/2}(z_0),$$

so by Rouché's theorem, h must have a zero inside $D_{r/2}(z_0)$, which means there exists $z \in D_{r/2}(z_0)$ such that $f(z) = w$. That is, for every $w \in D_C(w_0)$, w belongs to $f(\Omega)$. Hence the proof. □

As a direct consequence, we have the maximum modulus principle.

Theorem 3 (Maximum modulus principle)**.** *Let $f : \Omega \to \mathbb{C}$ holomorphic on Ω open. If $|f|$ achieves a local maximum at $z_0 \in \Omega$, then f is constant.*

Proof. By contraposition, suppose $z_0 \in \Omega$ and f nonconstant. Then there exists $r > 0$ such that $D_r(z_0) \subset \Omega$ and by the Open Mapping Theorem, $f(D_r(z_0))$ is open, that is, there exists $\rho > 0$ such that $D_\rho(w_0) \subset f(\Omega)$. In particular, for every $0 < \varepsilon < \rho$, the value $(|w_0| + \varepsilon)e^{i \arg w_0}$ is attained by f and has greater modulus than $|w_0|$, so $|f|$ cannot have a local maximum at z_0. □

As a corollary, if Ω is bounded, then $|f|$ can only achieve its maximum at $\partial\Omega = \overline{\Omega} \backslash \Omega$. This is not true if Ω is unbounded, as the following counterexample shows: for Ω the top-right quadrant and $f(z) = e^{-iz^2}$, we have $|f(z)| = 1$ everywhere on $\partial\Omega = \mathbb{R}_{\ge 0} \cup i\mathbb{R}_{\ge 0}$ and $f(te^{i\frac{\pi}{4}})$ is unbounded inside the domain for $t \in \mathbb{R}_{>0}$.

The proof of the Open Mapping Theorem not only shows that every w close enough to w_0 is attained, but that it is attained the *same number of times* as w_0 (with proper definition as below) !

Definition 1. *We say that*

(i) f attains m times the value w_0 at z_0 iff the function $f(z) - w_0$ has a zero of order m at $z = z_0$.

(ii) f attains m times the value w_0 on Ω iff the function $f(z) - w_0$ has m zeros on Ω, counting multiplicity.

As an example, the function $f(z) = z^5 + 1$ achieves 5 times the value 1 at $z = 0$.

In the proof of the Open Mapping Theorem, what we have in fact proved is the following:

Theorem 4. *If f achieves m times the value w_0 at z_0, then there exists $\rho > 0$ and $r > 0$ such that for every $w \in D_\rho(w_0)$, f achieves m times the value w on $D_r(z_0)$.*

Proof. Another proof of this is, in the setting as above, to introduce the *counting function*

$$N(w) := \frac{1}{2\pi i} \int_{C_{r/2}(z_0)} \frac{f'(z)}{f(z) - w}\, dz,$$

for w close enough to w_0. Namely, if $|f(z) - w_0| \geq C$ on $C_{r/2}(z_0)$, then $N(w)$ makes perfect sense for $|w - w_0| < C$, is an analytic function of w there (write a power series in $w - w_0$ for it !). Moreover, by the argument principle, N is integer-valued and counts the number of zeros of $f(z) - w$, i.e. the number of times f achieves the value w inside $D_{r/2}(z_0)$. Since N is continuous, N must be constant. $\square$

This has important consequences, namely: if $m = 1$, f is locally one-to-one and onto, and if $m > 1$, f is m-to-one in a punctured neighborhood of z_0. Note that when f achieves m times the value w_0 at z_0, the local representation of f gives

$$f(z) = w_0 + (z - z_0)^m g(z),$$

in a neighbourhood of z_0 with g analytic, non-vanishing. If $m = 1$, this is equivalent to $f'(z_0) \neq 0$ and if $m > 1$, this is equivalent to saying that

$$f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \qquad f^{(m)}(z_0) \neq 0.$$

In the latter case, we say that z_0 is a *critical point of order* $m - 1$. Since critical points are zeros of the analytic function f', they are isolated. Therefore in the case $m > 1$, it is immediate to see that the m preimages by f can be made distinct, by imposing that the neighborhood of z_0 contains no other critical point of f.

Local behavior near a non-critical point. In the complex analytic case, the case $m = 1$ can even be made more explicit as follows.

Theorem 5. *Let $f : U \to \mathbb{C}$, $z_0 \in \Omega$ non-critical (i.e. $f'(z_0) \neq 0$), set $w_0 = f(z_0)$. Then there exists $\rho > 0$ and $\delta > 0$ such that for every $w \in D_\delta(w_0)$, there exists a unique $z_w \in D_\rho(z_0)$ such that $f(z_w) = w$. Such a z_w is given by the explicit expression*

$$z_w = f^{-1}(w) = \frac{1}{2\pi i} \int_{|z - z_0| = \delta} \frac{z f'(z)}{f(z) - w}\, dz,$$

in particular, f^{-1} is analytic on its domain of definition.

Proof. Existence and uniqueness of z_w follow directly from Theorem 4. For fixed w, we define on $\overline{D_\rho(z_0)}$ the mapping $h(z) = \frac{zf'(z)}{f(z)-w}$. Clearly, h is analytic, its denominator does not vanish on $|z-z_0| = \rho$ and h only has a simple pole at $z = z_w$, then the residue theorem gives:

$$\text{Res }(h, z_w) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} h(z)\ dz.$$

Moreover, for simple poles, another way to compute the residue is

$$\text{Res }(h, z_w) = \lim_{z\to z_w}(z-z_w)h(z) = \lim_{z\to z_w}\frac{z-z_w}{f(z)-f(z_w)}zf'(z) = z_w.$$

Hence the result. Since f^{-1} is expressed as a paremeter-dependent integral with integrand analytic in w, f^{-1} is itself analytic. □

Logarithm and roots. The goal is to create locally analytic inverses of the function e^z. Judging by polar coordinates for $z \neq 0$, $z = \rho e^{i\theta} = e^{\log\rho + i\theta}$, this suggests that we define

$$\log z := \log|z| + i\arg z.$$

On the right-half plane, one may choose $\arg z = \tan^{-1}\frac{y}{x}$ and $\log|z| = \frac{1}{2}\log(x^2+y^2)$, and one may check that such functions satisfy the Cauchy-Riemann system, thus defining an analytic function. One may then extend this analytic function on the left-half plane from above and from below zero, and realize that there is no way that the extensions can agree on the negative real-axis. This is because, to some degree, the logarithm function is multiply-valued cannot be made globally analytic on $\mathbb{C}\backslash\{0\}$. For all practical purposes however, one may construct a single-valued **branch** of it on any disk included in $\mathbb{C}\backslash\{0\}$ or, later on, on any *simply connected* domain.

Theorem 6. *Let $z_0 \neq 0$, set $\Omega = D_{|z_0|}(z_0)$ and suppose $1 \in \Omega$. Then there exists a branch of the logarithm F analytic on Ω such that $F(1) = 0$ and $e^{F(z)} = z$ for every $z \in \Omega$. Moreover, $\log z$ agrees with the usual logarithm on $\Omega \cap \mathbb{R}_{>0}$.*

Proof. Since $w \mapsto \frac{1}{w}$ is analytic on Ω, we may construct a primitive unambiguously as $F(z) := \int_{[1,z]}\frac{dw}{w}$. F is clearly analytic, $F(1) = 0$ and $F'(z) = \frac{1}{z}$. In particular,

$$\frac{d}{dz}(ze^{-F(z)}) = (1 - F'(z)z)e^{-F(z)} = 0,$$

so $ze^{-F(z)}$ is a constant, equal to $1e^{-F(1)} = 1$. Hence the result. □

Theorem 7. *For $D \subset \mathbb{C}$ a disc and $f : D \to \mathbb{C}$ an analytic, non-vanishing function there exists $g : D \to \mathbb{C}$ analytic such that for every $z \in D$, $e^{g(z)} = f(z)$. With g as above, for any $m \in \mathbb{N}$, the function $h(z) = \exp\left(\frac{1}{m}g(z)\right)$ is analytic on D and satisfies $(h(z))^m = f(z)$.*

Proof. Similarly to Theorem 6, fix $z_0 \in \Omega$, $c_0 \in \mathbb{C}$ such that $e^{c_0} = f(z_0)$, and define $g(z) := \int_{[z_0,z]}\frac{f'(w)}{f(w)}\ dw + c_0$. The integrand has no poles and thus g is well-define and analytic, with $g'(z) = \frac{f'(z)}{f(z)}$. One may then check that $(e^{-g(z)}f(z))' = 0$, thus $e^{-g(z)}f(z)$ is a constant, equal to $e^{-c_0}f(z_0) = 1$. Hence the proof. □

Behavior of an analytic function near a critical point. Recall that in this case, we can write

$$f(z) = w_0 + (z - z_0)^m g(z), \qquad z \in D_r(z_0),$$

with g non-vanishing on $D_r(z_0)$. If $g(z) = 1$ is a constant, then it is easy to see what happens: write z in polar coordinates centered at z_0, then we have

$$f(z_0 + \rho e^{i\theta}) = w_0 + \rho^m e^{im\theta}.$$

In particular, if $\rho > 0$ and as θ traces the interval $[0, 2\pi]$, the image $f(z_0 + \rho e^{i\theta})$ traces the circle $C_{\rho^m}(w_0)$ counterclockwise m times, creating a genuine m-to-one map where any $w \neq w_0$ has m distinct preimages by f, given by

$$z_k(w) = e^{i\frac{2k\pi}{m}} (w - w_0)^{\frac{1}{m}}, \qquad 0 \le k \le m - 1,$$

once a branch of the function $z^{\frac{1}{m}}$ is given.

In the case where g is non-constant yet non-vanishing on $D_r(z_0)$, by Theorem 7, there exists $h : D_r(z_0) \to \mathbb{C}$ analytic such that $(h(z))^m = g(z)$ and $h(z_0) \neq 0$. Then f can be written as

$$f(z) = w_0 + ((z - z_0)h(z))^m = w_0 + (\ell(z))^m, \qquad \ell(z) := (z - z_0)h(z),$$

where the function $\ell(z)$ is analytic and satisfies $\ell(z_0) = 0$ and $\ell'(z_0) = h(z_0) \neq 0$. Thus by Theorem 5, it is a local diffeomorphism in a neighborhood of z_0, and maps bijectively and conformally a neighborhood of z_0 into a small disc.

Considering that $f = \phi \circ \ell$ with $\phi(z) := w_0 + z^m$, we see that f is locally a composition of a one-to-one map with an m-to-one map, and as such is an m-to-one map from a punctured neighborhood of z_0 to a punctured neighborhood of w_0, as in Fig. 1.

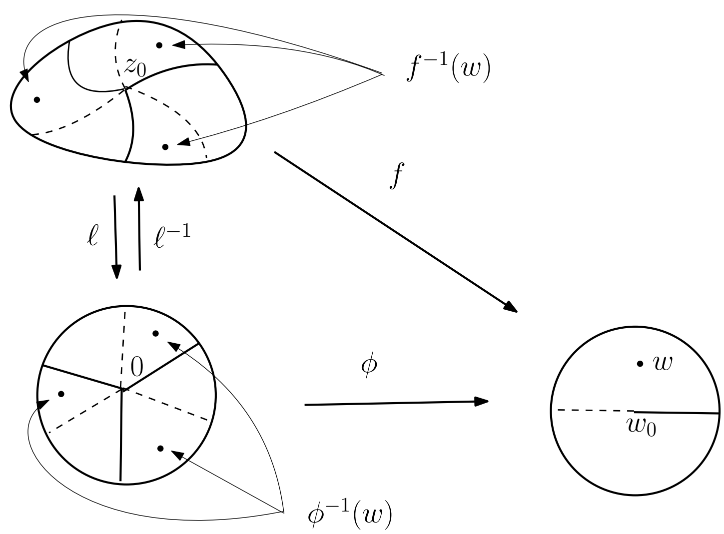

Figure 1: Example of local picture of f near a critical point with $m = 3$. Dashed curves are mapped into dashed curves and solid curves are mapped into solid curves.

References

[G] *Complex Analysis*, Theodore W. Gamelin. Undergraduate Texts in Mathematics, Springer. 1

[SS] *Complex Analysis*, Elias M. Stein and Rami Shakarchi, Princeton Lectures in Analysis II. 1

[T] *Complex Variables*, Joseph L. Taylor. Pure and Applied Undergraduate Texts Vol. 16, 2011.