

WHAT IS... G_2 ?

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Octonions

Let k be a field, with $\text{char}(k) \neq 2$. For example, k may be \mathbb{Q} or \mathbb{R} or \mathbb{C} (though the last case is not as interesting). A **Hurwitz algebra** over k is a finite-dimensional, k -algebra A with unit element, together with a quadratic form $N: A \rightarrow k$, such that the associated bilinear form is nondegenerate and:

$$N(xy) = N(x)N(y), \text{ for all } x, y \in A.$$

Every Hurwitz algebra¹ over k has dimension 1, 2, 4, or 8, as a k vector space.

In what follows, we write k, K, B, C for Hurwitz algebras of dimensions 1, 2, 4, 8 respectively; K is called a **quadratic étale k -algebra**, B is called a **quaternion algebra** over k , and C is called a **Cayley or octonion algebra** over k .

Hurwitz algebras become more pathological as their dimension rises: quadratic étale algebras are commutative and associative. However, quaternion algebras are associative but never commutative. Cayley algebras are neither commutative nor associative. However, Cayley algebras are **alternative**, so that for any two elements $x, y \in C$, the algebra generated by x and y is associative (e.g., $(xy)x = x(yx)$).

There are exactly two **complete chains** of Hurwitz algebras over the real numbers \mathbb{R} . First is the sequence $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, where \mathbb{C} denotes the complex numbers, \mathbb{H} Hamilton's quaternions, and \mathbb{O} Graves's octonions.² The other complete chain of Hurwitz algebras over \mathbb{R} is $\mathbb{R} \subset \mathbb{R} \times \mathbb{R} \subset M_2(\mathbb{R}) \subset \mathbb{O}_{spl}$, where $\mathbb{R} \times \mathbb{R}$ denotes the algebra of ordered pairs of real numbers (multiplication entry-wise), $M_2(\mathbb{R})$ denotes the two-by-two matrix algebra, and \mathbb{O}_{spl} denotes the "split octonions", which we discuss next.

For any field k (or even a commutative ring!), Zorn's algebra of split octonions over k is defined to be the set of all two-by-two "matrices":

$$\omega = \left\{ \begin{pmatrix} a & \vec{v} \\ \vec{w} & d \end{pmatrix} : a, d \in k, \vec{v}, \vec{w} \in k^3 \right\},$$

with composition given by the following formula:³

$$\begin{pmatrix} a & \vec{v} \\ \vec{w} & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & \vec{\phi} \\ \vec{\psi} & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + \vec{v} \cdot \vec{\psi} & a\vec{\phi} + d\vec{v} - \vec{w} \times \vec{\psi} \\ \alpha\vec{w} + d\vec{\psi} + \vec{v} \times \vec{\phi} & d\delta + \vec{w} \cdot \vec{\phi} \end{pmatrix}.$$

Resulting structures on a Hurwitz algebra A include a trace,

$$\text{Tr}(a) = N(a+1) - N(a) - N(1),$$

and involution $\bar{a} = \text{Tr}(a) - a$. Hurwitz algebras are quadratic; every element a satisfies the polynomial

$$a^2 - \text{Tr}(a)a + N(a) = 0.$$

¹ This was first proven over \mathbb{R} by Adolph Hurwitz, in *Über die Composition der quadratischen Formen von beliebig vielen Variablen*, found in *Math. Werke I*. For a general field, a proof can be found in Irving Kaplansky, *Infinite-dimensional quadratic forms admitting composition*, *Proc. Amer. Math. Soc.* 4, (1953).



Figure 1: A. Hurwitz (1859-1919), conducting, while his daughter plays violin with A. Einstein. Taken from David Rowe's *Intelligencer* article *Felix Klein, Adolph Hurwitz, and the "Jewish Question" in German academia* (original source, *Polya's Photo Album*).

² A nice historical overview of the quaternions and octonions can be found in the first few pages of Baez's article *The Octonions*, in *Bull. A.M.S.* Vol. 39, 2001.

Max Zorn, 1906-1993, a student of Artin, most famous probably for "Zorn's Lemma", but also active in algebra and analysis. His work on "octonions" can be found in *Theorie der Alternativen Ringe*, *Abh. Math. Sem. Hamburgischer Univ.*, vol. 8 (1930).

³ Here, we use the standard dot product, cross product, and scalar multiplication.

Automorphisms and Embeddings

Let us fix a base field k ($\text{char}(k) \neq 2$), and a complete chain⁴ of Hurwitz algebras $k \subset K \subset B \subset C$. One may consider the groups of k -linear automorphisms that preserve the algebra structure⁵; this yields groups:

$$\text{Aut}(k/k), \text{Aut}(K/k), \text{Aut}(B/k), \text{Aut}(C/k).$$

The group $\text{Aut}(k/k)$ is trivial, and the automorphism $\text{Aut}(K/k) = \{1, \sigma\}$, where $\sigma(a) = \bar{a}$ for all $a \in K$. The groups $\text{Aut}(B/k)$ and $\text{Aut}(C/k)$ are more complicated.

Suppose, for example, that $B = M_2(k)$ is the “split quaternion algebra”⁶. If $g \in GL_2(k)$, then define the **inner automorphism** $\text{Int}[g]$ by:

$$\text{Int}[g](b) = gbg^{-1}, \text{ for all } b \in B.$$

Then, Int is a homomorphism from $GL_2(k)$ to $\text{Aut}(B/k)$. Furthermore, if $z \in Z(GL_2(k))$ is a scalar matrix, then $\text{Int}[z] = 1$. Conversely, if $g \in GL_2(k)$, and $\text{Int}[g] = 1$, then g commutes with every $b \in B$, and hence $g \in Z(GL_2(k))$. It follows that Int descends to a unique injective homomorphism:

$$\overline{\text{Int}}: PGL_2(k) = GL_2(k)/k^\times \hookrightarrow \text{Aut}(B/k).$$

Conversely, if $\alpha \in \text{Aut}(B/k)$, then α is determined by the resulting linear automorphism α_\circ of the trace-zero subspace $B_\circ \subset B$. Moreover, α_\circ preserves the norm quadratic form N on B_\circ , so that $\text{Aut}(B/k)$ embeds into $O(B_\circ, N)$. In fact, a bit more work⁷ shows that:

$$PGL_2(k) \cong \text{Aut}(B/k) \cong SO(B_\circ, N).$$

AUTOMORPHISMS OF CAYLEY ALGEBRAS are more complicated than automorphisms of quaternion algebras. An automorphism α of (for example, the split) Cayley algebra C restricts to a linear, norm-preserving, automorphism α_\circ of the seven-dimensional trace-zero subspace C_\circ . Furthermore, α is determined from α_\circ . This provides an embedding $\text{Aut}(C/k) \hookrightarrow O(C_\circ, N)$.

The resulting group $\text{Aut}(C/k)$ is not easy to describe, but has another name:

$$G_{2,C} = \text{Aut}(C/k).$$

The answer to the title of this lecture is: G_2 is the automorphism group of a Cayley (a.k.a. octonion algebra); which Cayley algebra and which base field should be made clear.

⁴ It would not be too harmful to consider the chain

$$\mathbb{R} \subset \mathbb{R} \times \mathbb{R} \subset M_2(\mathbb{R}) \subset O_{\text{spl}}.$$

⁵ Since every element a of a Hurwitz algebra satisfies a quadratic identity $a^2 - \text{Tr}(a)a + N(a) = 0$, it can be seen that the algebra structure determines the quadratic form N , the trace map Tr , and hence the involution $\bar{a} = \text{Tr}(a) - a$.

⁶ The same methods work for nonsplit quaternion algebras as well; every automorphism of a quaternion algebra is inner, yielding an isomorphism from $B^\times/Z(B^\times)$ to $\text{Aut}(B/k)$. Indeed, every quaternion algebra splits over a Galois extension, and so descent applies to prove that $\overline{\text{Int}}$ is an isomorphism.

⁷ To show that the image of $\text{Aut}(B/k)$ in $O(B_\circ, N)$ is contained in $SO(B_\circ, N)$, one must check that no automorphism of B acts as a reflection of B_\circ . A reflection would preserve two orthogonal vectors x, y in B_\circ , and send a third vector z to its negative. But, a basis of B_\circ as a k -vector space is given by $x, y, (xy - \bar{y}\bar{x})$, so that the action of an algebra automorphism α on B_\circ is uniquely determined by its action on x and y . In particular, if α fixes x and y , then α fixes z . This demonstrates that $\text{Aut}(B/k)$ injectively maps to $SO(B_\circ, N)$. For surjectivity, a dimension argument suffices.

Subgroups of Automorphisms

Given a complete chain $k \subset K \subset B \subset C$ of Hurwitz algebras over k , we have considered groups:

$$\text{Aut}(k/k), \text{Aut}(K/k), \text{Aut}(B/k), \text{Aut}(C/k),$$

each arguably more interesting than the previous. The only group which is “really new” is $\text{Aut}(C/k)$. To “know” the group $\text{Aut}(C/k)$, it is most helpful to understand its subgroups⁸. Two subgroups which arise most generally are $\text{Aut}(C/K)$ and $\text{Aut}(C/B)$: the algebra automorphisms of C , which fix every element of K or of B , respectively.

First, let us consider $\alpha \in \text{Aut}(C/K)$; such an automorphism α preserves every element of the k -subspace K , and preserves the norm and trace, and hence stabilizes the subspace K^\perp of elements orthogonal to K :

$$K^\perp = \{\omega \in C : \text{Tr}(\omega\bar{z}) = 0 \text{ for all } z \in K\}.$$

The alternative property⁹ of C implies that K^\perp is a (left) K -module: in particular,

$$\omega \in K^\perp, z \in K \Rightarrow z \cdot \omega.$$

Now, every element of C can be expressed uniquely as a sum $z + \omega$ (or ordered pair (z, ω)), where $z \in K$ and $\omega \in K^\perp$. There is a unique Hermitian form¹⁰ $\Phi: K^\perp \times K^\perp \rightarrow K$, such that:

$$\text{proj}_K((z, \omega) \cdot (z', \omega')) = (zz' - \Phi(\omega, \omega')).$$

It follows that every automorphism $\alpha \in \text{Aut}(C/K)$ preserves this Hermitian form. More precisely, one arrives at an injective homomorphism $\text{Aut}(C/K) \hookrightarrow U(K^\perp, \Phi)$. The most precise possible result is that, in fact, $\text{Aut}(C/K)$ is isomorphic to the group $SU(K^\perp, \Phi)$.

A VERY GOOD EXAMPLE of $\text{Aut}(C/K)$ is provided by Zorn’s split octonions, and the algebra $K = k \times k$ embedded as the diagonal matrices. Then we find that:

$$K^\perp = \left\{ \begin{pmatrix} 0 & \vec{v} \\ \vec{w} & 0 \end{pmatrix} : \vec{v}, \vec{w} \in k^3 \right\}.$$

One can verify that $SU(K^\perp, \Phi)$ is isomorphic to $SL_3(k)$. Specifically, if $g \in SL_3(k)$, then the action of g on C is given by:

$$g \begin{pmatrix} a & \vec{v} \\ \vec{w} & d \end{pmatrix} = \begin{pmatrix} a & g\vec{v} \\ {}^t g^{-1}\vec{w} & d \end{pmatrix}.$$

This yields the **long root** embedding

$$e_{\text{long}}: SL_3(k) \hookrightarrow G_{2,C}.$$

⁸ A good reference for automorphisms of Cayley algebras is Jacobson’s *Composition algebras and their automorphisms*, Rend. Circ. Mat. Palermo 1958

⁹ The nontrivial fact is that if $z, w \in K$, and $\omega \in C$, then $z \cdot (w \cdot \omega) = (z \cdot w) \cdot \omega$. But the alternative property implies that such a triple z, w, ω lie in an associative subalgebra of C .

¹⁰ For all $z, z' \in K$, and $\omega, \omega' \in K^\perp$, $\Phi(z\omega, z'\omega') = zz'\Phi(\omega, \omega')$.

Here, we identify K^\perp with $k^3 \times k^3$, with K^3 , when $K = k \times k$. Note that “conjugation” in K is given by $\bar{z} = (y, x)$ if $z = (x, y)$. If $\vec{z} = (z_1, z_2, z_3) \in K^3$ and $\vec{w} = (w_1, w_2, w_3) \in K^3$, then the Hermitian form is given by:

$$\Phi(\vec{z}, \vec{w}) = z_1\bar{w}_1 + z_2\bar{w}_2 + z_3\bar{w}_3.$$

Quaternion subalgebras

One may also consider the automorphisms $Aut(C/B)$ fixing every element of the quaternion algebra B . Since $K \subset B$, we find that $Aut(C/B) \subset Aut(C/K)$. As in the case of quadratic subalgebras, we may consider the orthogonal complement B^\perp of B in C . There exists an element $\ell \in B^\perp$, such that every element of C has the form $a + b\ell$, for some $a, b \in B$. Moreover, if $\alpha \in Aut(C/B)$, then it is a theorem that there exists a unique element $u_\alpha \in B$ such that $N(u_\alpha) = 1$ and

$$\alpha(a + b\ell) = a + (u_\alpha b)\ell, \text{ for all } a, b \in B.$$

This provides an isomorphism:

$$Aut(C/B) \cong SB^\times = \{u \in B^\times : N(u) = 1\}.$$

When $B = M_2(k)$, $SB^\times = SL_2(k)$, providing the **highest root** embedding $e_{high} : SL_2(k) \hookrightarrow G_{2,C}$. Of course, since $Aut(C/B) \subset Aut(C/K)$, the image of e_{high} is contained in the image of e_{long} .

Another embedding of SB^\times into $G_{2,C}$ follows: if $g \in SB^\times$, and $a + b\ell \in C$, define:

$$g(a + b\ell) = (gag^{-1}) + (bg^{-1})\ell.$$

This defines the **short root** embedding $e_{short} : SL_2(k) \hookrightarrow G_{2,C}$. Observe that the images of e_{high} and e_{short} commute¹¹, and their intersection consists of $\{\pm 1\} \subset SB^\times$. This provides the embedding:¹²

$$e_{high} \times e_{short} : SB^\times \times_{\pm 1} SB^\times \hookrightarrow G_{2,C}.$$

Suppose C is Zorn's split Cayley algebra. If τ is a cyclic permutation of $\{1, 2, 3\}$, then τ naturally acts (by permuting basis elements) on k^3 . This yields an automorphism:

$$\tau \begin{pmatrix} a & \vec{v} \\ \vec{w} & d \end{pmatrix} = \begin{pmatrix} a & \tau\vec{v} \\ \tau\vec{w} & d \end{pmatrix}.$$

We find an embedding $w : A_3 \hookrightarrow G_{2,C}$ in this way.

There are three ways of embedding $M_2(k)$ into the split Cayley algebra C , via:

$$t_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & be_i \\ ce_i & d \end{pmatrix},$$

where (e_1, e_2, e_3) is the standard basis of k^3 . Altogether, we find three conjugate embeddings:

$$e_{short,i} : SL_2(k) \hookrightarrow G_{2,C}.$$

In fact, $G_{2,C}$ is generated by the images:

$$e_{short,i}(SL_2(k)), e_{long}(SL_3(k)).$$

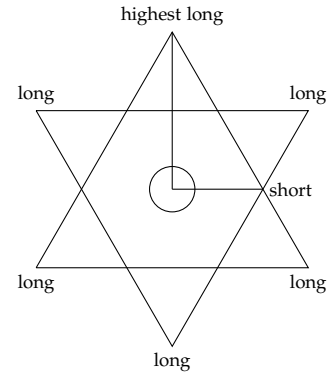


Figure 2: The long roots for G_2 . The other six intersection points are short roots.

¹¹ This reflects the orthogonality of the short root and highest long root in the diagram above

¹² This embedding is very important for the split real Lie group G_2 , where B is Hamilton's quaternion algebra and C is the split octonion algebra (such embeddings exist!). There, one finds an embedding $SU_2 \times_{\pm 1} SU_2$ as the maximal compact subgroup of the split Lie group G_2 . From this, one finds that the split G_2 is not simply-connected as a topological group; rather, it has a two-fold covering which is simply-connected. This leads to great confusion!