WELL-ROUNDED FACTS ABOUT SPHERES

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This handout accompanies a lecture on the volumes of n-dimensional spheres. The design and layout was inspired by the work of Edward Tufte, and typeset using the excellent tufte-LaTex class. Pictures were created using the PGF/TikZ package.

Computing the Volume

The closed n-dimensional ball of radius r, centered at the origin, is defined by:

$$B^n = \{\vec{x} \in \mathbb{R}^n \text{ such that } x_1^2 + x_1^2 + \cdots + x_n^2 \leq r^2\}.$$

As a bounded closed subset of $\mathbb{R}^n$, the n-ball has a well-defined volume, which we call $V_n(r)$. A table of volumes is given in the margin. Of course, the word "volume" might be a bit misleading in this degree of generality. In dimension 0, the "volume" $V_0(r)$ is the cardinality of the one-point set $\mathbb{R}^0$. In dimension 1, the "volume", $V_1(r) = 2r$, is simply the length of the line segment $[-r, r]$. In dimension 2, the "volume" of a circle is its area, and $V_2(r) = \pi r^2$. This, in essence, is the definition of π. In dimension 3, the "volume" of a sphere is its volume as the word is used by the English-speaking community at large.

n	$C_n = V_n(1)$	$V_n(r)$
0	1	1
1	2	$2r$
2	π	πr^2
3	$4/3 \cdot \pi$	$4/3 \cdot \pi r^3$
4	$1/2 \cdot \pi^2$	$1/2 \cdot \pi^2 r^4$
5	$8/15 \cdot \pi^2$	$8/15 \cdot \pi^2 r^5$

Don't confuse a definition with a computation. Of course, one could "compute" the area of a circle with an integral, but such an argument would necessarily be circular, pun intended. Perhaps the only fact that needs to be proven is that the circumference is the derivative of the area, as functions of the radius, which follows from Stokes theorem.

There is one fact about volumes of balls – the functions $V_n(r)$ – that can be deduced from the simplest change of variables: a ball of radius r can be obtained by scaling a unit ball by r. It follows that $V_n(r) = V_n(1)r^n$. For this reason, it is convenient to define $C_n = V_n(1)$, so that $V_n(r) = C_n r^n$.

Scaling is a particularly simple instance of the technique of change of variables. Scaling a measurable subset of $\mathbb{R}^n$ by r changes its volume by a factor of r^n

IN HIGH DIMENSIONS, volume is more difficult to imagine, but no easier and no more difficult to define mathematically. The basic techniques of calculus suffice to compute the volumes $V_n(r)$ in any dimension. It is a useful exercise to identify precisely the techniques required to compute these volumes.

Basic slicing

Slicing the n-dimensional ball like an egg is helpful for computing the volume $V_n(r)$:

$$V_n(r) = \int_{-r}^{r} V_{n-1}\left(\sqrt{r^2 - t^2}\right) dt.$$

Figure 1: Egg slicers, good for slicing eggs and teaching calculus.

With a substitution $t = rx$, a scaling, we find that:

The slices of an n-dimensional ball are $(n-1)$-dimensional balls, of various radii.

$$V_n(r) = r\int_{-1}^{1} V_{n-1}\left(r\sqrt{1-x^2}\right)dx.$$

Since $V_{n-1}(r) = C_{n-1}r^{n-1}$, it follows that:

$$C_n r^n = C_{n-1}r^n \int_{-1}^{1}(1-x^2)^{n-1/2}dx.$$

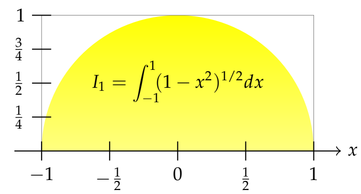
It follows that

Theorem 0.1 *Let $I_n = \int_{-1}^{1}(1-x^2)^{n/2}dx$. Then, $C_{n+1} = I_n C_n$.*

Plane geometry

It remains to explicitly determine the constants C_n, or at the very least, compute the integrals:

$$I_n = \int_{-1}^{1}(1-x^2)^{n/2}dx.$$



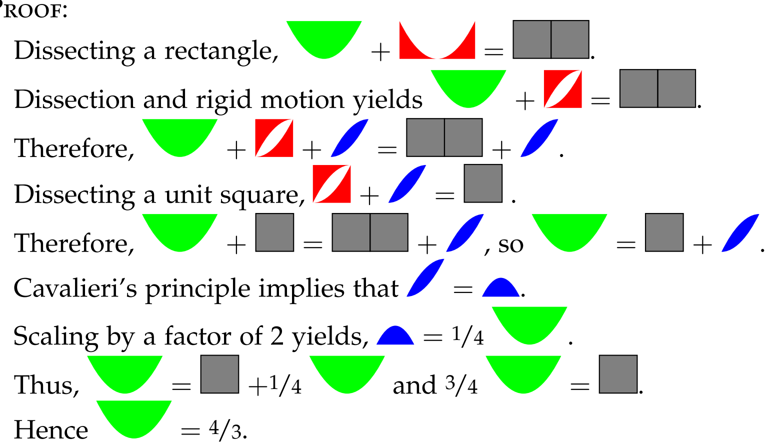
When $n = 1$, I_n can simply be *geometrically recognized* as the area of the unit semicircle: $I_1 = \pi/2$. When $n = 2$, I_2 can be computed using the fundamental theorem of calculus [1] since the integrand is a polynomial. It is remarkable that I_2 can be computed without recourse to the fundamental theorem of calculus; no antiderivatives are necessary as a Greek geometric argument suffices [2].

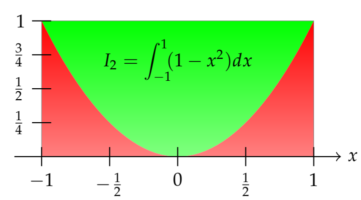
[1]
$$I_2 = \int_{-1}^{1}(1-x^2)dx = \left[x - \frac{1}{3}x^3\right]_{-1}^{1} = \frac{4}{3}.$$

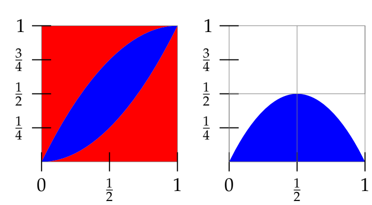
Theorem 0.2 *The fact that $\int_{-1}^{1}(1-x^2)dx = 4/3$ can be proven using only finite dissection, rigid motion, scaling, and Cavalieri's principle.*

[2] By "Greek geometric arguments", we mean to include finite dissection, rigid motion, scaling, and Cavalieri's principle. It is appropriate to view Cavalieri's principle as an element of Greek geometry, as scholars have asserted that it was used by Archimedes in *The Method*, where it was attributed also to Democritus

PROOF:







Q.E.D

The spherical two-step

When $n > 2$, a two-step recursion and integration by parts allows the computation of I_n.

Theorem 0.3 *Suppose that $n \geq 2$. Then, $I_n = \frac{n}{n+1} \cdot I_{n-2}$.*

PROOF: We relate the integral defining I_n to that defining I_{n-2} as follows, supposing that $n \geq 3$:

$$\begin{aligned} I_n &= \int_{-1}^{1} (1-x^2)^{n/2} dx \\ &= \left[x(1-x^2)^{n/2} \right]_{-1}^{1} + \int_{-1}^{1} nx^2 (1-x^2)^{n/2-1} dx \\ &= n \int_{-1}^{1} (1-x^2)^{(n-2)/2} - (1-x^2)^{n/2} dx \\ &= nI_{n-2} - nI_n. \end{aligned}$$

This utilizes integration by parts. Let $u = (1-x^2)^{n/2}$ and $dv = dx$. Then $v = x$ and

$$du = -nx(1-x^2)^{n/2-1}.$$

It follows that

$$I_n = \int_{-1}^{1} u dv = [uv]_{-1}^{1} - \int_{-1}^{1} v du.$$

It follows that:

$$I_n = \frac{n}{n+1} I_{n-2}.$$

Q.E.D

WE ARE INTERESTED in the sequence I_n, because $V_n(r) = C_n r^n$ and $C_1 = 2$ and for all $n \geq 1$, $C_{n+1} = C_n I_n$. The sequence of real numbers I_n, C_n, and the formulae $V_n(r)$, are given on the table at the right. We have now proven every one of these formulae.

n	I_n	C_n	$V_n(r)$
0	2	1	1
1	$\pi/2$	2	$2r$
2	$4/3$	π	πr^2
3	$3\pi/8$	$4/3 \cdot \pi$	$4/3 \cdot \pi r^3$
4	$16/15$	$1/2 \cdot \pi^2$	$1/2 \cdot \pi^2 r^4$
5	$15\pi/48$	$8/15 \cdot \pi^2$	$8/15 \cdot \pi^2 r^5$

Gamma and spheres

There is one more convenient way to express the volumes of spheres $V_n(R)$, or more accurately, the constants C_n. Consider the following integral:

$$G_n = \int_{\mathbb{R}^n} e^{-(x_1^2 + \cdots + x_n^2)} dx_1 \cdots dx_n.$$

On the one hand, the exponential can be separated as a product, and the integral can be expressed as: [3]

$$G_n = \left(\int_{\mathbb{R}} e^{-x^2} dx \right)^n = \pi^{n/2}.$$

On the other hand, one may use spherical coordinates, and express the integral as: [4]

$$G_n = \int_0^\infty \int_{S^{n-1}} e^{-r^2} r^{n-1} d\Theta dr = nC_n \int_0^\infty e^{-r^2} r^{n-1} dr.$$

[3] We utilize the standard, if misnamed, Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

[4] Here, $d\Theta$ is the volume form on the unit $(n-1)$-sphere obtained from its embedding in $\mathbb{R}^n$ and outward pointing orientation. For example, in polar coordinates, $d\Theta = d\theta$. In traditional spherical coordinates, $d\Theta = sin(\phi) d\phi d\theta$.

Substituting $r^2 = s$, $2rdr = ds$, [5]

$$G_n = (n/2)C_n \int_{s=0}^{\infty} e^{-s} s^{n/2} \frac{ds}{s} = (n/2)C_n \Gamma(n/2).$$

[5] We use the definition of the Γ function here:

$$\Gamma(z) = \int_0^{\infty} x^z e^{-x} \frac{dx}{x}.$$

It follows that:

$$Vol(S^{n-1}) = nC_n = 2\pi^{n/2}\Gamma(n/2)^{-1}.$$

The quantity nC_n is significant, as the "volume" of the $(n-1)$-dimensional unit sphere. Perhaps more remarkably, the Γ function can be uniquely meromorphically continued, so that $\Gamma(s)$ "makes sense" for all complex values of s except for negative integers (where Γ has poles). For this reason, it could be said that $Vol(S^{n-1})$ makes sense when n is a negative odd number. On the other hand, a simple-looking formula involving the Γ function is not useful for computation, unless one can compute $\Gamma(n/2)$ in the first place!

Euler and ζ

Volumes of spheres arise naturally across many fields of mathematics, but one of the more fascinating is within the realm of the Riemann's zeta function: for $n \geq 2$, it is defined by the series:[6]

$$\zeta(n) = 1^{-n} + 2^{-n} + 3^{-n} + \cdots.$$

[6] The series defining $\zeta(n)$ is absolutely convergent for $n \geq 2$

Better for convergence [7] is the alternating series of Euler:

$$\zeta^*(n) = 1^{-n} - 2^{-n} + 3^{-n} - \cdots,$$

related without loss of information to the zeta function by:

$$\zeta^*(n) = (1 - 2^{1-n}) \cdot \zeta(n).$$

[7] Note that $\zeta(1)$ is the divergent harmonic series. But, $\zeta^*(1) = \log(2)$, using the Taylor series for the logarithm, and Abel's 1826 theorem on Taylor series at the boundary of convergence.

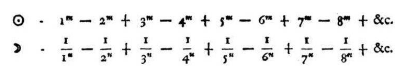


Figure 2: Euler's sun and moon, from "Remarques sur un Beau Rapport Entre Les Séries des Puissances tant Directes que Réciproques", published in 1768, and now available as E352. Euler's "sun" is the series we call $\zeta^*(n)$ for negative integers n. His "moon" corresponds to $\zeta^*(n)$ for positive integers n

To allow nonpositive [8] values of n, it is helpful to consider the series:

$$\zeta^*(n, x) = \sum_{m=0}^{\infty} (m+1)^{-n}(-x)^m,$$

and then define:

$$\zeta^*(n) = \lim_{x \to 1^-} \zeta^*(n, x).$$

[8] Note that for $|x| < 1$,

$$\zeta^*(0, x) = \sum_{m=0}^{\infty} (-x)^m = \frac{1}{1+x}.$$

Thus $\zeta^*(0) = 1/2$. Similarly,

$$\zeta^*(-1, x) = \sum_{m=0}^{\infty} (m+1)(-x)^m = \frac{1}{(1+x)^2}.$$

Thus $\zeta^*(-1) = 1/4$. Both of these facts and many more were observed by Euler

The zeta function can at last be defined for all integers (except $n = 1$) by:

$$\zeta(n) = (1 - 2^{1-n})^{-1} \cdot \lim_{x \to 1^-} \zeta^*(n, x).$$

Euler was able to tabulate many values of $\zeta^*(n)$. We prefer to tabulate values of $\zeta(n)$, as shown in the margin.

n	$\zeta(n)$	$\xi(n) = \frac{\zeta(n)}{Vol(S^{n-1})}$
$1-6$	$-16/63 \cdot 64$	$\pi^3/945$
$1-5$	0	$0 \cdot \infty$
$1-4$	$1/120$	$\pi^2/180$
$1-3$	0	$0 \cdot \infty$
$1-2$	$-1/12$	$\pi/12$
$1-1$	$-1/2$	∞
$0+1$	∞	∞
$0+2$	$\pi^2/6$	$\pi/12$
$0+3$	$\zeta(3)$	$\zeta(3)/4\pi$
$0+4$	$\pi^4/90$	$\pi^2/180$
$0+5$	$\zeta(5)$	$3\zeta(5)/8\pi^2$
$0+6$	$\pi^6/945$	$\pi^3/945$

But perhaps most remarkable is the functional equation, discovered by Euler, and written here using the volumes of spheres; define:

$$\xi(n) = \frac{\zeta(n)}{nC_n} = \frac{\zeta(n)}{Vol(S^{n-1})} = \frac{\Gamma(n/2)\zeta(n)}{2\pi^{n/2}}.$$

Visible on the right is the functional equation, observed first by Euler, and proven about a century later by Riemann:

$$\xi(n) = \xi(1-n).$$