CIA dynamics (chapter 3, 3rd ed.)

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cia_dynamics_3e.tex

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1 Introduction

This note provides more details on the derivation of the linear approximation used in Chapter 3 (3rd ed.) to study dynamics in the basic cash-in-advance (CIA) model. See **miu_dynamics_3e.pdf** for additional details on linearizing the equations common to both the CIA and MIU models and for a discussion of techniques for solving linear, rational expectations models.

2 The CIA model

The basic equilibrium conditions of the CIA model of Chapter 3 are given by

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}, \tag{1}$$

$$y_t = c_t + x_t, \tag{2}$$

$$x_t = k_t - (1-\delta)k_{t-1} \tag{3}$$

$$\Psi(1-n_t)^{-\eta} = (1-\alpha)\left(\frac{y_t}{n_t}\right)\lambda_t \tag{4}$$

$$R_t = \left[\alpha \mathrm{E}_t\left(\frac{y_{t+1}}{k_t}\right) + 1 - \delta\right] \tag{5}$$

$$\lambda_t = \beta \mathrm{E}_t R_t \lambda_{t+1} \tag{6}$$

$$c_t^{-\Phi} = \lambda_t(1+i_t) \tag{7}$$

$$\lambda_t = \beta \mathrm{E}_t\left(\frac{c_{t+1}^{-\Phi}}{\Pi_{t+1}}\right) \tag{8}$$

$$c_t = m_t \tag{9}$$

$$m_t = \left(\frac{1+u_t}{1+\pi_t}\right)m_{t-1}, \tag{10}$$

$$z_t = \rho_z z_{t-1} + e_t, \tag{11}$$

$$u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t, \quad 0 \le \gamma < 1, \tag{12}$$

3 The linear approximation

The next step is to derive first-order linear approximations to the model's equilibrium conditions.

3.1 Functional forms

The utility function:

$$u\left(c_t, m_t, 1-n_t\right) = \frac{\left[aC_t^{1-b} + (1-a)m_t^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi\left[\frac{(1-n_t)^{1-\eta}}{1-\eta}\right].$$

The production function:

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}$$

4 The linear approximation

The next step is to derive first-order linear approximations to the model's equilibrium conditions.

4.1 Functional forms

The utility function:

$$u\left(c_t, m_t, 1-n_t\right) = \frac{C_t^{1-\Phi}}{1-\Phi} + \Psi\left[\frac{(1-n_t)^{1-\eta}}{1-\eta}\right].$$

The production function:

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}$$

4.2 Linearization

The production function, the goods clearing condition, the capital accumulation equation, the labor-leisure optimality condition, the real return equation, and the Euler condition are the same as in the MIU model. See **miu_dynamics_3e.pdf** for details. The equations for the evolution of real money balances and for the exogenous disturbances are also identical to those used for the MIU model.

4.2.1 Marginal utility of consumption

We can rewrite

$$c_t^{-\Phi} = \lambda_t(1+i_t)$$

as

$$\left(c^{ss}\right)^{-\Phi}(1+\hat{c}_t)^{-\Phi} \approx \left(c^{ss}\right)^{-\Phi}(1-\Phi\hat{c}_t) = \lambda_t(1+i_t) = \lambda^{ss}(1+\hat{\lambda}_t)(1+i_t) \approx \lambda^{ss}\left(1+\hat{\lambda}_t+i_t\right).$$

Since $(c^{ss})^{-\Phi} = \lambda^{ss}(1+i^{ss})$, we have

$$-(c^{ss})^{-\Phi}\Phi\hat{c}_t \approx \lambda^{ss}\left(\hat{\lambda}_t + i_t - i^{ss}\right) \Rightarrow -\Phi\hat{c}_t = \hat{\lambda}_t + \hat{\imath}_t.$$

4.2.2 Money holdings

From the first order condition with respect to money holdings and the result that $c_t^{-\Phi} = \lambda_t(1+i_t)$,

$$\lambda_t = \beta \mathrm{E}_t\left(\frac{c_{t+1}^{-\Phi}}{\Pi_{t+1}}\right).$$

Rewrite this as

$$\lambda^{ss}\left(1+\hat{\lambda}_t\right) \approx \beta\mathrm{E}_t\left[\frac{(c^{ss})^{-\Phi}(1-\Phi\hat{c}_{t+1})}{(1+\pi_{t+1})}\right] \approx \beta(c^{ss})^{-\Phi}\mathrm{E}_t\left(1-\Phi\hat{c}_{t+1}-\pi_{t+1}\right).$$

Noting that

$$\lambda^{ss} = \beta\left[\frac{(c^{ss})^{-\Phi}}{1+\pi^{ss}}\right],$$

we have

$$\lambda^{ss}\left(1+\hat{\lambda}_t\right) \approx \lambda^{ss}(1+\pi^{ss})\mathrm{E}_t\left(1-\Phi\hat{c}_{t+1}-\pi_{t+1}\right) \approx \lambda^{ss}\mathrm{E}_t\left(1-\Phi\hat{c}_{t+1}-\pi_{t+1}+\pi^{ss}\right).$$

This implies that

$$\hat{\lambda}_t = -\mathrm{E}_t\left(\Phi\hat{c}_{t+1}+\hat{\pi}_{t+1}\right).$$

4.2.3 Cash-in-advance constraint

From

$$c_t = m_t$$

we obtain

$$\hat{c}_t = \hat{m}_t.$$

4.2.4 The Fisher equation

From

$$-\Phi\hat{c}_t = \hat{\lambda}_t + \hat{\imath}_t$$

and

$$\hat{\lambda}_t = -\mathrm{E}_t\left(\Phi\hat{c}_{t+1}+\hat{\pi}_{t+1}\right),$$

we can dervive the consumption Euler equation:

$$\hat{c}_t = \mathrm{E}_t\hat{c}_{t+1} - \left(\frac{1}{\Phi}\right)\left(\hat{\imath}_t - \mathrm{E}_t\hat{\pi}_{t+1}\right).$$

We can also use these same two equations, together with the Euler condition expressed in terms of $\hat{\lambda}_t$ to obtain

$$\hat{\lambda}_t = -\mathrm{E}_t\left(\Phi\hat{c}_{t+1} + \hat{\pi}_{t+1}\right) = \hat{\lambda}_t = \mathrm{E}_t\left(\hat{\lambda}_{t+1} + \hat{\imath}_{t+1} - \hat{\pi}_{t+1}\right),$$

or

$$\mathrm{E}_t\hat{\imath}_{t+1} = \hat{r}_t + \mathrm{E}_t\hat{\pi}_{t+1}.$$

4.3 Some simplifications

As before, we will eliminate expected future output from the equation for the real rate of return to obtain

$$\theta\hat{r}_t = \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left\{-(1-\alpha)\eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{k}_t + (1-\alpha)\lambda_t + \left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\rho_z z_t\right\}$$

where

$$\theta = \left[\alpha + \eta\left(\frac{n^{ss}}{l^{ss}}\right) + \alpha\left(1-\alpha\right)\left(\frac{y^{ss}}{k^{ss}}\right)\right]$$

Notice that in contrast to the MIU model, there remain three forward-looking variables: the Lagrangian multiplier λ, inflation $\hat{\pi}$, and consumption $\hat{c}$. However, we can use the cash-in-advance constraint to eliminate $\mathrm{E}_t\hat{c}_{t+1}$ by noting that

$$\mathrm{E}_t\hat{c}_{t+1} = \mathrm{E}_t\hat{m}_{t+1} = \mathrm{E}_t\left(u_{t+1} - \pi_{t+1} + \hat{m}_t\right) = \rho_u u_t + \hat{m}_t - \mathrm{E}_t\hat{\pi}_{t+1}.$$

Hence,

$$\hat{\lambda}_t = -\Phi\mathrm{E}_t\hat{c}_{t+1} - \mathrm{E}_t\hat{\pi}_{t+1} = -\Phi\hat{m}_t - (1-\Phi)\mathrm{E}_t\hat{\pi}_{t+1} - \Phi\rho_u u_t.$$

4.4 Collecting all equations

Unknowns: $\hat{y}_t$, $\hat{k}_t$, $\hat{n}_t$, $\hat{x}_t$, $\hat{c}_t$, $\hat{\lambda}_t$, $\hat{r}_t$, $\hat{\imath}_t$, π_t, $\hat{m}_t$ – 10 variables.

Ten equations, plus the specification of the processes governing the exogenous productivity and money growth disturbances.

$$z_t = \rho_z z_{t-1} + e_t$$

$$u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t.$$

$$\hat{y}_t = z_t + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{n}_t$$

$$\left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t = \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t + \delta\hat{x}_t$$

$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{x}_t$$

$$\left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\hat{n}_t = \hat{y}_t + \lambda_t$$

$$\theta\hat{r}_t = \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left\{-(1-\alpha)\eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{k}_t + (1-\alpha)\lambda_t + \left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\rho_z z_t\right\}$$

$$-\Phi\hat{c}_t = \hat{\lambda}_t + \hat{\imath}_t$$

$$\hat{c}_t = \hat{m}_t$$

$$\hat{m}_t = u_t - \pi_t + \hat{m}_{t-1}$$

$$\hat{\lambda}_t = \hat{r}_t + \mathrm{E}_t\hat{\lambda}_{t+1}$$

$$\hat{\lambda}_t = -\Phi\hat{m}_t - (1-\Phi)\mathrm{E}_t\hat{\pi}_{t+1} - \Phi\rho_u u_t.$$