The equivalence of robustly optimal targeting rules and robust control targeting rules: an extension

Carl E. Walsh*

May 2004

1 Introduction

Walsh (2004) demonstrated that the robustly optimal targeting rule of Giannoni and Woodford (2003a, 2003b) and the targeting rule of a policy maker concerned with robustness in the sense of Hansen and Sargent (2004) were identical in a purely forward-looking new Keynesian model. In this note, I first show that the equivalence result extends to a new Keynesian model with inflation inertia (Woodford 2003). Then, this equivalence is shown to apply in a wide class of models.

2 Basic model

The foundations of the benchmark new Keynesian model have been discussed extensively; I draw heavily on the formulation of Woodford (2003) to which the reader is referred.[1] The two equations of the model are an expectational IS curve given by (1) and an inflation adjustment equation given by (2):

$$x_t = E_t x_{t+1} - \left(\frac{1}{\sigma}\right)\left(i_t - E_t \pi_{t+1} - r_t^n\right), \tag{1}$$

where x_t is the output gap, i_t is the nominal rate of interest, π_{t+1} is the inflation rate from t to $t+1$, and r_t^n is the natural real rate of interest, and

$$\pi_t - \gamma\pi_{t-1} = \beta\left(E_t \pi_{t+1} - \gamma\pi_t\right) + \kappa x_t + e_t, \tag{2}$$

*University of California, Santa Cruz; walshc@ucsc.edu. I would like to thank Richard Dennis, Federico Ravenna, and John C. Williams for helpful conversations. All errors remain my own.

[1] See Walsh (2003a, Chapters 5 and 11) for further discussion of this model and for references to the literature.

where e_t is a cost shock that captures any factors that alter the relationship between real marginal costs and the output gap. I follow Woodford (2003) in assuming that with probability $1-\alpha$ firms optimally adjust their price and with probability α they simply index their price by a fraction γ of the most recent rate of inflation. For simplicity, assume the exogenous disturbances evolve according to

$$r_t^n = \rho_r r_{t-1}^n + v_t,\ 0 \leq \rho_r < 1 \tag{3}$$

$$e_t = \rho_e e_{t-1} + \varepsilon_t,\ 0 \leq \rho_e < 1. \tag{4}$$

where the innovations v and ε are white noise.

When $\gamma = 0$, one obtains the purely forward-looking model used in Walsh (2004). When $\gamma \neq 0$, lagged inflation becomes an endogenous state variable. The next section shows that the results in Walsh (2004) extend to this case.

3 Robustly optimal targeting rules

Svensson and Woodford (2004) and Giannoni and Woodford (2003a, 2003b) have analyzed a class of policy rules that represent *robustly optimal implicit instrument rules.*[2] The Giannoni-Woodford rules are optimal in that the rule supports the equilibrium consistent with an optimal commitment policy when evaluated from the timeless perspective (Woodford 2003), and they are robust in that the coefficients in the policy rule are independent of the parameters that characterize the behavior of the exogenous, stochastic disturbances.

Assume the central bank's objective is to minimize a loss function that depends on the variation of inflation, the output gap, and the nominal interest rate:

$$\left(\frac{1}{2}\right) E_t \sum_{i=0}^{\infty} \beta^i \left[z_{t+i}^2 + \lambda_x x_{t+i}^2 + \lambda_i (i_{t+i} - i^*)\right], \tag{5}$$

where $z_t \equiv \pi_t - \gamma\pi_{t-1}$. Woodford (2003) discussions the conditions under which (5) can be viewed as a second order approximation to the utility of the representative agent.

Under commitment, the Lagrangian for the policy maker's decision problem is

$$\begin{aligned} L_t \;=\; & E_t \sum_{i=0}^{\infty} \beta^i \left\{ \left(\frac{1}{2}\right) \left[z_{t+i}^2 + \lambda_x x_{t+i}^2 + \lambda_i (i_{t+i} - i^*)^2\right] \right. \\ & s_{1t+1+i} \left(z_{t+i} - \beta z_{t+1+i} - \kappa x_{t+i} - e_{t+i}\right) \\ & + s_{2t+1+i} \left[x_{t+i} - x_{t+1+i} + \left(\frac{1}{\sigma}\right)\left(i_{t+i} - \pi_{t+1+i} - r_{t+i}^n\right)\right] \\ & \left. + s_{3t+1+i} \left(z_{t+i} - \pi_{t+i} + \gamma\pi_{t-1+i}\right)\right\}, \end{aligned}$$

where s_1, s_2, and s_3 denote Lagrangian multipliers. The necessary first order conditions include (1) - (4) and the following four equations:

[2] These policy rules are also called targeting rules. See Woodford and Svensson (2004).

$$z_t + E_t s_{1t+1} - s_{1t} + E_t s_{3t+1} = 0, \tag{6}$$

$$\lambda_x x_t - \kappa E_t s_{1t+1} + E_t s_{2t+1} - \left(\frac{1}{\beta}\right) s_{2t} = 0, \tag{7}$$

$$-\left(\frac{1}{\beta\sigma}\right) s_{2t} - E_t s_{3t+1} + \beta\gamma E_t s_{3t+2} = 0, \tag{8}$$

$$\lambda_i (i_t - i^*) + \left(\frac{1}{\sigma}\right) E_t s_{2t+1} = 0. \tag{9}$$

Assume the optimal commitment policy from a timeless perspective (Woodford 2003) has been in place since at least $t-2$. This assumption allows one to use (9) to eliminate s_{2t+1} and s_{2t} from (7) and (8), obtaining an equilibrium expression for the nominal interest rate:

$$A(L) E_t i_{t+1} = \left(\frac{\kappa}{\beta\sigma}\right) i^* - \left(\frac{\kappa}{\sigma\lambda_i}\right) \left[q_t - \beta\gamma E_t q_{t+1}\right], \tag{10}$$

where

$$A(L) = \beta\gamma - (1 + \gamma + \beta\gamma) L + \left[1 + \gamma + \frac{1}{\beta}\left(1 + \frac{\kappa}{\sigma}\right)\right] L^2 - \left(\frac{1}{\beta}\right) L^3$$

and $q_t \equiv z_t + \left(\frac{\lambda_x}{\kappa}\right) \Delta x_t = z_t + (1/\psi)\Delta x_t$ and Δ denotes the first difference operator.

Equation (10) is the Giannoni-Woodford *robustly optimal, implicit instrument rule*. As they emphasize, the coefficients in (10) do not depend on either the serial correlation coefficients ρ_r and ρ_e or on the variances of the innovations to the disturbances.

The equilibrium under the optimal (timeless perspective) commitment policy is given by the rational expectations solution to equations (1) - (4) and (10).

3.1 Hansen and Sargent robust control

Hansen and Sargent, together with coauthors, have explored robust control approaches to the decision problem faced by agents who face model uncertainty (Hansen and Sargent 2003, 2004). In their approach, the central bank views its model as an approximation to the true model of the economy, knowing only that the true model is in a neighborhood around its approximating model.[3] Robust policies, in the sense of Hansen and Sargent, are min-max policies, designed to perform well in worst-case scenarios. Hansen and Sargent (2004, Chapter 15) apply their robust control methodology to Woodford's forward-looking new Keynesian model, and Giordani and Söderlind (2003) report some simulations

[3] Alternative approaches have been explored by Stock (1999), Giannoni (2002), Onatski and Stock (2002), and Onatski and Williams (2003).

of the Clarida, Galí, and Gertler (1999) new Keynesian model under robust min-max policies.[4]

As Hansen and Sargent (2004) explain, the policy maker's problem can be represented as a game between the policy maker who attempts to minimize the loss function (5) and nature (or an evil agent) who tries to maximize the same loss function. They show how this game can be represented in what they label the *multiplier version of the robust Stackelberg problem.* If the policy maker's approximating model is given by (1) and (2), this problem takes the form

$$\min_{z,x,i}\max_{w_1,w_2} E_t^{rc}\sum_{i=0}^{\infty}\beta^i\left[\left(\frac{1}{2}\right)\left[z_{t+i}^2+\lambda_x x_{t+i}^2+\lambda_i(i_{t+i}-i^*)^2\right]\right.$$
$$-\left(\frac{1}{2}\right)\beta\theta\left(w_{1t+1+i}^2+w_{2t+1+i}^2\right)+s_{1t+1+i}\left(z_{t+i}-\beta z_{t+1+i}-\kappa x_{t+i}-e_{t+i}\right)$$
$$+s_{2t+1+i}\left[x_{t+i}-x_{t+1+i}+\left(\frac{1}{\sigma}\right)\left(i_{t+i}-\pi_{t+1+i}-r_{t+i}^n\right)\right]$$
$$+s_{3t+1+i}\left(z_{t+i}-\pi_{t+i}+\gamma\pi_{t-1+i}\right)$$
$$+s_{4t+1+i}\left(\rho_e e_{t+i}+\varepsilon_{t+1+i}+w_{1t+1+i}-e_{t+1+i}\right)$$
$$\left.+s_{5t+1+i}\left(\rho_r r_{t+i}^n+\chi_{t+1+i}+w_{2t+1+i}-r_{t+1+i}^n\right)\right].$$

The new variables w_{1t+i} and w_{2t+i} represent the model mis-specifications that enter into the dynamics of the state variables. The new parameter θ reflects the policy maker's desire for robustness, with $0<\theta<\infty$. The standard rational expectations policy problem arises when $\theta=\infty$. The superscript rc on the expectations operator reflects the assumption that private agents share the policy maker's approximating model and degree of robustness; this will affect the way in which expectations are formed.

Again assuming a timeless perspective, the first order conditions for this problem include equations (6) - (9) and

$$-s_{1t+1}+\rho_e s_{4t+1}-\left(\frac{1}{\beta}\right)s_{4t}=0, \tag{11}$$

$$\left(\frac{1}{\sigma}\right)s_{2t+1}+\rho_r s_{5t+1}-\left(\frac{1}{\beta}\right)s_{5t}=0, \tag{12}$$

$$-\theta w_{1t+1}+s_{4t+1}=0, \tag{13}$$

$$-\theta w_{2t+1}+s_{5t+1}=0. \tag{14}$$

Note that the first order conditions (6) - (9) from the standard control problem are also necessary conditions for the robust control problem. Thus, proceeding as before, the Lagrangian multipliers s_1, s_2, and s_3 can be eliminated to yield an implicit instrument, or targeting,rule for the nominal rate of interest that is identical to (10). Thus, the implicit instrument or targeting rule that supports the optimal commitment policy from a timeless perspective is identical, except

[4] Some policy implications of this approach are discussed in Walsh (2003b).

for the manner in which expectations are formed, to the policy rule that would be adopted by a policy maker with a desire for robustness in the sense of Hansen and Sargent. Note that the coefficients in this resulting instrument rule are independent of θ, the policy maker's desired degree of robustness. The economy behaves differently under the two model specifications, however. This difference arises from the way in which expectations are formed. In the standard approach, expectations are based on the true processes governing the disturbance processes. In the robust control environment, expectations are based on distorted versions of the disturbance processes, where these distortions reflect the roles of w_1 and w_2.[5]

3.2 A generalization

The equivalence results obtained in the new Keynesian model are actually quite general. To see this, consider the following general Lagrangian version of the robust Stackelberg problem (see Hansen and Sargent, Ch. 15):

$$\begin{aligned}\mathcal{L}_t \quad = \quad & -E_t \sum_{i=0}^{\infty} \beta^i \left\{ y'_{1t+i} Q_{11} y_{1t+i} + y'_{1t+i} Q_{12} y_{2t+i} + y'_{2t+i} Q_{21} y_{1t+i} + y'_{2t+i} Q_{22} y_{2t+i} \right. \\ & + y'_{1t+i} S_1 u_{t+i} + y'_{2t+i} S_2 u_{t+i} + u'_{t+i} R u_{t+i} - \beta \Theta W'_{t+i+1} W_{t+i+1} \\ & + 2\beta \mu_{1t+1+i} \left[A_{11} y_{1t+i} + C_1 \left(\varepsilon_{t+i+1} + W_{t+i+1} \right) - y_{1t+i+1} \right] \\ & + 2\beta \mu_{2t+1+i} \left[A_{21} y_{1t+i} + A_{22} y_{2t+i} + B u_t \right. \\ & \left. \left. + C_2 \left(\varepsilon_{t+i+1} + W_{t+i+1} \right) - y_{2t+i+1} \right] \right\}, \end{aligned} \tag{15}$$

where the state vector $y' = [y'_1, y'_2]$ has been decomposed into a $k_1 \times 1$ vector of exogenous disturbances y_1 and a $k_2 \times 1$ vector of endogenous state and forward looking variables y_2. The control (instrument) variable is u_t; ε_t is a vector of exogenous stochastic innovations, and W represents the model mis-specification. The constraints in (15) are based on the transition equations

$$\begin{bmatrix} y_{1t+1} \\ y_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_t + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \left(\varepsilon_{t+1} + W_{t+1} \right), \tag{16}$$

where the zeros on the right side of (16) reflect the assumption that the elements of y_1 are exogenous processes. The first-order conditions with respect to y_2 and u are

$$Q'_{12} y_{1t} + Q_{21} y_{1t} + Q_{22} y_{2t} + S_2 u_t + \beta A'_{22} \mu_{2t+1} - \mu_{2t} = 0 \tag{17}$$

and

$$S'_1 y_{1t} + S'_2 y_{2t} + R u_t + \beta B'_2 \mu_{2t+1} = 0. \tag{18}$$

These conditions do not involve either W or Θ and are exactly the same as those obtained for the standard optimal commitment problem studied by Giannoni and Woodford. Thus, when the first order conditions are used to obtain a

[5] Walsh (2004) provides an example for the basic Clarida, Galí, and Gertler (1999) model in which $\lambda_i = \gamma = 0$.

targeting rule, this rule will be the same under the robust control approach as under the standard approach. In that sense, the policy maker concerned with robustness would implement the same rule as a policy maker concerned only with maximizing the expected value of the objective function.

Note, however, that (17) and (18) cannot in general be solved to obtain a robustly optimal rule, that is, a rule for u that is independent of the exogenous y_1 processes. Necessary conditions for doing so include $Q_{12} = Q_{21} = S_1 = 0$. That is, the exogenous disturbances cannot appear in the policy maker's objective function except in the form $y'_{1t+i}Q_{11}y_{1t+i}$ since this is independent of any policy choice. But these conditions hold for the standard monetary policy problem in which the loss function depends on endogenous target variables and not directly on the disturbance processes.

When $Q_{12} = Q_{21} = S_1 = 0$, (17) and (18) simplify to

$$Q_{22}y_{2t} + S_2u_t + \beta A'_{22}\mu_{2t+1} - \mu_{2t} = 0 \tag{19}$$

$$Ru_t + \beta B'_2\mu_{2t+1} = 0, \tag{20}$$

and these k_2+1 equations can be used to eliminate the k_2 Lagrangian multipliers to obtain a robustly optimal rule for u_t. Under the standard approach dealt with by Giannoni and Woodford, these equations are solved jointly with the transition equations in (16) with $W \equiv 0$ to obtain the equilibrium, while the Hansen-Sargent approach incorporates the first-order condition for the maximizing value of W.[6]

Inspection of (19) and (20) reveals that $Q_{12} = Q_{21} = S_1 = 0$ is necessary but not sufficient for obtaining an implicit instrument rule for u_t. Suppose $S_2 = R = 0$ so that the control instrument u_t also does not directly enter the objective function. Then, since u_t appears in neither (19) nor (20), its behavior must be obtained from one of the structural equations. But since these depend on the exogenous disturbances, optimal policy cannot be described in terms of a rule for the instrument that is robust to the behavior of the exogenous disturbances. It is still possible, however, to obtain a robustly optimal targeting rule from (19) and (20) that only involves the forward-looking variables y_2. Model specifications in which $S_2 = R = 0$ are frequently employed in monetary policy analysis in which the policy maker's loss function depends on inflation and output gap variability alone. This case corresponds to setting $\lambda_i = 0$ in (5) so that the nominal rate of interest does not appear in the loss function. The standard approach is to then treat the output gap as if it were the instrument, and a robustly optimal targeting rule involving the gap and inflation can be found. The behavior of the nominal interest rate is then obtained from (1) and so the resulting rule for the interest rate depends on the exogenous process governing the natural real rate r_t^n.

[6]This first order condition takes the form $\beta\Theta W_{t+1} - \beta C'\mu_t = 0$, where μ is the stacked vector of Lagrangian multipliers and $C' = [C'_1, C'_2]$.

References

[1] Clarida, R., J. Galí, and M. Gertler, "The Science of Monetary Policy: A New Keynesian Perspective," *Journal of Economic Perspectives*, 37(4), 1999, 1661-1707.

[2] Giannoni, M., "Does Model Uncertainty Justify Caution? Robust Optimal Monetary Policy in a Forward-Looking Model," *Macroeconomic Dynamics* 6(1): 111-144, February 2002.

[3] Giannoni, M. and M. Woodford, "Optimal Interest-Rate Rules: I. General Theory," NBER Working Paper No. 9419, Jan. 2003a.

[4] Giannoni, Marc and Michael Woodford, "Optimal Interest-Rate Rules: II. Applications," NBER Working Paper No. 9420, Jan. 2003b.

[5] Giordani, P. and P. Söderlind, "Solution of Macromodels with Hansen-Sargent Robust Policies: Summary and Some Extensions," May 2002.

[6] Hansen, Lars Peter and Thomas J. Sargent, "Robust Control of Forward-Looking Models," *Journal of Monetary Economics*, 50(3), April 2003, 581-604.

[7] Hansen, Lars Peter and Thomas J. Sargent, *Robust Control and Economic Model Uncertainty*, Princeton University Press, 2004.

[8] Onatski, Alexei and James H. Stock, "Robust Monetary Policy under Model Uncertainty in a Small Model of the U.S. Economy," *Macroeconomic Dynamics* 2002, 6, 85-110.

[9] Onatski, A. and N. Williams, "Modeling Model Uncertainty," Feb. 2003.

[10] Stock, James H., "Comment on 'Policy Rules for Inflation Targeting'," in J. B. Taylor, ed., *Monetary Policy Rules*, Chicago: University of Chicago Press, 1999, 253-259.

[11] Svensson, Lars E. O. and M. Woodford, "Implementing Optimal Policy through Inflation-Forecast Targeting," in Ben S. Bernanke and Michael Woodford (ed.), *The Inflation Targeting Debate*, Chicago: University of Chicago Press, 2004.

[12] Walsh, Carl E., *Monetary Theory and Policy*, 2nd. ed., Cambridge, MA: The MIT Press, 2003a.

[13] Walsh, Carl E. "Implications of a Changing Economic Structure for the Strategy of Monetary Policy," Monetary Policy Uncertainty: Adapting to a Changing Economy, Federal Reserve Bank of Kansas City, Jackson Hole Symposium, Aug. 2003b.

[14] Walsh, Carl E., "Robustly Optimal Instrument Rules and Robust Control: An Equivalence Result," May 2004, forthcoming, *Journal of Money, Credit, and Banking.*

[15] Woodford, Michael, *Interest and Prices*, Princeton University Press, 2003.