Welfare-based optimal monetary policy with unemployment and sticky prices: A linear-quadratic framework: Appendix

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1 The efficient equilibrium

Using the assumed functional form of the matching function, the social planner’s problem can be written as

\[
\max_{C, N, u, \theta} E_t \sum_{i=0}^{\infty} \beta^i \left\{ \left( \frac{C_{t+i}}{1 - \sigma} \right) + \lambda_{t+i} \left[ Z_{t+i}N_{t+i} - \kappa u_{t+i}\theta_{t+i} + w^a(1 - N_t) - C_{t+i} \right] \\
+ \psi_{t+i} \left[ (1 - \rho)N_{t+i-1} + \varphi^0_{t+i}u_{t+i} - N_{t+i} \right] + s_{t+i} \left[ u_{t+i} - 1 + (1 - \rho)N_{t+i-1} \right] \right\}.
\]

First order conditions are

- **C:** \( C_{t}^{-\sigma} - \lambda_t = 0; \)
- **\( \theta \):** \( -\lambda_t \kappa u_t + \psi_t \alpha \varphi \theta_t^{\alpha-1} u_t = 0; \)
- **\( u \):** \( -\lambda_t \kappa \theta_t + \psi_t \varphi \theta_t^\alpha + s_t = 0; \)
- **\( N \):** \( \lambda_t (Z_t - w^a) - \psi_t + (1 - \rho)\beta E_t (\psi_{t+1} + s_{t+1}) = 0. \)

The second of these first order conditions implies

\[
\frac{\psi_t}{\lambda_t} = \left( \frac{\kappa}{\alpha \varphi} \theta_t^{1-\alpha} \right),
\]

while the third then implies

\[
\frac{s_t}{\lambda_t} = \kappa \theta_t - \frac{\psi_t}{\lambda_t} \varphi \theta_t^\alpha = \left( \frac{\alpha - 1}{\alpha} \right) \kappa \theta_t.
\]
We can write the fourth of these first order conditions as

\[
Z_t = w^u + \frac{\psi t}{\lambda t} - (1 - \rho)\beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \left( \frac{\psi_{t+1} + s_{t+1}}{\lambda_{t+1}} \right)
\]

\[
= w^u + \left( \frac{\kappa}{\alpha \varphi} \theta_t^{1-\alpha} \right) - (1 - \rho)\beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \left[ \frac{\kappa}{\alpha \varphi} \theta_{t+1}^{1-\alpha} \left( \frac{1 - \alpha}{\alpha} \right) \kappa \theta_{t+1} \right].
\]

Rearranging this condition for efficiency and noting that \( \varphi \theta_t^{\alpha-1} = q(\theta_t) \) yields

\[
Z_t = w^u + \frac{\kappa}{\alpha q(\theta_t)} - (1 - \rho)\beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \left[ \frac{\kappa}{\alpha q(\theta_{t+1})} \right]
\]

\[
+ (1 - \rho) \left( \frac{1 - \alpha}{\alpha} \right) \beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \kappa \theta_{t+1}.
\]

(1)

In the steady-state, this condition becomes

\[
1 = w^u + \frac{\kappa}{\alpha q} - (1 - \rho)\beta \frac{\kappa}{\alpha q} + (1 - \rho) \left( \frac{1 - \alpha}{\alpha} \right) \beta \kappa \theta
\]

But as \( \bar{q} = \rho \bar{N}/\bar{V} \) and \( \theta = \bar{V}/\bar{U} \), we have

\[
\delta_1 \equiv 1 - w^u - \frac{\kappa \bar{V}}{\alpha \rho \bar{N}} = -\beta (1 - \rho) \frac{\kappa \bar{V}}{\alpha \bar{U}} \left( \alpha - 1 + \frac{\bar{U}}{\rho \bar{N}} \right) \equiv -\beta \delta_2.
\]

(2)

Note that \( \delta_1 \) is the net consumption generated by one additional match (the match produces 1 unit, but home production falls by \( w^u \), and to fill the march required posting \( 1/(\partial m/\partial v) = \bar{V}/\alpha \rho \bar{N} \) vacancies at a cost \( \kappa \) each).

Equations (11) and (12) imply that the job posting condition takes the form

\[
\frac{Z_t}{\mu_t} = w^u + \left( \frac{1}{1 - b_t} \right) \left[ \frac{\kappa}{q_t} - \beta (1 - \rho) E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) (1 - b_t p_{t+1}) \left( \frac{\kappa}{q_{t+1}} \right) \right].
\]

When prices are flexible, \( \mu_t = \mu > 1 \) for all \( t \). However, a tax-subsidy policy that offsets the allocative effects of the steady-state markup is not sufficient to ensure that the resulting flex-price equilibrium is efficient as is the case in the standard new Keynesian model. Inefficient job posting can lead to an inefficient level of vacancies and unemployment. Efficiency requires \( \mu_t = \mu = 1 \) \( \text{and} \ b_t = 1 - \alpha \). This second condition is the familiar the Hosios condition for efficient vacancy creation.\(^1\) Letting a superscript \( e \) denote the efficient equilibrium, the job posting condition takes

\(^1\)The Hosios condition requires that labor’s share of the surplus, \( b \), equal the elasticity of matches with respect to unemployment, \( 1 - \alpha \).
the form
\[ Z_t = w^u + \left( \frac{1}{\alpha} \right) \left\{ \frac{\kappa}{\varphi} (\theta^r_t)^{1-\alpha} - \beta (1-\rho) E_t \left( \frac{\lambda^r_{t+1}}{\lambda^r_t} \right) \kappa \left[ \frac{(\theta^r_{t+1})^{1-\alpha}}{\varphi} - (1-\alpha)\theta^r_{t+1} \right] \right\} . \]

When linearized, and expressed in terms of the efficient level of unemployment, this yields
\[ a_1 (\hat{u}^e_{t+1} - \rho \hat{u}^e_t) - \beta a_2 \left( E_t \hat{u}^e_{t+2} - \rho \hat{u}^e_{t+1} \right) - \beta a_3 \alpha \rho \eta \hat{z}^e_t = -\alpha \rho \eta \hat{z}_t, \]
where \( \hat{r}^e_t \) is the equilibrium real interest rate in the efficient equilibrium.

In addition, the IS relationship in the efficient equilibrium takes the form
\[ \hat{u}^e_{t+1} = \left( \frac{\beta}{1+\beta} \right) E_t \hat{u}^e_{t+2} + \left( \frac{1}{1+\beta} \right) \hat{u}^e_t \]
\[ - \left( \frac{1}{\sigma} \right) \left( \frac{1}{1+\beta} \right) \left( \frac{\eta^C}{\delta^2 Y} \right) r^e_t - \left( \frac{1}{1+\beta} \right) \left( \frac{Y^C}{\delta^2} \right) (1-\rho) \hat{z}_t, \]
where \( \delta^2 = (1-\rho) \left( \kappa \bar{V}/\alpha \bar{U} \right) (\alpha - 1 + \bar{U}/\rho \bar{N}) \). The responses of the efficient unemployment rate \( \hat{u}^e_{t+1} \) and the real interest rate in the face of productivity shocks can be found by jointly solving (3) and (4). Equations (3) and (4) imply that the equilibrium unemployment responds to productivity shocks even under flexible prices.

2 The linearized structural equations

The model consists of the following core equations (see paper for definitions of the variables and a discussion of the model equations):

\[ Y_t = C_t - w^u (1 - N_t) + \kappa V_t, \]
\[ \lambda_t = \beta E_t R_t \lambda_{t+1} \]
\[ u_t \equiv 1 - (1-\rho) N_{t-1}. \]
\[ N_t = (1-\rho) N_{t-1} + \chi v_t^\alpha u_t^{1-\alpha}. \]

In (4), we have followed Neiss and Nelson (2003) in defining \( \hat{u}^e_{t+1} \) relative to last-period’s efficient unemployment rate. Thus, the path of \( \hat{u}^e_{t+1} \) is that for an economy that has always been in an efficient equilibrium. Alternatively, Woodford (2003) defines the flex-price and efficient equilibria conditional on the actual outcomes in the previous period. In that case, \( \hat{u}^e_{t+1} \) would depend on \( \hat{u}_t \). Edge (2003) discusses these two alternative definitions in the context of a model in which the lagged capital stock is an endogenous state variable. We follow the Neiss-Nelson definition; as Edge shows, it proves more convenient for deriving the welfare approximation we use to characterize optimal monetary policy.
\[
\frac{Z_t}{\mu_t} = w_t + \frac{\kappa}{q_t} - \frac{(1 - \rho)}{\lambda_t} \beta E_t \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \left( \frac{\kappa}{q_{t+1}} \right) 
\]
(6)

\[
w_t = (1 - b_t) w^u + b_t \left[ \frac{Z_t}{\mu_t} + \frac{(1 - \rho)}{R_t} E_t \left( \frac{\kappa}{q_{t+1}} \right) p_{t+1} \right],
\]
(7)

\[
\tau_t \equiv w^u + \left( \frac{1}{1 - b_t} \right) \left\{ \frac{\kappa}{q_t} - \frac{(1 - \rho)}{R_t} E_t (1 - b_t p_{t+1}) \frac{\kappa}{q_{t+1}} \right\}.
\]
(8)

plus the Fisher equation linking the nominal and real interest rates, the first order condition for price adjusting firms, and the definitions of \( \lambda \) as the marginal utility of consumption, \( q_t \) as the probability a vacancy is filled, \( p_t \) as the probability an unemployment work finds a match, and \( b_t \) and \( z_t \) are exogenous bargaining and productivity shocks.

When linearized around the steady state of the model, one obtains

\[
\hat{y}_t = \left( \frac{C}{Y} \right) \hat{c}_t + w^u \hat{\bar{n}}_t + \left( \frac{\kappa V}{Y} \right) \hat{\bar{\pi}}_t.
\]
(9)

\[
\hat{c}_t = E_t \hat{c}_{t+1} - \left( \frac{1}{\sigma} \right) (i_t - E_t \pi_{t+1}).
\]
(10)

\[
\hat{\bar{n}}_t = - (1 - \rho) \left( \frac{\bar{N}}{\bar{u}} \right) \hat{\bar{n}}_{t-1} \equiv - \eta \hat{\bar{n}}_{t-1}.
\]
(11)

\[
\hat{\bar{n}}_t = \rho_u \hat{\bar{n}}_{t-1} + \alpha \rho \hat{\bar{\theta}}_t,
\]
(12)

where

\[
\hat{\bar{\theta}}_t = \hat{\bar{v}}_t - \hat{\bar{u}}_t.
\]
(13)

From (6) - (8) and the model of price adjustment,

\[
\hat{\mu}_t = z_t - \mu \left( a_1 \hat{\theta}_t - \beta a_2 E_t \hat{\theta}_{t+1} - \beta a_3 \hat{r}_t + B \hat{b}_t \right)
\]
(14)

\[
\pi_t = \beta E_t \pi_{t+1} - \delta \hat{\mu}_t.
\]
(15)

The Fisher equation is

\[
i_t = \hat{\bar{r}}_t - E_t \pi_{t+1}.
\]
(16)

These eight equations, when combined with the production function and a specification of monetary policy, can be solved for \( \hat{c}_t, \hat{\bar{y}}_t, \hat{\bar{n}}_t, \hat{\bar{u}}_t, \hat{\bar{v}}_t, \hat{\bar{\theta}}_t, \hat{\mu}_t, \hat{\bar{r}}_t, \pi_t, \) and \( i_t \). We proceed to obtain a version of the model that consists of three structural equations in \( \pi_t, \hat{\bar{u}}_t, \) and \( \hat{\bar{\theta}}_t \).

Define \( \eta \equiv (1 - \rho) \left( \bar{N}/\bar{u} \right) \) and \( \rho_u \equiv (1 - \rho) \left( 1 - \rho \bar{N}/\bar{u} \right) \). We can use (11), (12) and (13) to write

\[
\hat{\bar{\theta}}_t = - \left( \frac{1}{\alpha \rho \eta} \right) \left( \hat{\bar{u}}_{t+1} - \rho_u \hat{\bar{n}}_t \right).
\]
(17)
Since \( \hat{v}_t = \hat{\theta}_t + \hat{u}_t \),
\[
\hat{v}_t = -\left( \frac{1}{\alpha Y} \right) \hat{u}_{t+1} + \left[ 1 + \rho_u \left( \frac{1}{\alpha Y} \right) \right] \hat{u}_{t+1}.
\] (18)

The constant returns to scale technology for wholesale good’s production implies, when linearized, that \( \hat{y}_t = \hat{n}_t + \hat{z}_t \). Thus, \( \hat{y}_t \) can be eliminated from (9) to yield
\[
\hat{c}_t = \left( \frac{Y}{C} \right) (1 - w^n) \hat{n}_t + \left( \frac{Y}{C} \right) \hat{z}_t - \left( \frac{\kappa V}{C} \right) \hat{v}_t.
\] (19)

We can now use (18) to write the goods market clearing condition (19) as
\[
\hat{c}_t = -\left( \frac{Y}{C} \right) (1 - w^n) \hat{u}_{t+1} + \left( \frac{Y}{C} \right) \hat{z}_t + \left( \frac{\kappa V}{C} \right) \hat{u}_{t+1} - \left( \frac{\kappa V}{C} \right) \left[ 1 + \rho_u \left( \frac{1}{\alpha Y} \right) \right] \hat{u}_t,
\]

or
\[
\hat{c}_t = \varphi_1 \hat{u}_{t+1} - \varphi_2 \hat{u}_t + \left( \frac{Y}{C} \right) \hat{z}_t,
\] (20)

where, \( \varphi_1 \equiv -\left( \frac{Y}{\eta Y} \right) \left[ 1 - w^n - (\kappa V/\alpha Y) \right] = -\left( \frac{Y}{\eta Y} \right) \delta_1 \) and \( \varphi_2 \equiv (\kappa Y/\alpha \eta Y) \) \( (\alpha Y + \rho_u) = (\kappa Y/\eta Y) \delta_2 \).

Eliminating \( \hat{c}_t \) from (10) yields
\[
\hat{u}_{t+1} = \gamma E_t \hat{u}_{t+2} + (1 - \gamma) \hat{u}_t - \left[ \frac{1}{\sigma (\varphi_1 + \varphi_2)} \right] (i_t - E_t \pi_{t+1})
\]
\[
+ \left( \frac{1}{\varphi_1 + \varphi_2} \right) \left( \frac{Y}{C} \right) (E_t \hat{z}_{t+1} - \hat{z}_t),
\] (21)

where \( \gamma = \varphi_1/(\varphi_1 + \varphi_2) \). It is straightforward to show that when the efficiency condition (2) holds, \( \gamma = \beta/(1 + \beta) \) so that (21) becomes
\[
\hat{u}_{t+1} = \left( \frac{\beta}{1 + \beta} \right) E_t \hat{u}_{t+2} + \left( \frac{1}{1 + \beta} \right) \hat{u}_t - \left( \frac{1}{1 + \beta} \right) \left( \frac{\eta \bar{C}}{\delta_2 N} \right) \left( \frac{1}{\sigma} \right) (i_t - E_t \pi_{t+1})
\]
\[
+ \left( \frac{1}{1 + \beta} \right) \left( \frac{\eta}{\delta_2} \right) (E_t \hat{z}_{t+1} - \hat{z}_t)
\] (22)

Using (14) to eliminate real marginal cost from the inflation equation (15),
\[
\pi_t = \beta E_t \pi_{t+1} + \delta \left[ a_1 \hat{\theta}_t - \beta a_2 E_t \hat{\theta}_{t+1} + \beta a_3 (i_t - E_t \pi_{t+1}) + \beta b_t - \hat{z}_t \right].
\] (23)

Equations (17), (22), and (23) gives three equilibrium conditions in terms of unemployment,
labor market tightness, and inflation that could be combined with a specification of monetary policy.

A version of the model that is even more similar to a standard new Keynesian specification can be obtained by using (17) to eliminate $\dot{\theta}_t$ from (23) to yield two structural equations, an unemployment based IS relationship and an unemployment based Phillips curve. These two relationships are

$$\dot{u}_{t+1} = \gamma E_t \hat{u}_{t+2} + (1 - \gamma) \hat{u}_t - \left[ \frac{1}{\sigma(\varphi_1 + \varphi_2)} \right] \hat{r}_t - \left( \frac{1}{\varphi_1 + \varphi_2} \right) \left( \frac{\bar{Y}}{\bar{C}} \right) (1 - \rho_z) \dot{z}_t \tag{24}$$

and

$$\pi_t = \beta E_t \pi_{t+1} + \left( \frac{\delta \mu}{\alpha \rho \eta} \right) [a_2 \beta (E_t \hat{u}_{t+2} - \rho_u \hat{u}_{t+1}) - a_1 (\hat{u}_{t+1} - \rho_u \hat{u}_t)] + \beta \delta \mu a_3 \hat{r}_t + \delta \beta B \hat{b}_t - \delta \hat{z}_t \tag{25}$$

where

$$\gamma \equiv \varphi_1 / (\varphi_1 + \varphi_2)$$

$$\rho_u \equiv (1 - \rho)(1 - \rho \bar{N} / \bar{u})$$

$$\eta \equiv (1 - \rho) (\bar{N} / \bar{u})$$

$$\varphi_1 \equiv - (\bar{Y} / \eta \bar{C}) [1 - w^u - (\kappa \bar{V} / \alpha \rho \bar{Y})]$$

$$\varphi_2 \equiv (\kappa \bar{Y} / \alpha \rho \eta \bar{C}) (\alpha \rho \eta + \rho_u)$$

$$a_1 = [(1 - \alpha)/(1 - b)] (\kappa \bar{V} / \rho \bar{N})$$

$$a_2 = a_1 [(1 - \rho)/(1 - \alpha)] (1 - \alpha - \rho \bar{N} / \bar{u})$$

$$a_3 = a_1 [(1 - \rho)/(1 - \alpha)] (1 - \rho b \bar{N} / \bar{u})$$

$$B = [b/(1 - b)] [1 - \mu w^u + \mu \beta (1 - \rho) (\kappa \bar{V} / \bar{u})].$$

A primary difference from a standard Phillips curve is that inflation is affected by current and expected future values unemployment, as well as by the real rate of interest.\(^3\)

### 3 Natural levels (flexible prices)

From the marginal cost expression when prices are flexible,

\(^3\)To obtain a single equation representing the contraints on monetary policy (as long as interest rates do not appear in the central bank's objective function), the IS relationship can be used to eliminate the real interest rate from the Phillips curve. In this case, inflation will depend on current and expected future changes in the unemployment rate as well as its level.
\[
(1 - b_t) \frac{Z_t}{\mu} = (1 - b_t) w^a + \frac{\kappa}{\varphi} \theta_t^{1-\alpha} - (1 - \rho) \left( \frac{1}{R_t} \right) E_t \left( \frac{\kappa}{\varphi} \theta_{t+1}^{1-\alpha} - \kappa b_t \theta_{t+1} \right).
\]

(26)

Comparing this to (1) shows that the equilibrium will be efficient if \( b = 1 - \alpha \) (the Hosios condition), \( \hat{b}_t = 0, \) and \( \mu = 1. \) Linearizing (26) around the efficient steady-state, and letting a superscript \( e \) denote the efficient equilibrium, one obtains

\[
\dot{z}_t = (1 - \alpha) \frac{\kappa V}{\alpha \rho N} \left( \theta_t^e - \beta \rho_u E_t \theta_t^e \right) + \beta \delta e^e_t.
\]

(27)

Since \( \delta_1 = -\beta \delta_2, \) so we also can rewrite (27) as

\[
\dot{z}_t = (1 - \alpha) \left( \frac{\kappa V}{\alpha \rho N} \right) \left( \theta_t^e - \beta \rho_u E_t \theta_t^e \right) - \delta_1 e^e_t.
\]

(28)

The IS relationship (22) holds whether prices are sticky or flexible, so

\[
\begin{align*}
\dot{u}_{t+1}^e &= \left( \frac{\beta}{1 + \beta} \right) E_t \dot{u}_{t+2}^e + \left( \frac{1}{1 + \beta} \right) \dot{u}_{t}^e - \left( \frac{1}{1 + \beta} \right) \left( \frac{\eta}{\delta_2} \right) \left( \frac{C}{N} \right) \left( \frac{1}{\sigma} \right) \rho_t^e \\
&\quad + \left( \frac{1}{1 + \beta} \right) \left( \frac{\eta}{\delta_2} \right) (E_t z_{t+1} - \dot{z}_t).
\end{align*}
\]

It will be convenient to rewrite this in terms of employment as

\[
\dot{z}_t = \delta_2 \left[ (1 + \beta) \dot{n}_t^e - \beta E_t \dot{n}_{t+1}^e - \dot{n}_{t-1}^e \right] - \left( \frac{C}{Y} \right) \left( \frac{1}{\sigma} \right) \rho_t^e + E_t \dot{z}_{t+1}.
\]

(29)

4 Welfare

To derive an approximation to the representative agent’s utility, it is necessary to first introduce some additional notation. For any variable \( X_t, \) let \( \hat{X} \) be its steady-state value and let \( \dot{X}_t = \log \left( X_t / \hat{X} \right) \) be the log deviation of \( X_t \) around its steady-state value. Using a second order Taylor approximation,

\[
X_t - \hat{X} = \hat{X} \left( \frac{X_t}{\hat{X}} - 1 \right) \approx \hat{X} \left( X_t + \frac{1}{2} \dot{X}_t^2 \right).
\]

(30)

Employing this notation, we develop a second order approximation to the utility of the representative household. In doing so, we found the work of Edge (2003) very helpful.

4.1 Household utility

The second order approximation to household utility, which is a function of total consumption, is
\[
U(C_t) \approx U(\bar{C}) + U_c\bar{C}m \left( \hat{c}_t^m + \frac{1}{2} (\hat{c}_t^m)^2 \right) - w^u U_c \bar{N} \left( \hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) \\
+ \frac{1}{2} U_{cc} \left( \bar{C}m \right)^2 \left[ \hat{c}_t^m - \left( \frac{w^u \bar{N}}{\bar{C}m} \right) \hat{n}_t \right]^2.
\] (31)

If we define
\[
\hat{\sigma} = -\frac{U_{cc} \bar{C}m}{U_c} \left( \frac{\bar{C}m}{\bar{C}} \right) = \sigma \left( \frac{\bar{C}m}{\bar{C}} \right)
\]
Then (31) becomes
\[
U(C_t) \approx U(\bar{C}) + U_c\bar{C}m \left( \hat{c}_t^m + \frac{1}{2} (\hat{c}_t^m)^2 \right) - U_c w^u \bar{N} \left( \hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) \\
- \frac{1}{2} \hat{\sigma} U_c \bar{C}m \left[ \hat{c}_t^m - \left( \frac{w^u \bar{N}}{\bar{C}m} \right) \hat{n}_t \right]^2
\] (32)

4.1.1 Second order expansion for market consumption

Market clearing requires
\[
C_t^m D_t = Z_t N_t - \kappa V_t,
\]
where
\[
D_t = \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} \, di
\]
measures the dispersion of relative prices. So
\[
\frac{C_t^m D_t}{\bar{C}m} = \frac{Z_t N_t - \kappa V_t}{\bar{C}m} \\
\approx 1 + \frac{\bar{N}}{\bar{C}m} \left[ (\hat{z}_t + \hat{n}_t) + \frac{1}{2} (\hat{z}_t + \hat{n}_t)^2 \right] - \frac{\kappa \bar{V}}{\bar{C}m} \left( \hat{v}_t + \frac{1}{2} \hat{v}_t^2 \right).
\] (33)

4.1.2 Second order expansion for vacancies

Since \( V_t = \theta_t U_t \),
\[
\hat{v}_t = \hat{\theta}_t + \hat{u}_t
\]
is exact, but to express this in terms of \( \hat{n}_t \), we need to use the correct second order approximation for unemployment in terms of employment. Since \( U_t = 1 - (1 - \rho) N_{t-1} \),
\[
\hat{u}_t + \frac{1}{2} \hat{u}_t^2 = -(1 - \rho) \left( \frac{\bar{N}}{\bar{U}} \right) \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right) = -\eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right).
\] (34)
This then implies \( \hat{u}_t^2 = \eta^2 \hat{n}_{t-1}^2 \), so (34) becomes
\[
\hat{u}_t = -\eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right) - \frac{1}{2} \hat{u}_{t-1}^2 = -\eta \hat{n}_{t-1} - \frac{1}{2} \eta (1 + \eta) \hat{n}_{t-1}^2,
\]
and
\[
\hat{v}_t = \hat{\theta}_t + \hat{u}_t = \hat{\theta}_t - \eta \hat{n}_{t-1} - \frac{1}{2} \eta (1 + \eta) \hat{n}_{t-1}^2.
\]
while \( \hat{v}_t^2 = \left( \hat{\theta}_t - \eta \hat{n}_{t-1} \right)^2 \). Hence,
\[
\hat{v}_t + \frac{1}{2} \hat{v}_t^2 = \hat{\theta}_t - \eta \hat{n}_{t-1} - \eta \left( \hat{\theta}_t - \eta \hat{n}_{t-1} \right) + \frac{1}{2} \eta (1 + \eta) \hat{n}_{t-1}^2.
\]
(35)

4.1.3 Second order approximation for labor market tightness

Now we need a second order approximation for labor market tightness, \( \hat{\theta}_t \). From \( N_t = (1 - \rho)N_{t-1} + \varphi \theta_t^a U_t \), we obtain
\[
\hat{N} \left( 1 + \hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) = (1 - \rho) \hat{N} \left( 1 + \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right)
\]
\[
+ \varphi \hat{\theta}_t^a U_t \left( 1 + \alpha \hat{\theta}_t + \frac{1}{2} \alpha^2 \hat{\theta}_t^2 \right) \left( 1 - \eta \hat{n}_{t-1} - \frac{1}{2} \eta \hat{n}_{t-1}^2 \right).
\]
This implies
\[
\left( \hat{n}_t + \frac{1}{2} \hat{n}_t^2 \right) = (1 - \rho) \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right) + \rho \left( \alpha \hat{\theta}_t - \eta \hat{n}_{t-1} + \frac{1}{2} \alpha^2 \hat{\theta}_t^2 - \alpha \eta \hat{\theta}_t \hat{n}_{t-1} - \frac{1}{2} \eta \hat{n}_{t-1}^2 \right)
\]
which, since \( (1 - \rho) - \rho \eta = (1 - \rho) (1 - \rho \hat{N}/\hat{U}) = \rho_u \), can be written as
\[
\alpha \rho \left( \hat{\theta}_t + \frac{1}{2} \alpha \hat{\theta}_t^2 \right) = (\hat{n}_t - \rho_u \hat{n}_{t-1}) + \frac{1}{2} \left( \hat{n}_t^2 - \rho_u \hat{n}_{t-1}^2 \right) + \alpha \rho \eta \hat{\theta}_t \hat{n}_{t-1}.
\]
(36)
But (36) implies that
\[
\hat{\theta}_t \hat{n}_{t-1} = \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t - \rho_u \hat{n}_{t-1}) \hat{n}_{t-1},
\]
(37)
and
\[
\hat{\theta}_t^2 = \left( \frac{1}{\alpha \rho} \right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})^2.
\]
(38)
Now, using (36)-(38) in (35),

\[
\hat{v}_t + \frac{1}{2} \hat{v}_t^2 = \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t - \rho \hat{n}_{t-1}) + \frac{1}{2} \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t^2 - \rho \hat{n}_{t-1}^2) \\
- \eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right) + \frac{1}{2} (1 - \alpha) \left( \frac{1}{\alpha \rho} \right)^2 (\hat{n}_t - \rho \hat{n}_{t-1})^2
\]

(39)

4.1.4 The approximation for market consumption

We can now use the expressing for vacancies given by (39) in (33) to obtain

\[
\frac{C^m m D_t}{C^m} \approx 1 + \frac{\bar{N}}{C^m} \left[ (\hat{z}_t + \hat{n}_t) + \frac{1}{2} (\hat{z}_t + \hat{n}_t)^2 \right] \\
- \frac{\kappa V}{C^m} \left[ \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t - \rho \hat{n}_{t-1}) + \frac{1}{2} \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t^2 - \rho \hat{n}_{t-1}^2) \right] \\
- \frac{\kappa V}{C^m} \left[ \frac{1}{2} (1 - \alpha) \left( \frac{1}{\alpha \rho} \right)^2 (\hat{n}_t - \rho \hat{n}_{t-1})^2 \right] - \eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right)^2
\]

Using the approximation \( \ln (1 + x) \approx x - \frac{1}{2} x^2 \),

\[
\ln \left( \frac{C^m m D_t}{C^m} \right) = \hat{c}_t^m + \hat{d}_t = \left( \frac{\bar{N}}{C^m} \right) \left[ (\hat{z}_t + \hat{n}_t) - \left( \frac{\kappa V}{\alpha \rho N} \right) (\hat{n}_t - \rho \hat{n}_{t-1}) \right] \\
+ \frac{1}{2} \left( \frac{\bar{N}}{C^m} \right) (\hat{z}_t + \hat{n}_t)^2 - \frac{1}{2} \kappa V \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t^2 - \rho \hat{n}_{t-1}^2) \\
- \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{1}{\alpha \rho} \right)^2 (\hat{n}_t - \rho \hat{n}_{t-1})^2 - \eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right)^2
\]

Hence

\[
\hat{c}_t^m = - \hat{d}_t + \left( \frac{\bar{N}}{C^m} \right) \left[ (\hat{z}_t + \hat{n}_t) - \left( \frac{\kappa V}{\alpha \rho N} \right) (\hat{n}_t - \rho \hat{n}_{t-1}) \right] \\
+ \frac{1}{2} \left( \frac{\bar{N}}{C^m} \right) (\hat{z}_t + \hat{n}_t)^2 - \frac{1}{2} \kappa V \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t^2 - \rho \hat{n}_{t-1}^2) \\
- \frac{1}{2} \left( 1 - \alpha \right) \left( \frac{1}{\alpha \rho} \right)^2 (\hat{n}_t - \rho \hat{n}_{t-1})^2 - \eta \left( \hat{n}_{t-1} + \frac{1}{2} \hat{n}_{t-1}^2 \right)^2
\]

(40)
and

$$(\hat{c}_m^m)^2 = \left(\frac{\bar{N}}{C^m}\right)^2 \left[(\hat{z}_t + \hat{n}_t) - \left(\frac{kV}{\alpha \rho N}\right) (\hat{n}_t - \rho u \hat{n}_{t-1})\right]^2.$$  \hfill (41)

since $\hat{d}_t$ is already second order,

### 4.2 Evaluating utility

Repeating (32),

$$U(C_t) = U(\bar{C}) + U_c\bar{C}^{cm} \left(\hat{c}_t^m + \frac{1}{2} (\hat{c}_t^m)^2\right) - U_c w^n N \left(\hat{n}_t + \frac{1}{2} \hat{n}_t^2\right)$$

$$- \frac{1}{2} \hat{\sigma}_U \bar{C}^{cm} \left[\hat{c}_t^m - \left(\frac{w^n N}{\bar{C}^{cm}}\right) \hat{n}_t\right]^2.$$  

Recalling that

$$\delta_1 = 1 - w^u - \frac{kV}{\alpha \rho N}$$

and noting that

$$\frac{kV}{\alpha \rho N}(\rho_u + \alpha \rho \eta) = (1 - \rho) \frac{kV}{\alpha U} \left(\alpha - 1 + \frac{\bar{U}}{\rho N}\right) = \delta_2,$$

we can express utility, after some simplification, as

$$U(C_t) = U(\bar{C}) - U_c\bar{C}^{cm} \hat{d}_t + U_c\bar{C}^{cm} \left(\frac{\bar{N}}{C^m}\right) (\hat{z}_t + \delta_1 \hat{n}_t + \delta_2 \hat{n}_{t-1})$$

$$+ U_c\bar{C}^{cm} \left(\frac{\bar{N}}{C^m}\right) \left(\frac{1}{2} \hat{z}_t^2 + \hat{z}_t \hat{n}_t + \frac{1}{2} \delta_1 \hat{n}_t^2 + \frac{1}{2} \delta_2 \hat{n}_{t-1}^2\right)$$

$$- \frac{1}{2} U_c\bar{C}^{cm} (1 - \alpha) \frac{kV}{C^m} \left[\left(\frac{1}{\alpha \rho}\right)^2 (\hat{n}_t - \rho_u \hat{n}_{t-1})\right]^2$$

$$- \frac{1}{2} \hat{\sigma}_U \bar{C}^{cm} \left(\frac{\bar{N}}{C^m}\right)^2 (\hat{z}_t + \delta_1 \hat{n}_t + \delta_2 \hat{n}_{t-1})^2.$$  \hfill (42)
4.3 The present discounted value of utility

From (42),

\[ \sum_{i=0}^{\infty} \beta^i U(C_{t+i}) = \frac{U(C)}{1-\beta} - U_c \mathcal{C} \sum_{i=0}^{\infty} \beta^i \hat{d}_{t+i} + U_c N \sum_{i=0}^{\infty} \beta^i (\hat{z}_{t+i} + \delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1}) \]

\[ + \frac{1}{2} U_c N \sum_{i=0}^{\infty} \beta^i (\hat{z}_{t+i}^2 + \delta_1 \hat{n}_{t+i}^2 + \delta_2 \hat{n}_{t+i-1}^2) + U_c \mathcal{N} \sum_{i=0}^{\infty} \beta^i \hat{z}_{t+i} \hat{n}_{t+i} \]

\[ - \frac{1}{2} U_c (1-\alpha) \kappa \bar{V} \left( \frac{1}{\alpha \rho} \right)^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i} - \rho \hat{n}_{t+i-1})^2 \]

\[ - \frac{1}{2} \hat{\sigma} U_c \mathcal{C} \left( \frac{\hat{N}}{\mathcal{C} \mathcal{m}} \right)^2 \sum_{i=0}^{\infty} \beta^i (\hat{z}_{t+i} + \delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1})^2. \]

4.3.1 First order terms

In section 1, it is shown that in the efficient equilibrium, \( \delta_1 = -\beta \delta_2 \). Therefore, the first order terms become

\[ -U_c \hat{N} \delta_2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i} - \hat{n}_{t+i-1}) + \text{t.i.p.} \]

\[ = -U_c \hat{N} \delta_2 \{ \hat{n}_t - \hat{n}_{t-1} + \beta^2 \hat{n}_{t+1} - \beta \hat{n}_t + \ldots \} \]

\[ = U_c \hat{N} \delta_2 \hat{n}_{t-1} \]

which is independent of policy.

4.3.2 Second order terms

The second order terms are

\[ X_t \equiv -U_c \mathcal{C} \sum_{i=0}^{\infty} \beta^i \hat{d}_{t+i} + \frac{1}{2} U_c N \sum_{i=0}^{\infty} \beta^i (\delta_1 \hat{n}_{t+i}^2 + \delta_2 \hat{n}_{t+i-1}^2) \]

\[ + U_c N \sum_{i=0}^{\infty} \beta^i \hat{z}_{t+i} \hat{n}_{t+i} - \frac{1}{2} U_c (1-\alpha) \kappa \bar{V} \left( \frac{1}{\alpha \rho} \right)^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i} - \rho \hat{n}_{t+i-1})^2 \]

\[ - \frac{1}{2} \hat{\sigma} U_c \mathcal{C} \left( \frac{\hat{N}}{\mathcal{C} \mathcal{m}} \right)^2 \sum_{i=0}^{\infty} \beta^i (\hat{z}_{t+i} + \delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1})^2, \]

plus a term in \( \hat{z}_t^2 \) that is independent of policy.
**Price dispersion term** Up to second order,

\[
\sum_{i=0}^{\infty} \beta^i d_{t+i} = -\frac{\varepsilon}{2\delta} \sum_{i=0}^{\infty} \beta^i \pi_{t+i}^2
\]

where \(\delta\) is the standard new Keynesian coefficient giving the elasticity of inflation with respect to marginal cost.

**Term in employment squared** The next term is

\[
\frac{1}{2} U_c \tilde{N} \sum_{i=0}^{\infty} \beta^i (\delta_1 \hat{n}_{t+i}^2 + \delta_2 \hat{n}_{t+i-1}^2) = -\frac{1}{2} \delta_2 U_c \tilde{N} \sum_{i=0}^{\infty} \beta^i (\beta \hat{n}_{t+i}^2 - \hat{n}_{t+i-1}^2)
\]

which is independent of policy.

**Cross products with productivity** This leaves us to deal with

\[
X_t \equiv U_c \tilde{N} \sum_{i=0}^{\infty} \beta^i \tilde{z}_{t+i} \hat{n}_{t+i} - \frac{1}{2} U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa \tilde{V}}{N} \right) \left( \frac{1}{\alpha \rho} \right) \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i} - \rho_n \hat{n}_{t+i-1})^2
\]

\[
-\frac{1}{2} \delta \sigma U_c \bar{c} \beta \left( \frac{\tilde{N}}{\bar{C}_m} \right)^2 \sum_{i=0}^{\infty} \beta^i (\tilde{z}_{t+i} - \beta \delta_2 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1})^2,
\]

or

\[
X_t \equiv U_c \tilde{N} \sum_{i=0}^{\infty} \beta^i \tilde{z}_{t+i} \hat{n}_{t+i} - \frac{1}{2} U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa \tilde{V}}{N} \right) \left( \frac{1}{\alpha \rho} \right) \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i} - \rho_n \hat{n}_{t+i-1})^2
\]

\[
-\delta_2 \delta U_c \bar{c} \beta \left( \frac{\tilde{N}}{C_m} \right)^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i}) \tilde{z}_{t+i}
\]

\[
-\frac{1}{2} \delta \delta_2 \delta U_c \bar{c} \beta \left( \frac{\tilde{N}}{C_m} \right)^2 \sum_{i=0}^{\infty} \beta^i (\beta \hat{n}_{t+i} - \hat{n}_{t+i-1})^2.
\]

(43)

Start with the third term involving the cross product expression, \((\hat{n}_{t-1} - \beta \hat{n}_t) \tilde{z}_t\). Using (29),

\[
(\hat{n}_{t-1} - \beta \hat{n}_t) \tilde{z}_t = \hat{n}_{t-1} \tilde{z}_t - \beta \hat{n}_t \tilde{z}_t = \hat{n}_{t-1} \tilde{z}_t
\]

\[
-\beta \hat{n}_t \left\{ \delta_2 \left[ (1 + \beta) \hat{n}_t^* - \beta E_t \bar{n}_{t+1}^* - \hat{n}_{t+1}^* \right] - \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) \delta_1 \right\}
\]

\[
-\beta \hat{n}_t E_t \tilde{z}_{t+1}
\]
so

\[
\sum_{i=0}^{\infty} \beta^i \left( \hat{n}_{t+i-1} - \beta \hat{n}_{t+i} \right) \hat{z}_{t+i} = \hat{n}_{t-1} \hat{z}_t - \beta \hat{n}_t \hat{E}_t \hat{z}_{t+1} \\
- \beta \hat{n}_t \left\lbrace \delta_2 \left[ (1 + \beta) \hat{n}_{t+1}^c - \beta \hat{E}_t \hat{n}_{t+1}^c + \hat{n}_{t+1}^{e*} \right] - \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) \hat{r}_t^c \right\rbrace \\
+ \beta \hat{n}_t \hat{E}_t \hat{z}_{t+1} - \beta^2 \hat{E}_t \hat{n}_{t+1} \hat{z}_{t+2} \\
- \beta^2 \hat{E}_t \hat{n}_{t+1} \left\lbrace \delta_2 \left[ (1 + \beta) \hat{n}_{t+2}^c - \beta \hat{E}_t \hat{n}_{t+2}^c + \hat{n}_{t+2}^{e*} \right] - \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) \hat{r}_{t+1}^c \right\rbrace \\
+ \ldots
\]

= \hat{n}_{t-1} \hat{z}_t - \sum_{i=0}^{\infty} \beta^i \left\lbrace \delta_2 \left[ (1 + \beta) \hat{n}_{t+i}^c - \beta \hat{E}_t \hat{n}_{t+i}^c + \hat{n}_{t+i}^{e*} \right] - \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) \hat{r}_{t+i}^c \right\rbrace \hat{n}_{t+i}

But \( \hat{n}_{t-1} \hat{z}_t \) is independent of policy at time \( t \).

So the terms in \( X_t \) involving cross product terms with the shocks (the first and third terms in 43) become, in expected present value terms,\(^4\)

\[
U_c \bar{N} \left[ \sum_{i=0}^{\infty} \beta^i \hat{n}_{t+i} \hat{z}_{t+i} - \hat{\delta} \left( \frac{\bar{N}}{C_m} \right) \delta_2 \sum_{i=0}^{\infty} \beta^i \left( \hat{n}_{t+i-1} - \beta \hat{n}_{t+i} \right) \hat{z}_{t+i} \right]
= U_c \bar{N} \sum_{i=0}^{\infty} \beta^i \left\lbrace \hat{n}_{t+i} \hat{z}_{t+i} + \hat{\delta} \left( \frac{\bar{N}}{C_m} \right) \delta_2 \left[ (1 + \beta) \hat{n}_{t+i}^c - \beta \hat{E}_t \hat{n}_{t+i}^c + \hat{n}_{t+i}^{e*} \right] \hat{n}_{t+i} - \beta \delta_2 \hat{r}_{t+i} \hat{n}_{t+i} \right\rbrace.
\]

>From (28),

\[
\hat{n}_t \hat{z}_t = \left[ (1 - \alpha) \left( \frac{\kappa \bar{V}}{\alpha \rho \bar{N}} \right) \left( \hat{\theta}_t^e - \beta \rho_a \hat{E}_t \hat{\theta}_{t+1}^e \right) - \delta_1 \hat{r}_t^e \right] \hat{n}_t.
\]

Using this to replace \( \hat{n}_t \hat{z}_t \) in the expression for the present discounted value expression yields

\[
U_c \bar{N} \sum_{i=0}^{\infty} \beta^i \left[ (1 - \alpha) \left( \frac{\kappa \bar{V}}{\alpha \rho \bar{N}} \right) \left( \hat{\theta}_{t+i}^e - \beta \rho_a \hat{E}_t \hat{\theta}_{t+i+1}^e \right) - \delta_1 \hat{r}_{t+i}^e \right] \hat{n}_{t+i}
+ U_c \bar{N} \sum_{i=0}^{\infty} \beta^i \left\lbrace \beta \hat{\delta} \left( \frac{\bar{N}}{C_m} \right) \delta_2 \left[ (1 + \beta) \hat{n}_{t+i}^c - \beta \hat{E}_t \hat{n}_{t+i}^c + \hat{n}_{t+i}^{e*} \right] \hat{n}_{t+i} - \beta \delta_2 \hat{r}_{t+i} \hat{n}_{t+i} \right\rbrace.
\]

\(^4\)The coefficient on \( \hat{r}_t^e \hat{n}_t \) is \( \beta \hat{\delta} \left( \frac{\bar{N}}{C_m} \right) \delta_2 \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) \), but

\[
\beta \hat{\delta} \left( \frac{\bar{N}}{C_m} \right) \delta_2 \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right) = \beta \hat{\delta} \left( \frac{C_m}{C} \right) \left( \frac{\bar{N}}{C_m} \right) \delta_2 \left( \frac{\bar{C}}{Y} \right) \left( \frac{1}{\sigma} \right)
= \beta \delta_2.
\]
The real interest rate terms cancel out since \( \delta_1 - \beta \delta_2 = 0 \), so we are left with

\[
U_c \tilde{N} \sum_{i=0}^{\infty} \beta^i \left[ (1 - \alpha) \left( \frac{\kappa \tilde{V}}{\alpha \rho N} \right) \left( \hat{\theta}_t^e - \beta \rho_u \hat{\theta}_t^e \right) \right] \hat{n}_{t+i} \\
+ U_c \tilde{N} \sigma \left( \frac{\tilde{N}}{C_m} \right) \delta_2 \sum_{i=0}^{\infty} \beta^{i+1} \left[ (1 + \beta) \hat{n}_{t+i}^e - \beta E_i \hat{n}_{t+i+1}^e - \hat{n}_{t+i-1}^e \right] \hat{n}_{t+i}
\]

for the cross product terms involving \( \hat{n} \) and \( \hat{z} \). But

\[
\left( \hat{\theta}_t^e - \beta \rho_u \hat{\theta}_t^e \right) \hat{n}_t = \hat{\theta}_t^e \left( \rho_u \hat{n}_{t-1} + \alpha \rho \hat{\theta}_t \right) - \beta E_i \hat{\theta}_t^e \rho_u \hat{n}_t,
\]

so when we take present discounted values, the \( \hat{\theta}_t^e \rho_u \hat{n}_{t+i-1} \) and \( \beta E_i \hat{\theta}_t^e \rho_u \hat{n}_{t+i} \) terms will cancel (except for the first, \( \rho_u \hat{\theta}_t \hat{n}_{t-1} \), but that is independent of policy. Thus,

\[
\sum_{i=0}^{\infty} \beta^i \left( \hat{\theta}_t^e - \beta \rho_u \hat{\theta}_t^e \right) \hat{n}_{t+i} = \alpha \rho \sum_{i=0}^{\infty} \beta^i \hat{\theta}_t^e \hat{n}_{t+i} + \rho_u \hat{n}_{t-1} \hat{\theta}_t.
\]

Hence, for the PDV of the cross product terms we have

\[
\sum_{i=0}^{\infty} \beta^i B_{t+i} = U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa \tilde{V}}{N} \right) \sum_{i=0}^{\infty} \beta^i \hat{\theta}_t^e \hat{n}_{t+i} \\
+ U_c \tilde{N} \sigma \left( \frac{\tilde{N}}{C_m} \right) \delta_2 \sum_{i=0}^{\infty} \beta^{i+1} \left[ (1 + \beta) \hat{n}_{t+i}^e - \beta E_i \hat{n}_{t+i+1}^e - \hat{n}_{t+i-1}^e \right] \hat{n}_{t+i}
\]

Cross products with other variables

Define

\[
\sum_{i=0}^{\infty} \beta^i G_{t+i} = -\frac{1}{2} U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa \tilde{V}}{N} \right) \left( \frac{1}{\alpha \rho} \right) \sum_{i=0}^{\infty} \beta^i \left( \hat{n}_{t+i} - \rho_u \hat{n}_{t+i-1} \right)^2 \\
- \frac{1}{2} \tilde{U} \tilde{c} \left( \frac{\tilde{N}}{C_m} \right) \sum_{i=0}^{\infty} \beta^i \left( \delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1} \right)^2
\]

as the present discounted value of the cross product terms among the variables in \( X_t \) (i.e., terms 2 and 4 of 43). From (38), this can be written as

\[
\sum_{i=0}^{\infty} \beta^i G_{t+i} = -\frac{1}{2} U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa \tilde{V}}{N} \right) \sum_{i=0}^{\infty} \beta^i \hat{\theta}_t^2 \\
- \frac{1}{2} \tilde{U} \tilde{c} \left( \frac{\tilde{N}}{C_m} \right) \sum_{i=0}^{\infty} \beta^i \left( \delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1} \right)^2
\]

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For future reference, notice that

\[ \sum_{i=0}^{\infty} \beta^i (\delta_1 \hat{n}_{t+i} + \delta_2 \hat{n}_{t+i-1})^2 = \delta_2^2 \sum_{i=0}^{\infty} \beta^i (-\beta \hat{n}_{t+i} + \hat{n}_{t+i-1})^2 = \delta_2^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2. \]

### 4.3.3 Collecting results on all second order product terms

Adding together the PDVs of \( B_t \) and \( G_t \),

\[
X_t = (1 - \alpha) U_c \tilde{N} \left( \frac{\kappa V}{N} \right) \sum_{i=0}^{\infty} \beta^i \hat{\theta}_{t+i} \hat{\theta}_{t+i} \\
+ \sigma U_c \tilde{N} \left( \frac{\tilde{N}}{\tilde{C}^{mn}} \right) \delta_2^2 \sum_{i=0}^{\infty} \beta^{i+1} [(1 + \beta) \hat{n}^e_{t+i} - \beta E_t \hat{n}^e_{t+i+1} - \hat{n}^e_{t+i-1}] \hat{n}_{t+i} \\
- \frac{1}{2} \sigma U_c \tilde{N} (1 - \alpha) \left( \frac{\kappa V}{N} \right) \sum_{i=0}^{\infty} \beta^i \hat{\theta}_{t+i}^2 \\
- \frac{1}{2} \sigma U_c \tilde{N} \left( \frac{\tilde{N}}{\tilde{C}^{mn}} \right) \delta_2^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2 + t.i.p 
\]

or

\[
X_t = -\frac{1}{2} (1 - \alpha) U_c \left( \frac{\kappa V}{N} \right) \sum_{i=0}^{\infty} \beta^i \left( \hat{\theta}_{t+i} - \hat{\theta}_{t+i}^e \right)^2 \\
+ \sigma U_c \tilde{N} \left( \frac{\tilde{N}}{\tilde{C}^{mn}} \right) \delta_2^2 \sum_{i=0}^{\infty} \beta^{i+1} [(1 + \beta) \hat{n}^e_{t+i} - \beta E_t \hat{n}^e_{t+i+1} - \hat{n}^e_{t+i-1}] \hat{n}_{t+i} \\
- \frac{1}{2} \sigma U_c \tilde{N} \left( \frac{\tilde{N}}{\tilde{C}^{mn}} \right) \delta_2^2 \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2 + t.i.p 
\]

### Terms involving the CRRA

Start by focusing on all the terms that are multiplied by \( \sigma \).

These are \( \sigma U_c \tilde{N} \left( \frac{\tilde{N}}{\tilde{C}^{mn}} \right) \delta_2^2 \) times

\[
J_t = \sum_{i=0}^{\infty} \beta^{i+1} [(1 + \beta) \hat{n}^e_{t+i} - \beta E_t \hat{n}^e_{t+i+1} - \hat{n}^e_{t+i-1}] \hat{n}_{t+i} - \frac{1}{2} \beta (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2. 
\]

First, note that

\[
[(1 + \beta) \hat{n}^e_t - \beta E_t \hat{n}^e_{t+1} - \hat{n}^e_{t-1}] \hat{n}_t = [(\beta \hat{n}^e_t - \hat{n}^e_{t-1}) - (\beta E_t \hat{n}^e_{t+1} - \hat{n}^e_t)] \hat{n}_t.
\]
It follows that

\[
\sum_{i=0}^{\infty} \beta^{i+1} [(1 + \beta) \hat{n}_{t+i} - \beta E_t \hat{n}_{t+i+1} - \hat{n}_{t+i}] = \sum_{i=0}^{\infty} \beta^{i+1} \left[ (\beta \hat{n}_{t+i} - \hat{n}_{t+i-1}) - (\beta E_t \hat{n}_{t+i+1} - \hat{n}_{t+i}) \right] n_{t+i} = \\
\beta \left[ (\beta \hat{n}_{t} - \hat{n}_{t-1}) - (\beta E_t \hat{n}_{t+1} - \hat{n}_{t}) \right] n_t + \beta^2 \left[ (\beta \hat{n}_{t+1} - \hat{n}_{t}) - (\beta E_t \hat{n}_{t+2} - \hat{n}_{t+1}) \right] n_{t+1} + ... \\
= (\beta \hat{n}_{t} - \hat{n}_{t-1}) \hat{n}_{t-1} - (\beta \hat{n}_{t} - \hat{n}_{t-1}) (\hat{n}_{t-1} - \beta n_t) - \beta (\beta E_t \hat{n}_{t+1} - \hat{n}_{t}) (\hat{n}_{t} - \beta n_{t+1}) + ... + t.i.p. \\
= (\beta \hat{n}_{t} - \hat{n}_{t-1}) \hat{n}_{t-1} - \sum_{i=0}^{\infty} \beta^i (\beta \hat{n}_{t+i} - \hat{n}_{t+i-1}) (\hat{n}_{t+i-1} - \beta n_{t+i})
\]

Notice first term \((\beta \hat{n}_{t} - \hat{n}_{t-1}) \hat{n}_{t-1}\) is independent of policy.

Using these in (49) yields

\[
J_t = - \sum_{i=0}^{\infty} \beta^i (\beta \hat{n}_{t+i} - \hat{n}_{t+i-1}) (\hat{n}_{t+i-1} - \beta n_{t+i}) - \frac{1}{2} \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2 + t.i.p. \\
= \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i}) (\hat{n}_{t+i-1} - \beta n_{t+i}) - \frac{1}{2} \sum_{i=0}^{\infty} \beta^i (\hat{n}_{t+i-1} - \beta \hat{n}_{t+i})^2 + t.i.p. \\
= - \frac{1}{2} \sum_{i=0}^{\infty} \beta^i [(\hat{n}_{t+i-1} - \beta \hat{n}_{t+i}) - (\hat{n}_{t+i-1} - \beta n_{t+i})]^2 + t.i.p.
\]

So the terms involving \(\hat{\sigma}\) are equal to

\[
- \frac{1}{2} \sigma U_e \hat{N} \left( \frac{N}{C_m} \right) \delta^2 \sum_{i=0}^{\infty} \beta^i [(\hat{n}_{t+i-1} - \hat{n}_{t+i-1}) - \beta (\hat{n}_{t+i} - \hat{n}_{t+i})]^2. \tag{50}
\]

### 4.4 Final results

Collecting results, we can now write (48) as

\[
X_t = -(1 - \alpha) \frac{1}{2} U_e \hat{N} \left( \frac{N}{N} \right) \sum_{i=0}^{\infty} \beta^i \left( \hat{\theta}_{t+i} - \hat{\theta}_{t+i} \right)^2 - \frac{1}{2} \sigma U_e \hat{N} \left( \frac{N}{C_m} \right) \delta \sum_{i=0}^{\infty} \beta^i [(\hat{n}_{t+i-1} - \hat{n}_{t+i-1}) - \beta (\hat{n}_{t+i} - \hat{n}_{t+i})]^2 + t.i.p
\]

Recalling that \(\hat{\sigma} = \sigma C_m / \hat{C}\), the correct second order approximation for welfare is therefore
given by

\[
\sum_{i=0}^{\infty} \beta^i U(C_t) = \frac{U(\bar{C})}{1 - \beta} - \frac{\varepsilon}{2\delta} U_c \sum_{i=0}^{\infty} \beta^i \pi_t^2
\]

\[
- \left(1 - \alpha\right) \frac{1}{2} U_c, \bar{N} \left( \frac{\kappa V}{N} \right) \sum_{i=0}^{\infty} \beta^i \left( \hat{\theta}_{t+i} - \hat{\theta}_{t+i}^\varepsilon \right)^2
\]

\[
- \frac{1}{2} \sigma U_c, \bar{N} \left( \frac{\bar{N}}{\tilde{C}} \right) \delta_2 \sum_{i=0}^{\infty} \beta^i \left[ (\tilde{n}_{t+i-1} - \tilde{n}_{t+i-1}^\varepsilon) - \beta (\tilde{n}_{t+i} - \tilde{n}_{t+i}^\varepsilon) \right]^2 + t.i.p.
\]

It is useful at this point to recall that (20) implies we can write, in gap terms,

\[
\tilde{c}_t = - \left( \frac{\bar{N}}{\eta C} \right) \left( \delta_1 \tilde{u}_{t+1} + \delta_2 \tilde{u}_t \right) = \left( \frac{\bar{N}}{\tilde{C}} \right) \left( \delta_1 \tilde{n}_t + \delta_2 \tilde{n}_t - 1 \right).
\]

Using the efficient condition, this becomes

\[
\tilde{c}_t = - \left( \frac{\bar{N}}{\tilde{C}} \right) \delta_2 (\beta \tilde{n}_t - \tilde{n}_t - 1).
\]

Hence, the last term in the welfare approximation is equal to

\[
- \frac{1}{2} \sigma U_c, \bar{N} \sum_{i=0}^{\infty} \beta^i \tilde{c}_t^2.
\]

Thus, we have that

\[
\sum_{i=0}^{\infty} \beta^i U(C_t) = \frac{U(\bar{C})}{1 - \beta} - \frac{\varepsilon}{2\delta} U_c \sum_{i=0}^{\infty} \beta^i \left[ \pi_t^2 + \left(1 - \alpha\right) \left( \frac{\delta}{\varepsilon} \right) \left( \frac{\kappa V}{\tilde{C}} \right) \tilde{\theta}_{t+i}^2 + \sigma \left( \frac{\delta}{\varepsilon} \right) \tilde{c}_{t+i}^2 \right] + t.i.p.
\]

which is produces (23) of the text.

Since our structural model is expressed in terms of inflation and unemployment, it will be useful to similarly express the loss function in terms of these two variables. Since \( \tilde{u}_{t+1} = -\eta \tilde{n}_t \), and \( \tilde{\theta}_t = \left( \frac{1}{\alpha \rho} \right) (\hat{n}_t - \rho_a \hat{n}_{t-1}) \). In addition,

\[
\tilde{c}_t = - \left( \frac{\bar{N}}{\tilde{C}} \right) \delta_2 (\beta \tilde{n}_t - \tilde{n}_t - 1) = \left( \frac{\delta_2 \bar{N}}{\eta \tilde{C}} \right) (\beta \tilde{n}_{t+1} - \tilde{n}_t),
\]

so we can write the loss solely in terms of unemployment relative to the efficient level \( \hat{u}^\varepsilon \). Doing so yields (26) of the text.