# Appendix to "Monetary policy and resource mobility" prepared for the 200th Anniversary of the Bank of Finland, May 5-6, 2011.

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Preliminary and incomplete: April 2011

## 1 Introduction

This appendix to "Monetary policy and resource mobility" sets out the basic model of Lechthaler and Snower (2010) and derives the second-order approximation to the welfare of the representative household that is discussed in section 3 of the paper. It also describes the derivation of the two sector model with costly hiring that is discussed in section 6 of the paper. This discussion is preliminary.

## 2 Quadratic adjustment costs

This appendix to Monetary policy and resource mobility sets out the basic model of Lechthaler and Snower (2010) and derives the second-order approximation to the welfare of the representative household. The reader is referred to Lechthaler and Snower's paper for more details. This appendix corrects some errors in Lechthaler and Snower's Kiel WP1453, Jan. 2010 version.

### 2.1 The basic equations

The preferences of the representative household are given by Welfare is given by

$$W = \mathrm{E}_t \sum_{t=0} \beta^t \left[ U(C_t) - V(L_t) \right],$$

where $C$ is consumption, $L$ hours worked, and $0 < \beta < 1$ is a discount factor. The two key equations describing household behavior are the Euler condition and the labor-leisure choice. These are given by

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$$C_t^{-\sigma} = \beta \mathrm{E}_t \left( \frac{1+i_t}{\Pi_{t+1}} \right) C_{t+1}^{-\sigma}$$
$$\frac{L_t^{\varphi}}{C_t^{-\sigma}} = \frac{W_t}{P_t}.$$

### 2.1.1 Fintermediate Firm

Intermediate goods producing firms face costs of adjusting their labor, a simple production technology such that output $Y_t$ is $A_t L_t$, where $L_t$ is employment, and output is sold in a competitive goods market in which intermediate firms take the relative price of their output as given. If $Q_t$ is the price of intermdiate goods in terms of final goods, profits for the representative intermediate firm are

$$Q_t A_t L_t - \left( \frac{W_t}{P_t} \right) L_t - \frac{\Psi}{2} \left( \frac{L_t}{L_{t-1}} - 1 \right)^2 Y_t, \tag{1}$$

where $\Psi$ is a parameter that captures the costs of adjusting firm employment. These firms pick $L_t$ to maximize the expected discounted value of profits, taking aggregate output $Y_t$ in (1) as given.

The first order condition that determines labor demand is

$$Q_t A_t = \frac{W_t}{P_t} + \Psi \left( \frac{L_t}{L_{t-1}} - 1 \right) \frac{A_t L_t}{L_{t-1}} - \beta \mathrm{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \Psi \left( \frac{L_{t+1}}{L_t} - 1 \right) \frac{L_{t+1} Y_{t+1}}{L_t^2}. \tag{2}$$

### 2.1.2 Pricing decision

Final goods producing firms face a demand for their output with constant price elasticity $\varepsilon$. They also face quadratic costs of adjusting their price as in Rotemberg (1982). Thus, The expected profits of a final goods firm are

$$\mathrm{E}_t \sum_i \Delta_{t,t+i} \left[ \left( \frac{P_{i,t}}{P_t} \right)^{1-\varepsilon} Y_t - Q_t \left( \frac{P_{i,t}}{P_t} \right)^{-\varepsilon} Y_t - \frac{\Phi}{2} \left( \frac{P_t}{P_{t-1}} - 1 \right)^2 Y_t \right],$$

where $\Delta_{t,t+i}$ is the household's time $t$ discount factor for period $t+i$ payments. Recongizing that all final goods producing firms face an identical problem so that $P_{i,t} = P_t$ in equilibrium, the first order condition for the optimal choice of $P_{i,t}$ can be shown to imply

$$Q_t = \mu^{-1} + \frac{\Phi}{\varepsilon} \left( \Pi_t - 1 \right) \Pi_t - \beta \mathrm{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{\Phi}{\varepsilon} \left( \Pi_{t+1} - 1 \right) \Pi_{t+1} \left( \frac{Y_{t+1}}{Y_t} \right) \tag{3}$$

where

$$\mu \equiv \frac{\varepsilon}{\varepsilon - 1} > 1$$

is the markup.

For future reference, suppose the demand elasticity $\varepsilon$ (and so the markup) is stochastic and the firm also faces a tax/subsidy so that it's revenues after taxes are $(1+\tau)(P_{i,t}/P_t)^{1-\varepsilon_t} Y_t$. Then

$$Q_t = \mu_t^{-1} + \left[ \frac{\Phi}{\varepsilon_t} (\Pi_t - 1) \Pi_t - \beta \mathrm{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{\Phi}{\varepsilon_t} (\Pi_{t+1} - 1) \Pi_{t+1} \left( \frac{Y_{t+1}}{Y_t} \right) \right] \tag{4}$$

where $\mu_t = \varepsilon_t/(\varepsilon_t - 1)$.

### 2.1.3 Goods market clearing

Equilibrium in the goods market requires that total output be used for consumption or be absorbed in adjustment costs:

$$Y_t = C_t + \frac{\Psi}{2} \left( \frac{L_t}{L_{t-1}} - 1 \right)^2 Y_t + \frac{\Phi}{2} (\Pi_t - 1)^2 Y_t \tag{5}$$

From the production function, it also holds that

$$Y_t = A_t L_t. \tag{6}$$

## 2.2 The welfare approximation

Welfare is given by

$$W = \mathrm{E}_t \sum_{t=0} \beta^t \left[ U(C_t) - V(L_t) \right]$$

A second order approximation to $W$ around the steady state takes the form

$$\begin{aligned} W &= \bar{W} + \bar{U}_C \bar{C} \left( \hat{c}_t + \frac{1}{2} \hat{c}_t^2 \right) + \frac{1}{2} \bar{U}_{CC} \bar{C}^2 \hat{c}_t^2 \\ & \quad - \bar{V}_L \bar{L} \left( \hat{l}_t + \frac{1}{2} \hat{l}_t^2 \right) - \frac{1}{2} \bar{V}_{LL} \bar{L}^2 \hat{l}_t^2 \end{aligned}$$

In terms of deviations from the efficient Ramsey equilibrium (denoted by $^*$),

$$\begin{aligned} \bar{U}_C \bar{C} \left( \hat{c}_t + \frac{1}{2} \hat{c}_t^2 \right) + \frac{1}{2} \bar{U}_{CC} \bar{C}^2 \hat{c}_t^2 &= \bar{U}_C \bar{C} \left[ \hat{c}_t + \frac{1}{2} (1-\sigma) \hat{c}_t^2 \right] \\ &= \bar{U}_C \bar{C} \left[ \tilde{c}_t + \frac{1}{2} (1-\sigma) \tilde{c}_t^2 + (1-\sigma) c_t^* \tilde{c}_t \right] + t.i.p \end{aligned}$$

where

$$\tilde{c}_t = \hat{c}_t - c_t^*,$$

and

$$\begin{aligned} \bar{V}_L \bar{L} \left( \hat{l}_t + \frac{1}{2} \hat{l}_t^2 \right) + \frac{1}{2} \bar{V}_{LL} \bar{L}^2 \hat{l}_t^2 &= \bar{V}_L \bar{L} \left[ \hat{l}_t + \frac{1}{2} (1+\varphi) \hat{l}_t^2 \right] \\ &= \bar{V}_L \bar{L} \left[ \tilde{l}_t + \frac{1}{2} (1+\varphi) \tilde{l}_t^2 + (1+\varphi) l_t^* \tilde{l}_t \right] + t.i.p. \end{aligned}$$

where

$$\tilde{l}_t = \hat{l}_t - l_t^*.$$

Hence, expressed in terms of gaps from the efficient steady state,

$$\begin{aligned} W &= \bar{W} + \bar{U}_C\bar{C}\left[\tilde{c}_t - \left(\frac{\bar{V}_L\bar{L}}{\bar{U}_C\bar{C}}\right)\tilde{l}_t\right] + \bar{U}_C\bar{C}\left[\frac{1}{2}(1-\sigma)\tilde{c}_t^2 + (1-\sigma)c_t^*\hat{c}_t\right] \\ &\quad -\bar{V}_L\bar{L}\left[\frac{1}{2}(1+\varphi)\tilde{l}_t^2 + (1+\varphi)l_t^*\hat{l}_t\right] + t.i.p. \end{aligned}$$

**Remark 1** *Note that it is the linear terms I need to deal with since the market equilibrium is inefficient even if a subsidy eliminates the distortion due to imperfect competition.*

Define

$$A_t = \left[\tilde{c}_t + \frac{1}{2}(1-\sigma)\tilde{c}_t^2 + (1-\sigma)c_t^*\hat{c}_t\right]$$

$$B_t = \left[\tilde{l}_t + \frac{1}{2}(1+\varphi)\tilde{l}_t^2 + (1+\varphi)l_t^*\hat{l}_t\right]$$

### 2.2.1 Goods clearing/resource constraint

Using the goods clearing (5, LS eq 12) and the production function (6, LS eq. 11),

$$C_t = A_tL_t - \frac{\Psi}{2}\left(\frac{L_t}{L_{t-1}} - 1\right)^2 A_tL_t - \frac{\Phi}{2}\left(\Pi_t - 1\right)^2 A_tL_t.$$

In the zero inflation steady state, $\bar{C} = \bar{Y}$, and

$$\frac{L_t}{L_{t-1}} \approx 1 + \hat{l}_t - \hat{l}_{t-1} + \frac{1}{2}\hat{l}_t^2 - \frac{1}{2}\hat{l}_{t-1}^2 + \hat{l}_{t-1}^2 - \hat{l}_t\hat{l}_{t-1}.$$

Hence,

$$\begin{aligned} \left(\frac{L_t}{L_{t-1}} - 1\right)^2 Y_t &\approx \left[\hat{l}_t - \hat{l}_{t-1} + \frac{1}{2}\hat{l}_t^2 - \frac{1}{2}\hat{l}_{t-1}^2 + \hat{l}_{t-1}^2 - \hat{l}_t\hat{l}_{t-1}\right]^2 \bar{Y}\left(1 + \hat{y}_t + \frac{1}{2}\hat{y}_t^2\right) \\ &\approx \bar{Y}\left(\hat{l}_t - \hat{l}_{t-1}\right)^2 + \mathcal{O}\left(\| a^3 \|\right) \end{aligned}$$

Therefore, the goods clearing condition becomes, to second order,

$$\left(\hat{c}_t + \frac{1}{2}\hat{c}_t^2\right) \approx \left(\hat{y}_t + \frac{1}{2}\hat{y}_t^2\right) - \frac{\Psi}{2}\left(\hat{l}_t - \hat{l}_{t-1}\right)^2 - \frac{\Phi}{2}\hat{\pi}_t^2.$$

and up to second order,

$$\hat{c}_t^2 \approx \left(\frac{\bar{Y}}{\bar{C}}\right)^2 \hat{y}_t^2 - \left(\frac{\bar{G}}{\bar{C}}\right)^2 \hat{g}_t^2.$$

Hence,

$$\begin{aligned}\hat{c}_t &\approx -\frac{1}{2}\hat{c}_t^2 + \frac{\bar{Y}}{\bar{C}}\left(\hat{y}_t + \frac{1}{2}\hat{y}_t^2\right) - \frac{\Psi}{2}\frac{\bar{Y}}{\bar{C}}\left(\hat{l}_t - \hat{l}_{t-1}\right)^2 - \frac{\Phi}{2}\frac{\bar{Y}}{\bar{C}}\hat{\pi}_t^2 \\ &= \hat{y}_t - \frac{\Psi}{2}\Delta\hat{l}_t^2 - \frac{\Phi}{2}\hat{\pi}_t^2.\end{aligned}$$

Collecting results,

$$\begin{aligned}W &= \bar{W} + \bar{U}_C\bar{C}\left[\tilde{y}_t - \frac{\Psi}{2}\left[\Delta\hat{l}_t^2 - (\Delta l_t^*)^2\right] - \frac{\Phi}{2}\hat{\pi}_t^2\right] + \bar{U}_C\bar{C}\left[\frac{1}{2}(1-\sigma)\tilde{c}_t^2 + (1-\sigma)c_t^*\hat{c}_t\right] \\ &\quad -\bar{V}_L\bar{L}\left[\tilde{l}_t + \frac{1}{2}(1+\varphi)\tilde{l}_t^2 + (1+\varphi)l_t^*\hat{l}_t\right] + t.i.p.\end{aligned}$$

Since $x^2 - (x^*)^2 = (x - x^*)^2 + 2x^*x - 2(x^*)^2$, we can write welfare as

$$\begin{aligned}W &= \bar{W} + \bar{U}_C\bar{C}\left[\tilde{y}_t - \frac{\Psi}{2}\Delta\tilde{l}_t^2 - \frac{\Phi}{2}\hat{\pi}_t^2\right] \\ &\quad +\bar{U}_C\bar{C}\left[\frac{1}{2}(1-\sigma)\tilde{c}_t^2 + (1-\sigma)c_t^*\hat{c}_t - \Psi\Delta l_t^*\Delta\hat{l}_t\right] \\ &\quad -\bar{V}_L\bar{L}\left[\tilde{l}_t + \frac{1}{2}(1+\varphi)\tilde{l}_t^2 + (1+\varphi)l_t^*\hat{l}_t\right] + t.i.p.\end{aligned}$$

Assume the standard fiscal subsidy so that in the steady state, $\mu = 1$ so $\bar{U}_C\bar{Y} = \bar{V}_L\bar{L}$, then since $\tilde{y} = \tilde{l}$, the first order terms $\tilde{y}$ and $\tilde{l}$ drop out and

$$\begin{aligned}W &= \bar{W} - \left(\frac{1}{2}\right)\bar{U}_C\bar{Y}\left[\Psi\Delta\tilde{l}_t^2 + \Phi\hat{\pi}_t^2\right] \\ &\quad +\bar{U}_C\bar{C}\left[\frac{1}{2}(1-\sigma)\tilde{c}_t^2 + (1-\sigma)c_t^*\hat{c}_t - \Psi\frac{\bar{Y}}{\bar{C}}\Delta l_t^*\Delta\hat{l}_t\right] \\ &\quad -\bar{V}_L\bar{L}\left[\frac{1}{2}(1+\varphi)\tilde{l}_t^2 + (1+\varphi)l_t^*\hat{l}_t\right] + t.i.p.\end{aligned}$$

Because individual firms neglect the effect their choice of production levels has on aggregate income, they ignore the effect of $Y_t$ on the costs of labor adjustment terms. The solcial planner would not neglect this effect, and so the condition efficiency (LS, eq 14) takes the form

$$A_t = \frac{L_t^{\varphi}}{C_t^{-\sigma}} + \Psi\left(\frac{L_t}{L_{t-1}} - 1\right)\frac{Y_t}{L_{t-1}} - \beta\mathrm{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\Psi\left(\frac{L_{t+1}}{L_t} - 1\right)\frac{L_{t+1}Y_{t+1}}{L_t^2} - \Gamma_t, \tag{7}$$

where

$$\Gamma_t \equiv \frac{\Psi}{2}\left(\frac{L_t}{L_{t-1}} - 1\right)^2 A_t$$

is the term absent from the private market condition (2). Since $\Gamma_t$ is of second order, it drops out of the linearized model. Since $\Gamma$ is zero in the steady state, it is only of second order and so only contributes third order or igher order terms to the welfare approximation. Thus, up to second order, the contribution of $\Gamma_t$ drops out of the welfare approximation.

### 2.2.2 Collecting terms

Since to first order $\tilde{c}_t = \tilde{y} \ = \tilde{l}$, we can write welfare as

$$\begin{aligned} W &= \bar{W} - \left(\frac{1}{2}\right) \bar{U}_C \bar{Y} \left[(\sigma+\varphi)\tilde{y}_t^2 + \Psi \Delta \tilde{l}_t^2 + \Phi \hat{\pi}_t^2\right] \\ &\quad + \bar{U}_C \bar{C} \left[(1-\sigma)c_t^* \hat{c}_t - (1+\varphi) l_t^* \tilde{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t\right] \\ &\quad + t.i.p. \end{aligned}$$

where we have used the result that $\tilde{c}^2 \approx \tilde{y}_t^2 = \tilde{l}_t^2$. Or

$$W = \bar{W} - \left(\frac{1}{2}\right) \bar{U}_C \bar{Y} \mathcal{L}_t + \bar{U}_C \bar{C} K_t + t.i.p.$$

where

$$\mathcal{L}_t \equiv (\sigma+\varphi)\tilde{y}_t^2 + \Psi \Delta \tilde{l}_t^2 + \Phi \hat{\pi}_t^2$$

and

$$K_t \equiv (1-\sigma)c_t^* \hat{c}_t - (1+\varphi) l_t^* \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t.$$

To evaluation $c^* \hat{c}$ and the other terms, we only need a first order approximation to $\hat{c}$ and $\hat{l}$. From the goods clearing condition, $\hat{c} = \hat{y} = \hat{a} + \hat{l}$, so

$$\begin{aligned} K_t &\equiv (1-\sigma)c_t^* \left(\hat{a}_t + \hat{l}_t\right) - (1+\varphi) l_t^* \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t \\ &= \left[(1-\sigma)c_t^* - (1+\varphi) l_t^*\right] \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t + (1-\sigma) c_t^* \hat{a}_t \\ &= \left[(1-\sigma)\left(\hat{a}_t + l_t^*\right) - (1+\varphi) l_t^*\right] \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t + (1-\sigma) c_t^* \hat{a}_t \\ &= \left[(1-\sigma)\hat{a}_t - (\sigma+\varphi) l_t^*\right] \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t + t.i.p. \end{aligned}$$

### 2.2.3 Socially efficient condition

To a first order, the condition for efficiency (7, LS eq 14) is

$$\begin{aligned} \hat{a}_t &= \left[\varphi \hat{l}_t^* + \sigma\left(\hat{a}_t + \hat{l}_t^*\right)\right] + \Psi\left(\hat{l}_t^* - \hat{l}_{t-1}\right) - \beta \Psi \mathrm{E}_t \left(\hat{l}_{t+1}^* - \hat{l}_t^*\right) \\ &\Rightarrow (1-\sigma)\hat{a}_t = (\sigma+\varphi)\hat{l}_t^* + \Psi \Delta \hat{l}_t^* - \beta \Psi \mathrm{E}_t \Delta \hat{l}_{t+1}^* \end{aligned} \quad (8)$$

Therefore,

$$\begin{aligned} K_t &= \Psi\left(\Delta \hat{l}_t^* - \beta \mathrm{E}_t \Delta \hat{l}_{t+1}^*\right) \hat{l}_t - \Psi \Delta l_t^* \Delta \hat{l}_t \\ &= \Psi\left(\Delta l_t^* \hat{l}_{t-1} - \beta \mathrm{E}_t \Delta \hat{l}_{t+1}^* \hat{l}_t\right). \end{aligned}$$

### 2.2.4 Discounted welfare

$$W = \bar{W} - \left(\frac{1}{2}\right) \bar{U}_C \bar{Y} \mathcal{L}_t + \bar{U}_C \bar{C} K_t + t.i.p.$$

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i W_{t+i} = \frac{\bar{W}}{1-\beta} - \left(\frac{1}{2}\right) \bar{U}_C \bar{Y} \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i L_{t+i} + \mathrm{E}_t \bar{U}_C \bar{C} \sum_{i=0}^{\infty} \beta^i K_{t+i}.$$

But

$$\begin{aligned}
\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i K_{t+i} &= \Psi \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left( \Delta l_t^* \hat{l}_{t-1} - \beta \mathrm{E}_t \Delta \hat{l}_{t+1}^* \hat{l}_t \right) \\
&= \Psi \mathrm{E}_t \left[ \left( \Delta \hat{l}_t^* \hat{l}_{t-1} - \beta \Delta \hat{l}_{t+1}^* \hat{l}_t \right) + \beta \left( \Delta \hat{l}_{t+1}^* \hat{l}_t - \beta \Delta \hat{l}_{t+2}^* \hat{l}_{t+1} \right) + ... \right] \\
&= \Psi \Delta \hat{l}_t^* \hat{l}_{t-1}.
\end{aligned}$$

Therefore,

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i W_{t+i} = \frac{\bar{W}}{1-\beta} - \left(\frac{1}{2}\right) \bar{U}_C \bar{Y} \mathrm{E}_t \sum_{i=0}^{\infty} \beta_{t+i}^i \mathcal{L}_{t+i} + \Psi \Delta \hat{l}_t^* \hat{l}_{t-1}, \tag{9}$$

where

$$\mathcal{L}_t = (\sigma + \varphi) \tilde{y}_t^2 + \Psi \Delta \tilde{l}_t^2 + \Phi \hat{\pi}_t^2. \tag{10}$$

### 2.3 Linearized equations

From (2),

$$Q_t A_t = \frac{L_t^{\varphi}}{C_t^{-\sigma}} + \Psi \left( \frac{L_t}{L_{t-1}} - 1 \right) \frac{Y_t}{L_{t-1}} - \beta \Psi \mathrm{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \left( \frac{L_{t+1}}{L_t} - 1 \right) \frac{L_{t+1}}{L_t^2} Y_t$$

A first order approximation to the left side is

$$Q_t A_t \approx 1 + \hat{q}_t + \hat{a}_t.$$

A first order approximation to the terms on the right side yields

$$\frac{L_t^{\varphi}}{C_t^{-\sigma}} \approx \frac{L^{\varphi}}{C^{-\sigma}} \left( 1 + \varphi \hat{l}_t + \sigma \hat{c}_t \right);$$

$$\begin{aligned}
\Psi \left( \frac{L_t}{L_{t-1}} - 1 \right) \frac{Y_t}{L_{t-1}} &\approx \Psi \left( \hat{l}_t - \hat{l}_{t-1} \right) (1 + \hat{y}_t) \left( 1 - \hat{l}_{t-1} \right) \\
&= \Psi \left( \hat{l}_t - \hat{l}_{t-1}^2 \right);
\end{aligned}$$

$$\begin{aligned}
\beta \Psi \mathrm{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} \left( \frac{L_{t+1}}{L_t} - 1 \right) \frac{L_{t+1}}{L_t^2} Y_t &\approx \beta \Psi \mathrm{E}_t \left[ 1 - \sigma \left( \hat{c}_{t+1} - \hat{c}_t \right) \right] \\
&\quad \times \left( \hat{l}_{t+1} - \hat{l}_t \right) (1 + \hat{y}_t) \left( 1 + \hat{l}_{t+1} \right) \left( 1 - 2\hat{l}_t \right) \\
&\approx \beta \Psi \mathrm{E}_t \left( \hat{l}_{t+1} - \hat{l}_t \right).
\end{aligned}$$

Collecting reults,

$$
\begin{aligned}
1+\hat{q}_t+\hat{a}_t &= \frac{L^{\varphi}}{C^{-\sigma}}\left[1+\left(\varphi\hat{l}_t+\sigma\hat{c}_t\right)\right] \\
&\quad +\Psi\left(\hat{l}_t-\hat{l}_{t-1}\right)-\Psi \mathrm{E}_t\left(\hat{l}_{t+1}-\hat{l}_t\right)
\end{aligned}
$$

Since $\hat{c}_t=\hat{y}_t=\hat{a}_t+\hat{l}_t$, to a first order, this becomes

$$
\begin{aligned}
\hat{q}_t+\hat{a}_t &= \left(\varphi\hat{l}_t+\sigma\hat{c}_t\right)+\Psi\left(\hat{l}_t-\hat{l}_{t-1}\right)-\Psi \mathrm{E}_t\left(\hat{l}_{t+1}-\hat{l}_t\right) \\
&= \sigma\hat{a}_t+(\sigma+\varphi)\hat{l}_t+\Psi\Delta\hat{l}_t-\Psi \mathrm{E}_t\Delta\hat{l}_{t+1},
\end{aligned}
$$

and

$$
\hat{q}_t=(\sigma+\varphi)\hat{l}_t+\Psi\Delta\hat{l}_t-\Psi \mathrm{E}_t\Delta\hat{l}_{t+1}-(1-\sigma)\hat{a}_t
$$

But from (8),

$$
0=(\sigma+\varphi)\,\hat{l}_t^{*}+\Psi\Delta\hat{l}_t^{*}-\beta\Psi \mathrm{E}_t\Delta\hat{l}_{t+1}^{*}-(1-\sigma)\hat{a}_t
$$

so

$$
\tilde{q}_t=(\sigma+\varphi)\tilde{l}_t+\Psi\Delta\tilde{l}_t-\beta\Psi \mathrm{E}_t\Delta\tilde{l}_{t+1}.
$$

#### 2.3.1 Inflation

From the first order condition for pricing setting,

$$
Q_t=\mu_t^{-1}+\frac{\Phi}{\varepsilon_t}\left[\left(\Pi_t-1\right)\Pi_t-\beta \mathrm{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\Pi_{t+1}-1\right)\Pi_{t+1}\left(\frac{Y_{t+1}}{Y_t}\right)\right],
$$

one obtains to first order

$$
Q\left[1+\left(\hat{q}_t+\hat{\mu}_t\right)\right]=\mu^{-1}+\Phi\left[\begin{array}{c}\left(\pi_t\right)\left(1+\pi_t\right) \\ -\beta \mathrm{E}_t\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\pi_{t+1}\right)\left(1+\pi_{t+1}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\end{array}\right]
$$

so to a first approximation

$$
Q\left(1+\hat{q}_t\right)=1-\hat{\mu}_t+\Phi\left[\pi_t-\beta \mathrm{E}_t\pi_{t+1}\right]
$$

where $\tau$ is used to ensure $\mu=1$. Hence, since $Q=1$,

$$
\pi_t=\beta \mathrm{E}_t\pi_{t+1}+\left(\frac{\varepsilon}{\Phi}\right)\left(\hat{q}_t+\hat{\mu}_t\right). \tag{11}
$$

# 3 Sectoral reallocation and hiring costs

# 4 Introduction

# 5 Two sector model with firm specific labor

The model consists of a representative household and two consumption aggregates produced in different sectors. Each sector consists of a continuum of firms.

Firms in each sector employ labor supplied by households, and prices are sticky in both sectors. Sectors-specific variables are denoted by superscripts $s = 1, 2$. Individual firms are denoted by subscript $j$.

## 5.1 Households

Representative household consisting of a continuum of members. The household enters the period with $N^s_{j,t}$ workers employed at firm $j$ in sector $s$. Household utility is given by

$$\mathrm{E}_t \sum_{s=0}^{\infty} \beta^s \left[ \begin{array}{c} U(C_{t+s}) - \left(\frac{1}{1+\eta}\right) \int_0^1 \left(h^1_{j,t}\right)^{1+\eta} N^1_{j,t} dj \\ - \left(\frac{1}{1+\eta}\right) \int_0^1 \left(h^2_{j,t}\right)^{1+\eta} N^2_{j,t} dj \end{array} \right], \tag{12}$$

where $U(C_t)$ is the utility from consuming an aggregate basket of goods and terms of the form

$$\left(\frac{1}{1+\eta}\right) \int_0^1 \left(h^s_{j,t}\right)^{1+\eta} N^s_{j,t} dj$$

represent the disutility of work in sector $s$. The $N^s_{j,t}$ members of the household who are employed at firm $j$ in sector $s$ and each work $h^s_{j,t}$ hours.

Aggregate consumption is defined by

$$C_t = \left[ \left(\gamma^1\right)^{\frac{1}{a}} \left(C^1_t\right)^{\frac{a-1}{a}} + \left(\gamma^2\right)^{\frac{1}{a}} \left(C^2_t\right)^{\frac{a-1}{a}} \right]^{\frac{a}{a-1}}, \ a > 1, \ \gamma^1 + \gamma^2 = 1. \tag{13}$$

In (13), $C^s_t$, $s = 1, 2$ denotes the consumption of sector $s$ goods. In turn, $C^s_t$ is a Dixit-Stiglitz aggregate of the goods produced by individual firms in the sector:

$$C^s_t = \left[ \int_0^1 \left(C^s_{j,t}\right)^{\frac{\theta^s - 1}{\theta^s}} dj \right]^{\frac{\theta^s}{\theta^s - 1}}.$$

The domestic household's relative demand for $C^1_t$ and $C^2_t$ will depend on their relative prices. The problem of minimizing the cost $P^1_t C^1_t + P^2_t C^2_t$ of achieving a given level of $C_t$ yields the first order conditions

$$C^1_t = \gamma^1 \left(\frac{P^1_t}{P_t}\right)^{-a} C_t$$

and

$$C^2_t = \gamma^2 \left(\frac{P^2_t}{P_t}\right)^{-a} C_t.$$

The utility-based aggregate consumer price index is

$$P_t \equiv \left[ \gamma^1 \left(P^1_t\right)^{1-a} + \gamma^2 \left(P^2_t\right)^{1-a} \right]^{\frac{1}{1-a}}.$$

Given consumption $C_t^s$ of sector $s$ goods, a similar cost minimization problem implies the demand for individual goods is given by

$$C_{j,t}^s = \left(\frac{P_{j,t}^s}{P_t^s}\right)^{-\theta^s} C_t^s = \gamma^s \left(\frac{P_{j,t}^s}{P_t^s}\right)^{-\theta^s} \left(\frac{P_t^s}{P_t}\right)^{-a} C_t,$$

where $P_{j,t}^s$ is the price of firm $j$ in sector $s$ and

$$P_t^s = \left[\int_0^1 \left(P_{j,t}^s\right)^{1-\theta^s} dj\right]^{\frac{1}{1-\theta^s}}$$

Aggregate employment in sector $s$ is $N_t^s = \int_0^1 N_{j,t}^s dj$, $N_t \equiv N_t^1 + N_t^2$, and aggregate unemployment, defined as the number of unemployed members of the household, is

$$U_t = 1 - N_t^1 - N_t^2 = 1 - N_t.$$

Each unemployed worker is assumed to search for a job. Thus, the labor force participation margin is ignored.

The household's budget constraint in nominal terms takes the form

$$P_t C_t + \mathrm{E}_t D_{t,t+1} B_{t+1} \le P_t \left[\int_0^1 w\left(h_{j,t}^1\right) N_{j,t}^1 dj + \int_0^1 w\left(h_{j,t}^2\right) N_{j,t}^2 dj\right] + P_t b U_t + \Pi_t + B_t - T_t$$

where $w\left(h_{j,t}^s\right)$ is the real wage per worker at fime $j$ in sector $s$, firm $j$ working $h_t^s(j)$ hours. The term $bU_t$ is a real unemployment benefit. Households can purchase bonds whose time $t$ price is $\mathrm{E}_t D_{t,t+1}$ and which pay off one dollar at $t+1$.

Letting $\lambda_t$ be the Lagrangian multiplier on the budget constraint, utility maximization implies

$$U_{C,t} = P_t \lambda_t$$

and

$$\begin{aligned} \mathrm{E}_t D_{t,t+1} &= \beta \mathrm{E}_t \left(\frac{\lambda_{t+1}}{\lambda_t}\right) \\ &= \beta \mathrm{E}_t \left(\frac{P_t}{P_{t+1}}\right)\left(\frac{U_{C,t+1}}{U_{C,t}}\right) \end{aligned}$$

Defining

$$R_t \equiv \frac{1}{\mathrm{E}_t D_{t,t+1}},$$

we have

$$U_{C,t} = \beta \mathrm{E}_t \left[\left(\frac{P_t}{P_{t+1}}\right) R_t U_{C,t+1}\right].$$

The first order condition for labor hours in sector $s$ is

$$-\left(h_{j,t}^s\right)^{\eta} N_{j,t}^s + \lambda_t P_t w'\left(h_{j,t}^s\right) N_{j,t}^s = 0,$$

so hours must satisfy the standard conditions that require the marginal rate of substitution between leisure and consumption to equal the marginal wage:

$$w'\left(h_{j,t}^{s}\right)=\frac{\left(h_{j,t}^{s}\right)^{\eta}}{U_{C,t}};\ s=1,2.$$

Define $W_{j,t}^{s}$ as the surplus to the household of having a member employed at firm $j$ in sector $s$. This surplus is

$$\begin{aligned} W_{j,t}^{s} &= w\left(h_{j,t}^{s}\right)-\frac{\left(h_{j,t}^{s}\right)^{1+\eta}}{(1+\eta)U_{C,t}}+(1-\rho^{s})\mathrm{E}_{t}D_{t,t+1}W_{j,t+1}^{s} \\ &\quad -\left(\frac{b}{U_{C,t}}\right)-\sum_{k=1,2}\left[r_{t}^{k}\int_{0}^{1}\left(\frac{v_{i,t}^{k}}{v_{t}^{k}}\right)\mathrm{E}_{t}D_{t,t+1}W_{i,t+1}^{k}di\right]. \end{aligned} \tag{14}$$

The first term is the wage the worker earns from working $h_{j,t}^{s}$ hours. The second term is the disutility of earning this wage. The third term is the continuation value of being employed at this firm in the following period, where $\rho^{s}$ is a (sector specific) exogenous separation rate. The final two terms measure the opportunity cost of having a worker employed as opposed to unemployed.[1] These include the utility value of any unemployment benefit $b/U_{C,t}$ and the possibility, if unemployed, of finding employment during the period. The value of this is equal to the probably of finding a job in sector $s$, denoted by $r_{t}^{s}$, times the expected value of the subsequent job. With probability $r_{t}^{s}\left(v_{i,t}^{s}/v_{t}^{s}\right)$ the worker is hired by firm $i$ in sector $s$, where $v_{i,t}^{s}$ is the number of vacancies that firm has posted and $v_{t}^{s}$ is the aggregate number of vacancies in that sector.

## 5.2 Firms

The profits (expressed in terms of the final consumption basket) of firm $j$ in sector $s$ are given by

$$\left(\frac{P_{j,t}^{s}}{P_{t}}\right)C_{j,t}^{s}-w(h_{j,t}^{s})N_{j,t}^{s}-\kappa v_{j,t}^{s}-\Phi_{t}^{s}H_{j,t}^{s}, \tag{15}$$

where $w(h_{j,t}^{s})$ is the wage schedule at firm $j$, $h_{j,t}^{s}$ equals hours per worker at the firm, $N_{j,t}^{s}$ is the number of employees, $v_{j,t}^{s}$ is the number of job vacancies the firm has posted, and $H_{j,t}^{s}$ are the number of new hires by the firm. There is a fixed cost $\kappa$ per vacancy posted, as well as costs associated with hiring new workers equal to $\Phi_{t}^{s}$. To capture costs of shifting resources (labor) between sectors, we assume that there are adjustment costs associated with new hires that are related to the number of workers hired in sector $s$ that previously worked in sector $r\neq s$. One approach would be to model labor as heterogenous so that the costs of recruitment/training in sector $s$ would be increasing in the fraction of the unemployment pool who have a comparative advantage in working in

[1] This assumes workers cannot immediately switch from a job in sector $s$ to a job in sector $r\neq s$.

sector $r$ (as in Epstein 2010). Instead, I assume hiring costs in sector $s$ are increasing in the fraction of the unemployed (and therefore of new matches) who were last employed in sector $r \neq s$. These are assumed to be independent of the firm but are sector specific.[2] Specifically, assume

$$\Phi_t^s = \bar{\Phi}^s + \frac{\vartheta}{1+\epsilon}\left(\lambda_t^{k\neq s} - \lambda_t^s\right)^{1+\epsilon};\ \epsilon > 0, \tag{16}$$

where $\lambda_t^{k\neq s}$ is the share of the unemployed job seekers last employed in sector $k \neq s$. So in the steady state, $\Phi_t^s = \bar{\Phi}^s$

Firms maximize profits subject to a technology given by

$$C_{j,t}^s = A_t^s h_{j,t}^s N_{j,t}^s;\ s = 1, 2,$$

and a demand curve given by

$$C_{j,t}^s = \left(\frac{P_{j,t}^s}{P_t^s}\right)^{-\theta^s} C_t^s.$$

In addition, labor is specifc to the firm, and the firm's work force evolves according to

$$N_{j,t+1}^s = (1-\rho^s)N_{j,t}^s + H_{j,t}^s,$$

where $H_{j,t}^s$ is equal to new hires by firm $j$. New hiring is equal to

$$H_{j,t}^s = q_t^s v_{j,t}^s,$$

where $q_t^s$ is the fraction of the firm's vacancies that result in a new hire.

With firm specific labor, the firm does not face a fixed wage per hour of labor. To induce more hours, it needs to initially induce more effort on the part of its existing workers. Therefore, the expected real value of firm $j$ in sector $s$ is

$$\begin{aligned}\Pi_{j,t}^s \quad = \quad & (1+\tau^s)\left(\frac{P_{j,t}^s}{P_t}\right) C_{j,t}^s - w(h_{j,t}^s)N_{j,t}^s - \left(\frac{\kappa}{U_{C,t}}\right) v_{j,t}^s \\ & - \left(\frac{1}{U_{C,t}}\right)\Phi_t^s q_t^s v_{j,t}^s + \mathrm{E}_t D_{t,t+1}\Pi_{j,t+1}^s\end{aligned}$$

where $\tau^s$ is standard subsidy to offset the mark up in the steady-state, $\kappa$ represents the cost of posting job vacancies, and $\Phi_t^s q_t^s v_{j,t}^s$ represents hiring costs.

Demand facing the firm is

$$C_{j,t}^s = \left(\frac{P_{j,t}^s}{P_t^s}\right)^{-\theta^s} C_t^s = \gamma^s \left(\frac{P_{j,t}^s}{P_t^s}\right)^{-\theta^s} \left(\frac{P_t^s}{P_t}\right)^{-a} C_t,$$

while from the production function,

$$C_{j,t}^s = A_t^s h_{j,t}^s N_{j,t}^s,$$

[2] For a model with quadratic costs of labor adjustment, see Lechthaler and Snower (2011).

and

$$N^s_{j,t+1} = (1-\rho)N^s_{j,t} + q_t v^s_{j,t}.$$

Rewrite the firm's problem as

$$\max \mathrm{E}_t \sum_{t=0}^{\infty} D_{0,t} \begin{bmatrix} (1+\tau_1)\left(\frac{P^s_{j,t}}{P_t}\right) C^s_{j,t} - w(h^s_{j,t})N^s_{j,t} - \left(\frac{\kappa}{U_{C,t}} + \Phi^s_t q^s_t\right) v^s_{j,t} \\ +mc^s_{j,t}\left[A^s_t h^s_{j,t} N^s_{j,t} - C^s_{j,t}\right] \\ +\varphi^s_{j,t}\left[(1-\rho^s)N^s_{j,t} + q_t v^s_{j,t} - N^s_{j,t+1}\right] \end{bmatrix}$$

where $mc^s_{j,t}$ is real marginal cost for firm $j$ in sector $s$ while $\varphi^s_{j,t}$ is the marginal value of an additional employee.

The first order conditions for hours, vacancies, and employment are

$$h^s_{j,t}\colon - w'(h^s_{j,t})N^s_{j,t} + mc^s_{j,t}A^s_t N^s_{j,t} = 0 \Rightarrow mc^s_{j,t} = \frac{w'(h^s_{j,t})}{A^s_t}$$

$$v^s_{j,t}\colon - \left(\frac{1}{U_{C,t}}\right)(\kappa + \Phi^s_t q^s_t) + \varphi^s_{j,t} q^s_t = 0 \Rightarrow \varphi^s_{j,t} = \left(\frac{1}{U_{C,t}}\right)\left(\frac{\kappa}{q^s_t} + \Phi^s_t\right) \tag{17}$$

$$N^s_{j,t+1}\colon\ -\varphi^s_{j,t} + \mathrm{E}_t D_{t,t+1}\left[mc^s_{j,t+1}A^s_{t+1}h^s_{j,t+1} - w(h^s_{j,t+1}) + (1-\rho^s)\varphi^s_{j,t+1}\right] \tag{18}$$

With sticky prices, firms produce to meet demand, so

$$h^s_{j,t} = \frac{C^s_{j,t}}{A^s_t N^s_{j,t}}.$$

Thus, real marginal costs can be written as

$$mc^s_{j,t} = w'\left(\frac{C^s_{j,t}}{A^s_t N^s_{j,t}}\right)\frac{1}{A^s_t} \tag{19}$$

### 5.3 Wage determination

To determine the real wage, we assume Nash bargaining with fixed shares of the joint surplus to a match going to workers and firms. From (14), the surplus of a match for workers is

$$\begin{aligned} W^s_{j,t} &= w\left(h^s_{j,t}\right) - \frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} + (1-\rho^s)\mathrm{E}_t D_{t,t+1} W^s_{j,t+1} \\ &\quad - \left(\frac{b}{U_{C,t}}\right) - \sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \mathrm{E}_t D_{t,t+1} W^k_{i,t+1} di\right], \end{aligned}$$

where $r^k_t$ is the probability the worker finds a job in sector $k$ and $v^k_{i,t}/v^k_t$ is the probability that job is with firm $i$. From the firm's perspective, the surplus value of additional worker is

$$S^s_{j,t} = mc^s_{j,t}A^s_t h^s_{j,t} - w(h^s_{j,t}) + (1-\rho^s)\mathrm{E}_t D_{t,t+1} S^s_{j,t+1}. \tag{20}$$

Let $\chi$ be the firm's share of the total surplus. Then, under Nash bargaining,

$$(1-\chi)S^s_{j,t} = \chi W^s_{j,t},\ s = 1,2.$$

Using the expressions for $S^s_{j,t}$ and $W^s_{j,t}$, one obtains

$$(1-\chi)\begin{bmatrix} mc^s_{j,t}A^s_t h^s_{j,t} - w(h^s_{j,t}) \\ +(1-\rho^s)\mathrm{E}_t D_{t,t+1} S^s_{j,t+1} \end{bmatrix} = \chi \begin{bmatrix} w\left(h^s_{j,t}\right) - \frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} - \left(\frac{b}{U_{C,t}}\right) \\ -\sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \mathrm{E}_t D_{t,t+1} W^k_{i,t+1} di\right] \\ +(1-\rho^s)\mathrm{E}_t D_{t,t+1} W^s_{j,t+1} \end{bmatrix}.$$

Then, since

$$(1-\chi)S^s_{j,t} = \chi W^s_{j,t} \Rightarrow (1-\chi)\mathrm{E}_t D_{t,t+1} S^s_{j,t+1} = \chi \mathrm{E}_t D_{t,t+1} W^s_{j,t+1} = 0,$$

this expression becomes

$$(1-\chi)\left[mc^s_{j,t}A^s_t h^s_{j,t} - w(h^s_{j,t})\right] = \chi \begin{bmatrix} w\left(h^s_{j,t}\right) - \frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} - \left(\frac{b}{U_{C,t}}\right) \\ -\sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \mathrm{E}_t D_{t,t+1} W^k_{i,t+1} di\right] \end{bmatrix}.$$

Solving for the wages, the wage schedule at firm $j$ in sector $s$ is

$$\begin{aligned} w(h^s_{j,t}) &= (1-\chi)mc^s_{j,t}A^s_t h^s_{j,t} + \chi\left[\frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad +\chi \sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \mathrm{E}_t D_{t,t+1} W^k_{i,t+1} di\right] \end{aligned}$$

From (18) and (20),

$$\begin{aligned} \varphi^s_{j,t} &= \mathrm{E}_t D_{t,t+1}\left[mc^s_{j,t+1}A^s_{t+1}h^s_{j,t+1} - w(h^s_{j,t+1}) + (1-\rho^s)\varphi^s_{j,t+1}\right] \\ &= \mathrm{E}_t D_{t,t+1} S^s_{j,t+1} \end{aligned}$$

This means that Nash bargaining implies

$$\mathrm{E}_t D_{t,t+1} W^k_{i,t+1} = \left(\frac{1-\chi}{\chi}\right)\mathrm{E}_t D_{t,t+1} S^s_{j,t+1} = \left(\frac{1-\chi}{\chi}\right)\varphi^s_{j,t}.$$

Using this result,

$$\begin{aligned} w(h^s_{j,t}) &= (1-\chi)mc^s_{j,t}A^s_t h^s_{j,t} + \chi\left[\frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad +\chi\left(\frac{1-\chi}{\chi}\right) \sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \mathrm{E}_t D_{t,t+1} W^k_{i,t+1} di\right] \\ &= (1-\chi)mc^s_{j,t}A^s_t h^s_{j,t} + \chi\left[\frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad +(1-\chi) \sum_{k=1,2} r^k_t \left[\int_0^1 \left(\frac{v^k_{i,t}}{v^k_t}\right) \varphi^k_{i,t+1} di\right] \end{aligned}$$

Earlier, the vacancy posting condition (17) was shown to imply

$$\varphi_{i,t}^{s} = \left(\frac{1}{U_{C,t}}\right)\left(\frac{\kappa}{q_t^s} + \Phi_t^s\right)$$

so using this to eliminate $\varphi_{i,t}^{s}$ from the wage expression yields

$$\begin{aligned} w(h_{j,t}^s) &= (1-\chi) mc_{j,t}^s A_t^s h_{j,t}^s + \chi\left[\frac{\left(h_{j,t}^s\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\sum_{k=1,2} r_t^k \left[\int_0^1 \left(\frac{v_{i,t}^k}{v_t^k}\right)\left(\frac{\kappa}{q_{t+1}^k} + \Phi_{t+1}^k\right) di\right]. \end{aligned}$$

But since neither $q^s$ nor $\Phi^s$ depend in $j$, and

$$\int_0^1 \left(\frac{v_{i,t}^k}{v_t^k}\right) di = 1,$$

the wage becomes

$$\begin{aligned} w(h_{j,t}^s) &= (1-\chi) mc_{j,t}^s A_t^s h_{j,t}^s + \chi\left[\frac{\left(h_{j,t}^s\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\sum_{k=1,2} r_t^k \left(\frac{\kappa}{q_{t+1}^k} + \Phi_{t+1}^k\right). \end{aligned}$$

Then, because (see below)

$$\frac{r^k}{q^k} = \theta_t \left(\frac{v_t^s}{v_t}\right),$$

$$\begin{aligned} w(h_{j,t}^s) &= (1-\chi) mc_{j,t}^s A_t^s h_{j,t}^s + \chi\left[\frac{\left(h_{j,t}^s\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\kappa\theta_t \sum_{k=1,2}\left(\frac{v_t^k}{v_t}\right) + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\sum_{k=1,2} r_t^k \Phi_t^k \\ &= (1-\chi) mc_{j,t}^s A_t^s h_{j,t}^s + \chi\left[\frac{\left(h_{j,t}^s\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\kappa\theta_t + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\sum_{k=1,2} r_t^k \Phi_t^k. \end{aligned}$$

since

$$\int \left(\frac{v_{i,t}^k}{v_t}\right) di = \left(\frac{v_t^k}{v_t}\right),\ r_t \frac{\kappa}{q_t} = \kappa\theta_t, \text{ and } \sum_{k=1,2}\left(\frac{v_t^k}{v_t}\right) = 1.$$

Notice that the last term is independent of $i$. It does depend on the relationship between a sector's $\Phi^s$ and its relative share of vacancies.

Using the expression for wages,

$$w'(h^s_{j,t}) = (1-\chi)mc^s_{j,t}A^s_t + \frac{\chi\left(h^s_{j,t}\right)^\eta}{U_{C,t}}$$

But since

$$mc^s_{j,t} = \frac{w'(h^s_{j,t})}{A^s_t}$$

we would have

$$A^s_t mc^s_{j,t} = (1-\chi)mc^s_{j,t}A^s_t + \frac{\chi\left(h^s_{j,t}\right)^\eta}{U_{C,t}}$$

or

$$mc^s_{j,t} = \frac{\left(h^s_{j,t}\right)^\eta}{A^s_t U_{C,t}} \tag{21}$$

which agrees with the earlier result.

#### 5.3.1 Vacancy posting

From the wage equation and the first order conditions for $v^s_t(j)$ and $N^s_{t+1}(j)$ reported above, one obtains

$$\left(\frac{1}{U_{C,t}}\right)\left(\frac{\kappa}{qhs_t} + \Phi^s_t\right) = \mathrm{E}_t D_{t,t+1}\left[\begin{array}{c} mc^s_{j,t+1}A^s_{t+1}h^s_{j,t+1} - w(h^s_{j,t+1}) \\ +(1-\rho^s)\left(\frac{1}{U_{C,t+1}}\right)\left(\frac{\kappa}{q^s_t} + \Phi^s_{t+1}\right)\end{array}\right].$$

So using the wage expression,

$$\begin{aligned} w(h^s_{j,t}) &= (1-\chi)mc^s_{j,t}A^s_t h^s_{j,t} + \chi\left[\frac{\left(h^s_{j,t}\right)^{1+\eta}}{(1+\eta)U_{C,t}} + \left(\frac{b}{U_{C,t}}\right)\right] \\ &\quad +(1-\chi)\left(\frac{1}{U_{C,t}}\right)\kappa\theta_t + (1-\chi)\left(\frac{1}{U_{C,t}}\right)\sum_{k=1,2} r^k_t \Phi^k_t, \end{aligned}$$

the vacancy posting condition is

$$\begin{aligned} \left(\frac{\kappa}{q^s_t} + \Phi^s_t\right) &= \beta\left[U_{C,t+1}mc^s_{j,t+1}A^s_{t+1}h^s_{j,t+1} - w(h^s_{j,t+1}) + (1-\rho^s)\left(\frac{\kappa}{q_{t+1}} + \Phi^1_{t+1}\right)\right] \\ &= \beta\mathrm{E}_t\chi\left[U_{C,t+1}mc^s_{j,t+1}A^s_{t+1}h^s_{j,t+1} - \left(\frac{\left(h^s_{j,t+1}\right)^{1+\eta}}{(1+\eta)} + b\right)\right] \\ &\quad -\beta(1-\chi)\kappa\mathrm{E}_t\theta_{t+1} - (1-\chi)\,\beta\mathrm{E}_t\sum_{k=1,2} r^k_{t+1}\Phi^k_{t+1} \\ &\quad +\beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q^s_{t+1}} + \Phi^s_{t+1}\right) \end{aligned}$$

The new terms relative to a standard specification are $\Phi^1$ and $\Phi^2$ and the sector specific $q's$.

Equation (21) allows marginal cost to be eliminated,

$$\begin{aligned}\left(\frac{\kappa}{q_t^s}\right) &= \beta \mathrm{E}_t \chi \left[\left(\frac{\eta}{1+\eta}\right)\left(h_{j,t+1}^s\right)^{1+\eta} - b\right] - \beta\left(1-\chi\right)\kappa \mathrm{E}_t \theta_{t+1} \\ &\quad + \beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}^s}\right) + \beta(1-\rho^s)\mathrm{E}_t\left(\Phi_{t+1}^s\right) - \Phi_t^s \\ &\quad - \beta \mathrm{E}_t \sum_{k=1,2}\left[(1-\chi)\, r_{t+1}^k\right]\Phi_{t+1}^k \qquad (22)\end{aligned}$$

which depends on both adjustment costs in the own sector and in the other sector.

Note that in the absence of hiring costs, (22) would take the form

$$\begin{aligned}\left(\frac{\kappa}{q_t^s}\right) &= \chi\beta \mathrm{E}_t \left[\left(\frac{\eta}{1+\eta}\right)\left(h_{j,t+1}^s\right)^{1+\eta} - b\right] \\ &\quad + \beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}^s}\right)\end{aligned}$$

which would imply all firms in sector $s$ have the same expected hours. Hence, we could write this as

$$\chi\beta \mathrm{E}_t\left(\frac{\eta}{1+\eta}\right)\left(h_{t+1}^s\right)^{1+\eta} = X_t^s$$

where

$$X_t^s \equiv \left(\frac{\kappa}{q_t^s}\right) + \beta\chi b - \beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}^s}\right)$$

which does differ by sector.

## 5.4 Labor flows

Employment in sector $s$ is given by

$$N_t^s = \int N_{j,t}^s dj.$$

$$N_{j,t+1}^s = (1-\rho^s)N_{j,t}^s + q_t v_{j,t}^s.$$

$$N_{t+1}^s = (1-\rho^s)N_t^s + H_t^s.$$

New hires in sector $s$ are $q_t^s v_t^s$

$$N_{t+1}^s = (1-\rho^s)N_t^s + q_t^s v_t^s \Rightarrow H = \rho N$$

so

$$\hat{n}_{t+1}^s = (1-\rho^s)\hat{n}_t^s + \rho^s\left(\hat{q}_t^s + \hat{v}_t^s\right)$$

Aggregate employment is

$$N_t = N_t^1 + N_t^2.$$

The number of job seekers is

$$U_t = 1 - N_t + \rho N_t.$$

Job seekers last employed in sector $s$ evolve as

$$U_t^s = \rho N_t^s + U_{t-1}^s - r_{t-1} U_{t-1}^s = \rho N_t^s + (1 - r_{t-1}) U_{t-1}^s.$$

Then

$$\begin{aligned} U_t &= U_t^1 + U_t^2 = \rho N_t + (1 - r_{t-1}) U_{t-1} \\ &= \rho N_{t-1} + (1 - r_t)(1 - N_{t-1} + \rho N_{t-1}) \\ &= (1 - r_t) - (1 - r_t)(1 - \rho) N_{t-1} \end{aligned}$$

### 5.4.1 The matching function

Total matches in sector $s$ are a function of the number of vacancies in the sector and the number of unemployed workers. We assume that the number of matches actually made depends on the fraction of the unemployed who worked in sector $s$. Specifically,

$$M_t^s = (v_t^s)^a \left[\zeta (u_t^s)^\delta + (1 - \zeta) \left(u_t^{k \neq s}\right)^\delta\right]^{\frac{1-a}{\delta}}$$

with $0 < \mu < 1$. If we define $\lambda^s = u^s / u$ as the fraction of the unemployed who last worked in sector $s$, then

$$\begin{aligned} M_t^s &= (v_t^s)^a u_t^{1-a} \left[\zeta v \left(\frac{u_t^s}{u_t}\right)^\delta + (1 - \zeta)\left(\frac{u_t^{k \neq s}}{u_t}\right)^\delta\right]^{\frac{1-a}{\delta}} \\ &= (v_t^s)^a u_t^{1-a} \left[\zeta (\lambda_t^s)^\delta + (1 - \zeta)(1 - \lambda_t^s)^\delta\right]^{\frac{1-a}{\delta}} \\ &= (v_t^s)^a u_t^{1-a} g(\lambda_t^s) \\ &= (v_t)^a u_t^{1-a} g(\lambda_t^s) \left(\frac{v_t^s}{v_t}\right)^a \end{aligned}$$

Total matches are

$$M_t = M_t^1 + M_t^2 = (v_t)^a u_t^{1-a} \left[g(\lambda_t^1)\left(\frac{v_t^1}{v_t}\right)^a + g(\lambda_t^2)\left(\frac{v_t^2}{v_t}\right)^a\right].$$

Then

$$\begin{aligned} q_t^s &= \frac{M_t^s}{v_t^s} = \frac{(v_t)^a u_t^{1-a} g(\lambda_t^s)}{v_t^s} \left(\frac{v_t^s}{v_t}\right)^a \\ &= \theta_t^{a-1} g(\lambda_t^s) \left(\frac{v_t^s}{v_t}\right)^{a-1} \end{aligned}$$

where $\theta_t = v_t/u_t$ is the aggregate measure of labor market tightness.

For a worker, the probability of finding a job is

$$\begin{aligned} r_t &= \frac{M_t}{u_t} = (v_t)^a u_t^{1-a} \frac{g(\lambda_t^1)\left(\frac{v_t^1}{v_t}\right)^a + g(\lambda_t^2)\left(\frac{v_t^2}{v_t}\right)^a}{u_t} \\ &= \theta_t^a \left[ g(\lambda_t^1)\left(\frac{v_t^1}{v_t}\right)^a + g(\lambda_t^2)\left(\frac{v_t^2}{v_t}\right)^a \right] \end{aligned}$$

while the probability of finding a job in sector $s$ is

$$r_t^s = r_t \left(\frac{M_t^s}{M_t}\right)$$

and

$$\begin{aligned} \frac{r_t^s}{q_t^s} &= r_t \left(\frac{M_t^s}{M_t}\right)\left(\frac{v_t^s}{M_t^s}\right) = r_t \left(\frac{v_t}{M_t}\right)\left(\frac{v_t^s}{v_t}\right) \\ &= \left(\frac{r_t}{q_t}\right)\left(\frac{v_t^s}{v_t}\right) = \theta_t \left(\frac{v_t^s}{v_t}\right). \end{aligned}$$

## 5.5 Price setting

Following Thomas (2009), define $mc_{j,t/T}^s$ as the real marginal cost of firm $j$ in sector 2 at time $T$ if that firm last reset price in period $t$.

The pricing decision at time $t$ for firm $j$ in sector 2 maximizes (once demand curve has been used to eliminate $Y_{j,t}^s$)

$$\max_{P_{j,t}^s} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} D_{t,T} \left\{ \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_{j,t}^s}{P_T^s}\right]^{1-\theta^s} C_T^s - mc_{j,t/T}^s \left[\frac{P_{j,t}^s}{P_T^s}\right]^{-\theta^s} C_T^s \right\}$$

First order condition is

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left\{ \left(\frac{P_T^s}{P_T}\right) \left[\frac{1}{P_T^s}\right]^{1-\theta^s} \left(P_{j,t}^{*s}\right)^{-\theta^s} - \mu_s mc_{j,t/T}^s \left(P_{j,t}^{*s}\right)^{-1-\theta^s} \left[\frac{1}{P_T^s}\right]^{-\theta^s} \right\} C_T^s = 0$$

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left\{ \left(\frac{P_T^s}{P_T}\right) \left[\frac{1}{P_T^s}\right]^{1-\theta^s} - \mu_s mc_{j,t/T}^s \left(P_{j,t}^{*s}\right)^{-1} \left[\frac{1}{P_T^s}\right]^{-\theta^s} \right\} C_T^s = 0$$

where

$$\mu_s \equiv \left(\frac{\theta^s}{\theta^s - 1}\right) > 1$$

is the markup in sector $s$.

We can also write this first order condtion as

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left\{ \left(\frac{P_T^s}{P_T}\right) \left[\frac{1}{P_T^s}\right]^{1-\theta^s} P_{j,t}^{*s} - \mu_s mc_{j,t/T}^s \left[\frac{1}{P_T^s}\right]^{-\theta^s} \right\} C_T^s = 0$$

or

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{1}{P_T^s}\right]^{1-\theta^s} P_{j,t}^{*s} C_T^s = \mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{1}{P_T^s}\right]^{-\theta^s} C_T^s$$

So the LHS is

$$\begin{aligned}
\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{1}{P_T^s}\right]^{1-\theta^s} P_{j,t}^{*s} C_T^s &= \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_t^s}{P_T^s}\right]^{1-\theta^s} \left[\frac{1}{P_t^s}\right]^{1-\theta^s} P_{j,t}^{*s} C_T^s \\
&= \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_t^s}{P_T^s}\right]^{1-\theta^s} \left[\frac{1}{P_t^s}\right]^{-\theta^s} \frac{P_{j,t}^{*s}}{P_t^s} C_T^s \\
&= \left[\frac{1}{P_t^s}\right]^{-\theta^s} \frac{P_{j,t}^{*s}}{P_t^s} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_t^s}{P_T^s}\right]^{1-\theta^s} C_T^s
\end{aligned}$$

and the RHS becomes

$$\begin{aligned}
\mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{1}{P_T^s}\right]^{-\theta^s} C_T^s &= \mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{P_t^s}{P_T^s}\right]^{-\theta^s} \left[\frac{1}{P_t^s}\right]^{-\theta^s} C_T^s \\
&= \mu_s \left[\frac{1}{P_t^s}\right]^{-\theta^s} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{P_t^s}{P_T^s}\right]^{-\theta^s} C_T^s
\end{aligned}$$

so equating the LHS and RHS and canceling $(1/P_t^s)^{-\theta^s}$,

$$\frac{P_{j,t}^{*s}}{P_t^s} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_t^s}{P_T^s}\right]^{1-\theta^s} C_T^s = \mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{P_t^s}{P_T^s}\right]^{-\theta^s} C_T^s$$

or

$$\frac{P_{j,t}^{*s}}{P_t^s} = \frac{\mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{P_T^s}{P_t^s}\right]^{\theta^s} C_T^s}{\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_T^s}{P_t^s}\right]^{\theta^s-1} C_T^s}$$

**Remark 2** *Under flexible prices,*

$$\frac{P_{j,t}^{*s}}{P_t^s} = 1 = \left(\frac{P_T}{P_T^s}\right) \mu_s mc_{j,t}^s \Rightarrow mc_{j,t}^s = \left(\frac{P_T^s}{P_T}\right)\left(\frac{1}{\mu_s}\right),$$

*so marginal cost depends on the markup and the relative price of the sector.*

We can also write this first order condtion as

$$\frac{P_{j,t}^{*s}}{P_t^s} \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left(\frac{P_T^s}{P_T}\right) \left[\frac{P_T^s}{P_t^s}\right]^{\theta^s-1} C_T^s = \mu_s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} mc_{j,t/T}^s \left[\frac{P_T^s}{P_t^s}\right]^{\theta^s} C_T^s$$

**Remark 3** *Linearizing this – start with LHS:*

$$\left(\frac{P^s}{P}\right) C^s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} D_{t,T} \left[1 + p_{j,t}^{*s} - p_t^s + p_T^s - p_T - (1-\theta^s)\left(p_T^s - p_t^s\right) + c_t^s\right]$$

**Remark 4** *Now RHS:*

$$\mu_s mc^s C^s \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \left[1 + \widehat{mc_{j,t/T}^s} + \theta^s \left(p_T^s - p_t^s\right) + c_T^s\right]$$

Equating the two sides and canceling the $c_T^s$, the $\theta^s \left(p_T^s - p_t^s\right)$ and the 1 terms,

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} D_{t,T} \left[p_{j,t}^{*s} - p_t^s + p_T^s - p_T - \left(p_T^s - p_t^s\right)\right] = \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \widehat{mc_{j,t/T}^s}$$

$$\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} D_{t,T} \left(p_{j,t}^{*s} - p_T\right) = \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_2^{T-t} D_{t,T} \widehat{mc_{j,t/T}^s}$$

$$\left(\frac{1}{1-\alpha_s\beta}\right) p_{j,t}^{*s} = \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} \beta^{T-t} (\widehat{mc_{j,t/T}^s} + p_T)$$

$$\begin{aligned} p_{j,t}^{*s} &= (1-\alpha_s\beta)\, \mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t} \beta^{T-t} (\widehat{mc_{j,t/T}^s} + p_T) \\ &= (1-\alpha_s\beta)\,(\widehat{mc_{j,t}^s} + p_t) + (1-\alpha_s\beta)\,\alpha_s\beta \mathrm{E}_t \sum_{T=t+1}^{\infty} \alpha_s^{T-t} \beta^{T-t} (\widehat{mc_{j,t/T}^s} + p_T) \\ &= (1-\alpha_s\beta)\,(\widehat{mc_{j,t}^s} + p_t) + \alpha_s\beta \mathrm{E}_t p_{j,t+1}^{*s} \end{aligned}$$

From (19),

$$mc_{j,t/T}^s = w' \left(\frac{C_{j,T}^s}{A_T^s N_{j,T}^s}\right) \frac{1}{A_T^s} = \frac{\left(h_{j,t/T}^s\right)^{\eta}}{A_T^s U_{C,T}}$$

and

$$C_{j,t}^s = A_t^s h_{j,t}^s N_{j,t}^s$$

so

$$mc_{j,t/T}^s = \frac{\left(h_{j,t/T}^s\right)^{\eta}}{A_T^s U_{C,T}} = \left[\frac{C_{j,t/T}^s}{A_T^s N_{j,t/T}^s}\right]^{\eta} \left(\frac{1}{A_T^s U_{C,T}}\right)$$

Log linearizing,

$$\widehat{mc_{j,t/T}^s} = \widehat{mc}_T + \eta \left(\hat{y}_{j.t/T}^s - \hat{y}_T^s\right) - \eta \left(\hat{n}_{j,t/T}^s - \hat{n}_T^s\right).$$

Furthermore,

$$\hat{y}^s_{j.t/T} = \hat{y}^s_T - \theta^s \left(p^{*s}_{j,t} - p^s_T\right)$$

so

$$\widehat{mc^s_{j,t/T}} = \widehat{mc}_T - \eta\theta^s \left(p^{*s}_{j,t} - p^s_T\right) - \eta \left(\hat{n}^s_{j,t/T} - \hat{n}^s_T\right).$$

Hence, we obtain

$$\begin{aligned} p^{*s}_{j,t} &= (1-\alpha_s\beta)\,\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t}\beta^{T-t}(\widehat{mc^s_{j,t/T}} + p_T) \\ &= (1-\alpha_s\beta)\,\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T - \eta\theta^s \left(p^{*s}_{j,t} - p^s_T\right) - \eta\left(\hat{n}^s_{j,t/T} - \hat{n}^s_T\right) + p_T\right] \end{aligned}$$

or

$$(1+\eta\theta^s)\,p^{*s}_{j,t} = (1-\alpha_s\beta)\,\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T + (1+\eta\theta^s)\,p^s_T - \eta\left(\hat{n}^s_{j,t/T} - \hat{n}^s_T\right) + p_T - p^s_T\right]$$

**Remark 5** *Compare to (27) in Thomas (2009) – new term is the relative price effect on the end.*

Solution method employed by Thomas (2009) follows Woodford (2005), and begins by guessing that

$$p^{*s}_{j,t} = p^{*s}_t - \tau^{*s}\tilde{n}_{j,t} \Rightarrow \tilde{p}^{*s}_{j,t} = -\tau^{*s}\tilde{n}_{j,t}, \tag{23}$$

where $\tilde{p}_j$ $(\tilde{n}_j)$ denotes the price (employment) of firm $j$ relative to sector average.

From the production function,

$$\hat{h}^s_{j,t} = \hat{y}^s_{j,t} - a^s_t - \hat{n}^s_{j,t}$$

so if $\tilde{h}^s_{j,t}$ is the hours at firm $j$ relative to sector average,

$$\begin{aligned} \tilde{h}^s_{j,t} &= \left(\hat{y}^s_{j,t} - \hat{y}^s_t\right) - \left(\hat{n}^s_{j,t} - \hat{n}^s_t\right) \\ &= -\theta^s\left(\hat{p}^s_{j,t} - p^s_t\right) - \left(\hat{n}^s_{j,t} - \hat{n}^s_t\right) \\ &= -\theta^s\tilde{p}^s_{j,t} - \tilde{n}^s_{j,t} \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{E}_t\tilde{p}^s_{j,t+1} &= \alpha_s\left(\hat{p}^s_{j,t} - \hat{p}^s_{t+1}\right) + (1-\alpha_s)\left(\hat{p}^{*s}_{j,t+1} - \hat{p}^s_{t+1}\right) \\ &= \alpha_s\left(\hat{p}^s_{j,t} - \hat{p}^s_{t+1} + \hat{p}^s_t - \hat{p}^s_t\right) + (1-\alpha_s)\left(\hat{p}^{*s}_{j,t+1} - \hat{p}^{*s}_{t+1} + \hat{p}^{*s}_{t+1} - \hat{p}^s_{t+1}\right) \\ &= \alpha_s\left(\hat{p}^s_{j,t} - \hat{p}^s_t - \pi^s_{t+1}\right) + (1-\alpha_s)\left(\hat{p}^{*s}_{j,t+1} - \hat{p}^{*s}_{t+1} + \hat{p}^{*s}_{t+1} - \hat{p}^s_{t+1}\right) \\ &= \alpha_s\left(\tilde{p}^s_{j,t} - \pi^s_{t+1}\right) + (1-\alpha_s)\left(\hat{p}^{*s}_{j,t+1} - \hat{p}^{*s}_{t+1} + \hat{p}^{*s}_{t+1} - \hat{p}^s_{t+1}\right) \end{aligned}$$

Then since

$$\begin{aligned} \hat{p}^s_t &= (1-\alpha_s)\hat{p}^{*s}_t + \alpha_s\hat{p}^s_{t-1} \\ &\Rightarrow \alpha_s\left(\hat{p}^s_t - \hat{p}^s_{t-1}\right) = (1-\alpha_s)\left(\hat{p}^{*s}_t - \hat{p}^s_t\right) \\ &\Rightarrow \pi^s_t = \left(\frac{1-\alpha_s}{\alpha_s}\right)\left(\hat{p}^{*s}_t - \hat{p}^s_t\right) \end{aligned}$$

we have

$$
\begin{aligned}
\mathrm{E}_t\tilde{p}^s_{j,t+1} &= \alpha_s\left(\tilde{p}^s_{j,t}-\pi^s_{t+1}\right)+(1-\alpha_s)\left[\hat{p}^{*s}_{j,t+1}-\hat{p}^{*s}_{t+1}+\left(\frac{\alpha_s}{1-\alpha_s}\right)\pi^{*s}_{t+1}\right] \\
&= \alpha_s\tilde{p}^s_{j,t}-(1-\alpha_s)\left(\hat{p}^{*s}_{j,t+1}-\hat{p}^{*s}_{t+1}\right) \\
&= \alpha_s\tilde{p}^s_{j,t}-(1-\alpha_s)\tau^{*s}\tilde{n}_{j,t+1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{E}_t\tilde{h}^s_{j,t+1} &= -\theta^s\mathrm{E}_t\tilde{p}^s_{j,t+1}-\tilde{n}^s_{j,t+1} \\
&= -\theta^s\left[\alpha_s\tilde{p}^s_{j,t}-(1-\alpha_s)\tau^{*s}\tilde{n}^s_{j,t+1}\right]-\tilde{n}^s_{j,t+1} \\
&= -\alpha_s\theta^s\tilde{p}^s_{j,t}-\left[1-(1-\alpha_s)\tau^{*s}\theta^s\right]\tilde{n}^s_{j,t+1}=0
\end{aligned}
$$

so

$$
\tilde{n}^s_{j,t+1}=-\left[\frac{\alpha_s\theta^s}{1-(1-\alpha_s)\tau^{*s}\theta^s}\right]\tilde{p}^s_{j,t}=-\tau^{ns}\tilde{p}^s_{j,t} \tag{24}
$$

which is Thomas's Proposition 1 (p. 13).

From the equation for $p^{*s}_{j,t}$, the term

$$
\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left[\left(\hat{n}^s_{j,t/T}-\hat{n}^s_T\right)\right]=\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\tilde{n}^s_{j,t/T},
$$

and

$$
\begin{aligned}
\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\tilde{n}^s_{j,t/T} &= \tilde{n}^s_{j,t}+\alpha_s\beta\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\tilde{n}^s_{j,t/T+1} \\
&= \tilde{n}^s_{j,t}-\tau^{ns}\alpha_s\beta\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left(p^{*s}_{j,t}-p^s_T\right)
\end{aligned}
$$

Using this result, we can re-write

$$
(1+\eta\theta^s)\,p^{*s}_{j,t}=(1-\alpha_s\beta)\,\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T+(1+\eta\theta^s)\,p^s_T-\eta\left(\hat{n}^s_{j,t/T}-\hat{n}^s_T\right)+p_T-p^s_T\right]
$$

as

$$
\begin{aligned}
(1+\eta\theta^s)\,p^{*s}_{j,t} &= (1-\alpha_s\beta)\,\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T+(1+\eta\theta^s)\,p^s_T+p_T-p^s_T\right] \\
&\quad -(1-\alpha_s\beta)\,\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\eta\left(\hat{n}^s_{j,t/T}-\hat{n}^s_T\right)
\end{aligned}
$$

**Remark 6** *This deviation is actually provided below. See (25).*

**Log linearize the vacancy posting condition** From (22),

$$\begin{aligned}
\left(\frac{\kappa}{q_t^s}\right) &= \beta \mathrm{E}_t \chi \left[\left(\frac{\eta}{1+\eta}\right)\left(h_{j,t+1}^s\right)^{1+\eta} - b\right] - \beta\left(1-\chi\right)\kappa \mathrm{E}_t \theta_{t+1} \\
&+\beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}^s}\right) + \beta(1-\rho^s)\mathrm{E}_t\left(\Phi_{t+1}^s\right) - \Phi_t^s \\
&-\beta \mathrm{E}_t \sum_{k=1,2}\left[\left(1-\chi\right)r_{t+1}^k\right]\Phi_{t+1}^k
\end{aligned}$$

Notice that nothing depends on $j$ except $h_{j,t+1}^s$, so expected hours must be independent of $j$. Therefore

$$\begin{aligned}
\left(\frac{\kappa}{q_t^s}\right) &= \beta \mathrm{E}_t \chi \left[\left(\frac{\eta}{1+\eta}\right)\left(h_{t+1}^s\right)^{1+\eta} - b\right] - \beta\left(1-\chi\right)\kappa \mathrm{E}_t \theta_{t+1} \\
&+\beta(1-\rho^s)\mathrm{E}_t\left(\frac{\kappa}{q_{t+1}^s}\right) + \beta(1-\rho^s)\mathrm{E}_t\left(\Phi_{t+1}^s\right) - \Phi_t^s \\
&-\beta \mathrm{E}_t \sum_{k=1,2}\left[\left(1-\chi\right)r_{t+1}^k\right]\Phi_{t+1}^k
\end{aligned}$$

$$\begin{aligned}
\frac{\kappa}{q^s}\left(1-\hat{q}_t^s\right) &= \beta\chi\left[\left(\frac{\eta}{1+\eta}\right)\left(h^s\right)^{1+\eta}\left(1+(1+\eta)\mathrm{E}_t\hat{h}_{t+1}^s\right)\right] - \beta\chi b \\
&+\beta(1-\rho^s)\left(\frac{\kappa}{q^s}\right)\mathrm{E}_t\left(1-\hat{q}_{t+1}^s\right) + \beta(1-\rho^s)\Phi^s\mathrm{E}_t\left(1+\hat{\phi}_{t+1}^s\right) - \Phi^s(1+\hat{\phi}_t^s) \\
&-\beta\left(1-\chi\right)\mathrm{E}_t\sum_{k=1,2}\left[\frac{1}{2}r^k\Phi^k(1+\hat{r}_{t+1}^k)\left(1+\hat{\phi}_{t+1}^k\right)\right].
\end{aligned}$$

In the symmetric steady state,

$$\begin{aligned}
\frac{\kappa}{q} &= \beta\chi\left[\left(\frac{\eta\left(h^s\right)^{1+\eta}}{1+\eta}\right) - b\right] \\
&+\beta(1-\rho)\frac{\kappa}{q} + \beta(1-\rho^s)\Phi - \Phi \\
&-\beta\left(1-\chi\right)r\Phi
\end{aligned}$$

So when the steady-state is symmetric,

$$\begin{aligned}
-\left(\frac{\kappa}{q}\right)\hat{q}_t^s &= \beta\chi\eta\left(h^s\right)^{1+\eta}\mathrm{E}_t\hat{h}_{t+1}^s - \beta(1-\rho)\left(\frac{\kappa}{q}\right)\mathrm{E}_t\hat{q}_{t+1}^s \\
&+\beta(1-\rho^s)\Phi\mathrm{E}_t\hat{\phi}_{t+1}^s - \Phi\hat{\phi}_t^s \\
&-\beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\sum_{k=1,2}(\hat{r}_{t+1}^k + \hat{\phi}_{t+1}^k).
\end{aligned}$$

From the matching function,

$$q_t^s = \theta_t^{a-1} g\left(\lambda_t^s\right) \left(\frac{v_t^s}{v_t}\right)^{a-1}$$

where

$$g(\lambda_t^s) = \left[\zeta \left(\lambda_t^s\right)^\delta + (1-\zeta)\left(1-\lambda_t^s\right)^\delta\right]^{\frac{1-a}{\delta}}$$

so

$$\hat{q}_t^s = (a-1)\hat{\theta}_t + (a-1)\left(\hat{v}_t^s - \hat{v}_t\right) + \hat{g}_t^s$$

and in a symmetric strady state,

$$\hat{\lambda}_t^1 = -\hat{\lambda}_t^2$$

$$\begin{aligned} g^{\frac{\delta}{1-a}}\left(1+\hat{g}_t^s\right)^{\frac{\delta}{1-a}} &= \left[\zeta(\lambda^s)^\delta\left(1+\delta\hat{\lambda}_t^s\right) + (1-\zeta)\left(\lambda^{k\neq s}\right)^\delta\left(1+\delta\lambda_t^{k\neq s}\right)\right] \\ &= (\lambda)^\delta\left[1+\zeta\delta\hat{\lambda}_t^s + (1-\zeta)\delta\hat{\lambda}_t^{k\neq s}\right] \\ &= (\lambda)^\delta\left[1+\zeta\delta\hat{\lambda}_t^s - (1-\zeta)\delta\hat{\lambda}_t^s\right] \\ &= (\lambda)^\delta - (\lambda)^\delta(1-2\zeta)\,\delta\hat{\lambda}_t^s \end{aligned}$$

Since in the symmetric steady state,

$$\begin{aligned} g^{\frac{\delta}{1-a}} &= \left[\zeta\lambda^\delta + (1-\zeta)\lambda^\delta\right] \\ &= \left[\zeta + (1-\zeta)\right]\lambda^\delta = \lambda^\delta \end{aligned}$$

we have

$$\left(\frac{\delta}{1-a}\right)\hat{g}_t^s = -(1-2\zeta)\,\delta\hat{\lambda}_t^s \Rightarrow \hat{g}_t^s = (a-1)(1-2\zeta)\,\hat{\lambda}_t^s$$

Therefore, since

$$\hat{\lambda}_t^s = \hat{u}_t^s - \hat{u}_t$$

$$\begin{aligned} \hat{q}_t^s &= (a-1)\hat{\theta}_t + (a-1)\left(\hat{v}_t^s - \hat{v}_t\right) + (a-1)(1-2\zeta)\,\hat{\lambda}_t^s \\ &= (1-a)\hat{u}_t + (a-1)\hat{v}_t^s + (a-1)(1-2\zeta)\,\hat{\lambda}_t^s \\ &= (a-1)\left[\hat{v}_t^s - \hat{u}_t + (1-2\zeta)\left(\hat{u}_t^s - \hat{u}_t\right)\right] \end{aligned}$$

so

$$\hat{q}_t^s = (a-1)\left[\hat{\theta}_t^s + (1-2\zeta)\left(\hat{u}_t^s - \hat{u}_t\right)\right].$$

**Remark 7** *If $\zeta = 1/2$, get standard result that $\hat{q}^s = (a-1)\hat{\theta}^s$.*

**Remark 8** *If $\zeta > 1/2$, there is "home bias" in employment and as $\zeta$ increases, the effects of $\hat{u}$ ($\hat{u}^s$) decrease (increase) on the probability a firm in sector $s$ can fill a vacancy. For example, if $\zeta = 1/2$,*

$$\hat{q}_t^s = (1-a)\hat{u}_t + (a-1)\hat{v}_t^s$$

*and only total unemployment matters, while if $\zeta = 1$,*

$$\hat{q}_t^s = (1-a)\hat{u}_t^s + (a-1)\hat{v}_t^s$$

*and only sector $s$ unemployment matters.*

We also have

$$\frac{r_t^s}{q_t^s} = \theta_t \left( \frac{v_t^s}{v_t} \right),$$

so

$$\hat{r}_t^s = \hat{q}_t^s + \hat{\theta}_t + \hat{v}_t^s - \hat{v}_t = \hat{q}_t^s + \hat{v}_t^s - \hat{u}_t$$

Therefore

$$\begin{aligned} \hat{q}_t^s &= -\left[ \frac{q\beta\chi\eta\,(h^s)^{1+\eta}}{\kappa} \right] \mathrm{E}_t \hat{h}_{t+1}^s + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^s \\ &\quad +\beta \left[ \frac{1}{2}(1-\chi)\,r - (1-\rho^s) \right] \Phi \mathrm{E}_t \hat{\phi}_{t+1}^s + \beta(1-\chi)\frac{1}{2} r\Phi \mathrm{E}_t \hat{\phi}_{t+1}^{k\neq s} + \Phi\hat{\phi}_t^s \\ &\quad +\beta(1-\chi)\frac{1}{2} r\Phi \mathrm{E}_t \sum_{k=1,2} (\hat{q}_{t+1}^k + \hat{v}_{t+1}^k - \hat{u}_{t+1}). \end{aligned}$$

$$\begin{aligned} \hat{q}_t^s &= -\left[ \frac{q\beta\chi\eta\,(h^s)^{1+\eta}}{\kappa} \right] \mathrm{E}_t \hat{h}_{t+1}^s + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^s \\ &\quad +\beta \left[ \frac{1}{2}(1-\chi)\,r - (1-\rho^s) \right] \Phi \mathrm{E}_t \hat{\phi}_{t+1}^s + \beta(1-\chi)\frac{1}{2} r\Phi \mathrm{E}_t \hat{\phi}_{t+1}^{k\neq s} \\ &\quad +\Phi\hat{\phi}_t^s + \beta(1-\chi)\frac{1}{2} r\Phi \mathrm{E}_t \sum_{k=1,2} (\hat{q}_{t+1}^k + \hat{v}_{t+1}^k) \\ &\quad -\beta(1-\chi)\frac{1}{2} r\Phi \mathrm{E}_t \hat{u}_{t+1}. \end{aligned}$$

From the production function,

$$\hat{h}_t^s = \hat{y}_t^s - a_t^s - \hat{n}_t^s$$

so

$$\begin{aligned}
\hat{q}_t^s &= -\left[\frac{q\beta\chi\eta\left(h^s\right)^{1+\eta}}{\kappa}\right]\mathrm{E}_t\left(\hat{y}_{t+1}^s - a_{t+1}^s - \hat{n}_{t+1}^s\right) + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^s \\
&+\beta\left[\frac{1}{2}\left(1-\chi\right)r - \left(1-\rho^s\right)\right]\Phi\mathrm{E}_t\hat{\phi}_{t+1}^s + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{\phi}_{t+1}^{k\neq s} \\
&+\Phi\hat{\phi}_t^s + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\sum_{k=1,2}\left(\hat{q}_{t+1}^k + \hat{v}_{t+1}^k\right) \\
&-\beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{u}_{t+1}.
\end{aligned}$$

**Linearization of job finding rate**

$$r_t = \theta_t^a\left[g\left(\lambda_t^1\right)\left(\frac{v_t^1}{v_t}\right)^a + g\left(\lambda_t^2\right)\left(\frac{v_t^2}{v_t}\right)^a\right]$$

$$r = 2\left(\frac{1}{2}\right)^a\theta^a g\left(\lambda\right)$$

$$\begin{aligned}
r\left(1+\hat{r}_t\right) &= \left(\frac{1}{2}\right)^a\theta^a\left(1+a\hat{\theta}_t\right)\left\{g\left(\lambda_t^1\right)\left[1+a\left(\hat{v}_t^1-\hat{v}_t\right)\right] + g\left(\lambda_t^2\right)\left[1+a\left(\hat{v}_t^2-\hat{v}_t\right)\right]\right\} \\
&= \left(\frac{1}{2}\right)^a\theta^a g\left(\lambda\right)\left(1+a\hat{\theta}_t\right)\left\{\begin{array}{c}\left(1+\hat{g}_t^1\right)\left[1+a\left(\hat{v}_t^1-\hat{v}_t\right)\right] \\ +\left(1+\hat{g}_t^2\right)\left[1+a\left(\hat{v}_t^2-\hat{v}_t\right)\right]\end{array}\right\}
\end{aligned}$$

$$\begin{aligned}
\left(1+\hat{r}_t\right) &= \left(\frac{1}{2}\right)^a\theta^a\left(1+a\hat{\theta}_t\right)\left\{g\left(\lambda_t^1\right)\left[1+a\left(\hat{v}_t^1-\hat{v}_t\right)\right] + g\left(\lambda_t^2\right)\left[1+a\left(\hat{v}_t^2-\hat{v}_t\right)\right]\right\} \\
&= \left(\frac{1}{2}\right)\left(1+a\hat{\theta}_t\right)\left\{1+\hat{g}_t^1+a\left(\hat{v}_t^1-\hat{v}_t\right)+1+\hat{g}_t^2+a\left(\hat{v}_t^2-\hat{v}_t\right)\right\} \\
&= \left(\frac{1}{2}\right)\left\{2+2a\hat{\theta}_t+\hat{g}_t^1+a\left(\hat{v}_t^1-\hat{v}_t\right)+\hat{g}_t^2+a\left(\hat{v}_t^2-\hat{v}_t\right)\right\}
\end{aligned}$$

$$\hat{r}_t = \left(\frac{1}{2}\right)\left[2a\hat{\theta}_t+\hat{g}_t^1+a\left(\hat{v}_t^1-\hat{v}_t\right)+\hat{g}_t^2+a\left(\hat{v}_t^2-\hat{v}_t\right)\right]$$

$$\hat{g}_t^s = \left(a-1\right)\left(1-2\zeta\right)\left(\hat{u}_t^s-\hat{u}_t\right)$$

which implies

$$\begin{aligned}
\left(\frac{1}{2}\right)\left(\hat{g}_t^1+\hat{g}_t^2\right) &= \left(a-1\right)\left(1-2\zeta\right)\left(\frac{1}{2}\right)\left(\hat{u}_t^1-\hat{u}_t+\hat{u}_t^2-\hat{u}_t\right) \\
&= \left(a-1\right)\left(1-2\zeta\right)\left[\left(\frac{1}{2}\right)\left(\hat{u}_t^1+\hat{u}_t^2\right)-\hat{u}_t\right] \\
&= 0
\end{aligned}$$

So

$$
\begin{aligned}
\hat{r}_t &= \left(\frac{1}{2}\right)\left[2a\hat{\theta}_t + a\left(\hat{v}_t^1 - \hat{v}_t\right) + a\left(\hat{v}_t^2 - \hat{v}_t\right)\right] \\
&= \left(\frac{1}{2}\right)\left[2a\hat{\theta}_t + a\left(\hat{v}_t^1 + \hat{v}_t^2\right)\right] - a\hat{v}_t \\
&= a\hat{\theta}_t
\end{aligned}
$$

which is just as in a single-sector model.

**Labor adjustment costs** Labor adjustment costs are

$$\Phi_t^s = \bar{\Phi}^s - \frac{\vartheta}{1+\epsilon}\left(\lambda_t^s\right)^{1+\epsilon},$$

$$\Phi(1+\hat{\phi}_t^s) = \Phi^s - \frac{\vartheta}{1+\epsilon}\left(\bar{\lambda}\right)^{1+\epsilon}\left[1 + (1+\epsilon)\,\hat{\lambda}_t^s\right]$$

$$\hat{\phi}_t^s = -\frac{\vartheta\left(\bar{\lambda}\right)^{1+\epsilon}}{\Phi}\left(\hat{\lambda}_t^s\right),\ \Phi = \Phi^s - \frac{\vartheta}{1+\epsilon}\left(\bar{\lambda}\right)^{1+\epsilon} = \Phi^s - \frac{\vartheta}{1+\epsilon}\left(\frac{1}{2}\right)^{1+\epsilon}$$

So

$$\hat{\phi}_t^s = -\varepsilon\left(\hat{u}_t^s - \hat{u}_t\right);\ \varepsilon = \frac{\vartheta\left(\bar{\lambda}\right)^{1+\epsilon}}{\Phi}$$

**Log linear system** Complete pricing system:

$$
\begin{aligned}
\left(1+\eta\theta^s\right)p_{j,t}^{*s} &= \left(1-\alpha_s\beta\right)\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left(\widehat{mc}_T + \left(1+\eta\theta^s\right)p_T^s - \eta\left(\hat{n}_{j,t/T}^s - \hat{n}_T^s\right) + p_T - p_T^s\right) \\
&= \left(1-\alpha_s\beta\right)\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T + \left(1+\eta\theta^s\right)p_T^s - \eta\tilde{n}_{j,t/T}^s + p_T - p_T^s\right]
\end{aligned}
$$

Since

$$\tilde{n}_{j,t+1}^s = -\tau^{ns}\tilde{p}_{j,t}^s$$

$$
\begin{aligned}
\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\tilde{n}_{j,t/T}^s &= \tilde{n}_{j,t}^s + \alpha_s\beta\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\tilde{n}_{j,t/T+1}^s \\
&= \tilde{n}_{j,t}^s - \tau^{ns}\alpha_s\beta\mathrm{E}_t\sum_{T=t}^{\infty}\alpha_s^{T-t}\beta^{T-t}\left(\hat{p}_{j,t}^s - \hat{p}_T^s\right)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(1+\eta\theta^{s})\,p_{j,t}^{*s} &= (1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(1+\eta\theta^{s})\,p_{T}^{s}-\eta\tilde{n}_{j,t/T}^{s}+p_{T}-p_{T}^{s}\right] \\
&= (1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(1+\eta\theta^{s})\,p_{T}^{s}+p_{T}-p_{T}^{s}\right] \\
&\quad +\eta\tau^{ns}\alpha_{s}\beta\,(1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left(\hat{p}_{j,t}^{s}-\hat{p}_{T}^{s}\right)-(1-\alpha_{s}\beta)\,\eta\tilde{n}_{j,t}^{s}
\end{aligned}
$$

$$
\begin{aligned}
(1+\eta\theta^{s})\,p_{j,t}^{*s} &= (1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(1+\eta\theta^{s})\,p_{T}^{s}+p_{T}-p_{T}^{s}\right] \\
&\quad -(1-\alpha_{s}\beta)\,\eta\tilde{n}_{j,t}^{s}+\eta\tau^{ns}\alpha_{s}\beta\hat{p}_{j,t}^{s}-\eta\tau^{ns}\alpha_{s}\beta\,(1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\hat{p}_{T}^{s} \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
(1+\eta\theta^{s}-\eta\tau^{ns}\alpha_{s}\beta)\,p_{j,t}^{*s} &= (1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(p_{T}-p_{T}^{s})\right]-(1-\alpha_{s}\beta)\,\eta\tilde{n}_{j,t}^{s} \\
&\quad +(1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[(1+\eta\theta^{s}-\eta\tau^{ns}\alpha_{s}\beta)\right]\hat{p}_{T}^{s}
\end{aligned}
$$

$$
(1+\phi)\,p_{j,t}^{*s}=(1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(1+\bar{\eta}^{s})\,\hat{p}_{T}^{s}+(p_{T}-p_{T}^{s})\right]-(1-\alpha_{s}\beta)\,\eta\tilde{n}_{j,t}^{s}. \qquad (26)
$$

where

$$
\bar{\eta}^{s}\equiv\eta\,(\theta^{s}-\tau^{ns}\alpha_{s}\beta)\,.
$$

Averaging across $j$ and since adjusters are chosen at random, we obtain

$$
(1+\phi^{s})\,p_{t}^{*s}=(1-\alpha_{s}\beta)\,\mathrm{E}_{t}\sum_{T=t}^{\infty}\alpha_{s}^{T-t}\beta^{T-t}\left[\widehat{mc}_{T}+(1+\bar{\eta}^{s})\,\hat{p}_{T}^{s}+(p_{T}-p_{T}^{s})\right]. \qquad (27)
$$

Subtracting (27) from (26),

$$
(1+\bar{\eta}^{s})\,\tilde{p}_{j,t}^{*s}=-\eta\tilde{n}_{j,t}^{s},
$$

which is consistent with (23) iff

$$
\tilde{p}_{j,t}^{*s}=-\tau^{*s}\hat{n}_{j,t}=-\left[\frac{(1-\alpha_{s}\beta)\,\eta}{1+\bar{\eta}^{s}}\right]\tilde{n}_{j,t}^{s}
$$

or

$$
\tau^{*s}=\left[\frac{(1-\alpha_{s}\beta)\,\eta}{1+\bar{\eta}^{s}}\right]=\left[\frac{(1-\alpha_{s}\beta)\,\eta}{1+\eta\theta^{s}-\eta\tau^{ns}\alpha_{s}\beta}\right]
$$

From (24),

$$\tau^{ns} = \left[\frac{\alpha_s \theta^s}{1-(1-\alpha_s)\tau^{*s}\theta^s}\right]$$

so these two equations need to be jointly solved for $\tau^{*s}$ and $\tau^{ns}$.

**Phillips curve** From (26) and $\hat{p}_t^{*s} - \hat{p}_t^s = [\alpha_s/(1-\alpha_s)]\,\pi_t^s$, and

$$(1-\alpha_s)\left(\hat{p}_{t+1}^{*s} - \hat{p}_t^s\right) = \pi_{t+1}^s,$$

and

$$\begin{aligned}
(1+\bar{\eta}^s)\,\hat{p}_t^{*s} &= (1-\alpha_s\beta)\,\mathrm{E}_t \sum_{T=t}^{\infty} \alpha_s^{T-t}\beta^{T-t}\left[\widehat{mc}_T + (1+\bar{\eta}^s)\,\hat{p}_T^s + (p_T - p_T^s)\right] \\
&= (1-\alpha_s\beta)\left[\widehat{mc}_t + (1+\bar{\eta}^s)\,\hat{p}_t^s + (p_t - p_t^s)\right] \\
&\quad + \alpha_s\beta\mathrm{E}_t\,(1+\bar{\eta}^s)\,\hat{p}_{t+1}^{*s}
\end{aligned}$$

we can get

$$\begin{aligned}
(1+\bar{\eta}^s)\,(\hat{p}_t^{*s} - \hat{p}_t^s) &= (1-\alpha_s\beta)\left[\widehat{mc}_t + (p_t - p_t^s)\right] \\
&\quad + \alpha_s\beta\mathrm{E}_t\,(1+\bar{\eta}^s)\left(\hat{p}_{t+1}^{*s} - \hat{p}_t^s\right) \\
&\quad + (1-\alpha_s\beta)\,(1+\bar{\eta}^s)\,\hat{p}_t^s + \alpha_s\beta(1+\bar{\eta}^s)\hat{p}_t^s - (1+\bar{\eta}^s)\,\hat{p}_t^s
\end{aligned}$$

$$\begin{aligned}
(1+\bar{\eta}^s)\left(\frac{\alpha_s}{1-\alpha_s}\right)\pi_t^s &= (1-\alpha_s\beta)\left[\widehat{mc}_t + (p_t - p_t^s)\right] \\
&\quad + \alpha_s\beta\mathrm{E}_t\,(1+\bar{\eta}^s)\left(\frac{1}{1-\alpha_s}\right)\pi_{t+1}^{*s}
\end{aligned}$$

$$\pi_t^s = \left(\frac{\kappa^s}{1+\bar{\eta}^s}\right)\left[\widehat{mc}_t + (p_t - p_t^s)\right] + \beta\mathrm{E}_t\pi_{t+1}^{*s}$$

where

$$\kappa^s \equiv \frac{(1-\alpha_s)\,(1-\alpha_s\beta)}{\alpha_s}.$$

## 5.6 Linearized model

Euler condition:

$$\hat{y}_t = \mathrm{E}_t \hat{y}_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t \pi_{t+1})$$

Goods market:

$$\hat{y}_t^1 = -a\left(\hat{p}_t^1 - \hat{p}_t\right) + \hat{y}_t$$

$$\hat{y}_t^2 = -a\left(\hat{p}_t^2 - \hat{p}_t\right) + \hat{y}_t$$

Inflation/prices:

$$\pi_t^1 = \left(\frac{\kappa^1}{1+\phi^1}\right)\left[\widehat{mc}_t^1 + \left(p_t - p_t^1\right)\right] + \beta \mathrm{E}_t \pi_{t+1}^{*1}$$

$$\pi_t^2 = \left(\frac{\kappa^2}{1+\phi^2}\right)\left[\widehat{mc}_t^2 + \left(p_t - p_t^2\right)\right] + \beta \mathrm{E}_t \pi_{t+1}^{*2}$$

$$\hat{p}_t = \left(\frac{P^1}{P}\right)\hat{p}_t^1 + \left(\frac{P^2}{P}\right)\hat{p}_t^2$$

$$\hat{p}_t = \hat{p}_{t-1} + \pi_t$$

$$\hat{p}_t^1 = \hat{p}_{t-1}^1 + \pi_t^1$$

$$\hat{p}_t^2 = \hat{p}_{t-1}^2 + \pi_t^2$$

$$\pi_t = \hat{p}_t - \hat{p}_{t-1}$$

Employment:

$$\hat{h}_t^1 = \hat{y}_t^1 - a_t^1 - \hat{n}_t^1$$

$$\hat{h}_t^s = \hat{y}_t^s - a_t^s - \hat{n}_t^s$$

$$\hat{n}_{t+1}^1 = (1-\rho^1)\hat{n}_t^1 + \rho^1\left(\hat{q}_t^1 + \hat{v}_t^1\right)$$

$$\hat{n}_{t+1}^2 = (1-\rho^2)\hat{n}_t^2 + \rho^1\left(\hat{q}_t^2 + \hat{v}_t^2\right)$$

$$\hat{q}_t^1 = (a-1)\hat{\theta}_t^1 + 2\,(1-\zeta)\,\hat{u}_t^1.$$

$$\hat{q}_t^2 = (a-1)\hat{\theta}_t^2 + 2\,(1-\zeta)\,\hat{u}_t^2.$$

$$\hat{r}_t^1 = \hat{q}_t^1 + \hat{v}_t^1 - \hat{u}_t$$

$$\hat{r}_t^2 = \hat{q}_t^2 + \hat{v}_t^2 - \hat{u}_t$$

$$\hat{\lambda}_t^1 = \hat{u}_t^1 - \hat{u}_t$$

$$\hat{\lambda}_t^2 = \hat{u}_t^2 - \hat{u}_t$$

$$\hat{\theta}_t^1 = \hat{v}_t^1 - \hat{u}_t$$

$$\hat{\theta}_t^2 = \hat{v}_t^2 - \hat{u}_t$$

$$\begin{aligned}
\hat{q}_t^1 &= -\left[\frac{q\beta\chi\eta\left(h^s\right)^{1+\eta}}{\kappa}\right]\mathrm{E}_t\hat{h}_{t+1}^1 + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^1 \\
&\quad +\beta\left[\frac{1}{2}\left(1-\chi\right)r - \left(1-\rho^s\right)\right]\Phi\mathrm{E}_t\hat{\phi}_{t+1}^1 + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{\phi}_{t+1}^2 \\
&\quad +\Phi\hat{\phi}_t^1 + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\sum_{k=1,2}\left(\hat{q}_{t+1}^k + \hat{v}_{t+1}^k\right) \\
&\quad -\beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{u}_{t+1}.
\end{aligned}$$

$$\begin{aligned}
\hat{q}_t^2 &= -\left[\frac{q\beta\chi\eta\left(h^s\right)^{1+\eta}}{\kappa}\right]\mathrm{E}_t\hat{h}_{t+1}^2 + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^2 \\
&\quad +\beta\left[\frac{1}{2}\left(1-\chi\right)r - \left(1-\rho^s\right)\right]\Phi\mathrm{E}_t\hat{\phi}_{t+1}^2 + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{\phi}_{t+1}^1 \\
&\quad +\Phi\hat{\phi}_t^2 + \beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\sum_{k=1,2}\left(\hat{q}_{t+1}^k + \hat{v}_{t+1}^k\right) \\
&\quad -\beta\left(1-\chi\right)\frac{1}{2}r\Phi\mathrm{E}_t\hat{u}_{t+1}.
\end{aligned}$$

**Remark 9** *Note that each of these can be rewritten as*

$$\begin{aligned}
\hat{q}_t^1 &= -\left[\frac{q\beta\chi\eta\left(h^s\right)^{1+\eta}}{\kappa}\right]\mathrm{E}_t\hat{h}_{t+1}^1 + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^1 \\
&\quad -\beta(1-\rho^s)\Phi\mathrm{E}_t\hat{\phi}_{t+1}^1 + \Phi\hat{\phi}_t^1 + \mathrm{E}_t\hat{x}_{t+1}.
\end{aligned}$$

$$\begin{aligned}
\hat{q}_t^2 &= -\left[\frac{q\beta\chi\eta\left(h^s\right)^{1+\eta}}{\kappa}\right]\mathrm{E}_t\hat{h}_{t+1}^2 + \beta(1-\rho)\mathrm{E}_t\hat{q}_{t+1}^2 \\
&\quad -\beta(1-\rho^s)\Phi\mathrm{E}_t\hat{\phi}_{t+1}^2 + \Phi\hat{\phi}_t^2 + \mathrm{E}_t\hat{x}_{t+1}.
\end{aligned}$$

$$\begin{aligned}
\hat{x}_t &\equiv \frac{1}{2}\beta\left(1-\chi\right)r\Phi\left[\sum_{k=1,2}\left(\hat{\phi}_t^k + \hat{q}_t^k + \hat{v}_t^k\right) - \hat{u}_t\right] \\
&= \beta\left(1-\chi\right)r\Phi\left[\frac{1}{2}\sum_{k=1,2}\left(\hat{\phi}_t^k + \hat{q}_t^k\right) + \hat{\theta}_t\right].
\end{aligned}$$

$$\hat{u}_t^1 = \rho\left(\frac{N^1}{U^1}\right)\hat{n}_{t-1}^1 + \left(1-r\right)\hat{u}_{t-1}^1 - r\hat{r}_t$$

$$\hat{u}_t^2 = \rho \left( \frac{N^2}{U^2} \right) \hat{n}_{t-1}^2 + (1 - r) \, \hat{u}_{t-1}^2 - r \hat{r}_t$$

$$r_t = a \hat{\theta}_t = a \left( \hat{v}_t - \hat{u}_t \right)$$

$$\hat{u}_t = \left( \frac{1}{2} \right) \left( \hat{u}_t^1 + \hat{u}_t^2 \right)$$

$$\hat{n}_t = \left( \frac{1}{2} \right) \left( \hat{n}_t^1 + \hat{n}_t^2 \right)$$

$$\hat{v}_t = \left( \frac{1}{2} \right) \left( \hat{v}_t^1 + \hat{v}_t^2 \right)$$

$$\hat{\theta}_t = \hat{v}_t - \hat{u}_t$$

Marginal costs:

$$\widehat{mc}_t^1 = (\eta + \sigma) \, \hat{y}_t^1 - (1 + \eta) a_t^1 - \eta \hat{n}_t^1$$

$$\widehat{mc}_t^2 = (\eta + \sigma) \, \hat{y}_t^2 - (1 + \eta) a_t^2 - \eta \hat{n}_t^2$$

Management costs:

$$\hat{\phi}_t^1 = (1 + \epsilon) \hat{\lambda}_t^2$$

$$\hat{\phi}_t^2 = (1 + \epsilon) \hat{\lambda}_t^1$$

Policy rule:

$$i_t = \phi_\pi \pi_t$$