Optimal Monetary Policy with the Cost Channel: Appendix (not for publication)

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1 Derivations for section 2.1

1.1 The flexible-price equilibrium output (eq. 9)

When price are flexible, equations (4) and (7) of the text imply that equilibrium requires

$$\frac{\chi N_t^{\eta}}{\xi_t C_t^{-\sigma}} = w_t = \left(\frac{\theta - 1}{\theta}\right)\frac{A_t}{R_t} = \frac{A_t}{\Phi R_t}, \tag{1}$$

where $\Phi = \theta/(\theta - 1)$ is the markup. From the resource constraint, $C_t = \gamma_t Y_t$, and from teh production function, $Y_t = A_t N_t$. Thus, we can rewrite (1) as

$$\frac{\chi (Y_t/A_t)^{\eta}}{\xi_t (\gamma_t Y)_t^{-\sigma}} = \frac{A_t}{\Phi R_t}.$$

Solving for Y yields the expression for equilibrium output under flexible prices:

$$Y_t^f = \left(\frac{\xi_t \gamma_t^{-\sigma} A_t^{1+\eta}}{\chi \Phi R_t^f}\right)^{\frac{1}{\sigma+\eta}},$$

which is equation (9) of the text.

1.2 The social planner's problem under flexible prices (eq. 12)

We solve a simple one-period version of the social planner's problem to show how the fiscal variable γ_t affects the efficient level of output.

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The social planner's problem is

$$\max U(C,N) + \lambda\left[AN - C - G\right] + \mu\left[(1-\gamma)AN - G\right].$$

First order conditions are

$$U_c - \lambda = 0$$
$$U_N + \lambda A + \mu(1-\gamma)A = 0$$
$$-\lambda - \mu = 0$$

Eliminating the Lagrangian multipliers implies

$$\frac{U_N}{U_c} = \frac{\lambda A + \mu(1-\gamma)A}{\lambda} = \frac{\lambda A - \lambda(1-\gamma)A}{\lambda} = \gamma A. \tag{2}$$

In teh text, the utility function is assumed to take the form

$$\frac{\xi_{t+i}C_{t+i}^{1-\sigma}}{1-\sigma} - \chi\frac{N_{t+i}^{1+\eta}}{1+\eta}.$$

Evaluating (2) yields

$$\frac{\chi N^{\eta}}{\xi C^{-\sigma}} = \frac{\chi(Y/A)^{\eta}}{\xi\gamma^{-\sigma}Y^{-\sigma}} = \gamma A \Rightarrow Y^{**} = \left(\frac{\xi\gamma^{1-\sigma}A^{1+\eta}}{\chi}\right)^{\frac{1}{\eta+\sigma}} = \gamma^{\frac{1}{\eta+\sigma}}Y^{f},$$

where Y^f is the flex-price output level when $\Phi = R^f = 0$.

1.3 The real interest rate in the Euler equation (eq. 14)

Using the resource constraint, the Euler condition (3) can be linearized around the steady state to yield

$$\hat{\xi}_t - \sigma\hat{C}_t = \mathrm{E}_t\left(\hat{\xi}_{t+1} - \sigma\hat{C}_{t+1}\right) + \hat{r}_t.$$

Using the resource contraint ($C_t = \gamma_t Y_t$), this can be written as

$$\hat{Y}_t = -\left(\frac{1}{\sigma}\right)\mathrm{E}_t\left(\hat{\xi}_{t+1} - \hat{\xi}_t\right) + \mathrm{E}_t(\hat{\gamma}_{t+1} - \hat{\gamma}_t) + \mathrm{E}_t\hat{Y}_{t+1} - \left(\frac{1}{\sigma}\right)\hat{r}_t.$$

Expressed in terms of the output gap, the Euler condition becomes

$$\hat{Y}_t - \hat{Y}_t^f = -\left(\frac{1}{\sigma}\right)\mathrm{E}_t\left(\hat{\xi}_{t+1} - \hat{\xi}_t\right) + \mathrm{E}_t(\hat{\gamma}_{t+1} - \hat{\gamma}_t) + \mathrm{E}_t\left(\hat{Y}_{t+1} - \hat{Y}_{t+1}^f\right) + \mathrm{E}_t\left(\hat{Y}_{t+1}^f - \hat{Y}_t^f\right) - \left(\frac{1}{\sigma}\right)\hat{r}_t.$$

Finally, defining

$$\hat{r}_t^f = \sigma\left(\mathrm{E}_t\hat{Y}_{t+1}^f - \hat{Y}_t^f\right) - \left(\mathrm{E}_t\hat{\xi}_{t+1} - \hat{\xi}_t\right) + \sigma\left(\mathrm{E}_t\hat{\gamma}_{t+1} - \hat{\gamma}_t\right), \tag{3}$$

the Euler condition can be written as

$$\hat{Y}_t - \hat{Y}_t^f = \mathrm{E}_t\left(\hat{Y}_{t+1} - \hat{Y}_{t+1}^f\right) - \left(\frac{1}{\sigma}\right)\left(r_t - r_t^f\right),$$

where $r_t = \hat{R}_t - \mathrm{E}_t\pi_{t+1}$. This is equation (14) of the text.

Equation (3) shows that the real interest rate in the flexible-price equilibrium will be affected by productivity (via $\mathrm{E}_t\hat{Y}_{t+1}^f - \hat{Y}_t^f$), taste, and fiscal shocks, unless these shocks follow random walk processes.

2 Welfare approximation

2.1 The loss function

In this appendix, the approximation to the welfare of the representative household is derived. In doing so, we follow Woodford (1999). Our model differs from his in three ways. First, we assume the utility of the representative household depends on consumption and leisure, while Woodford assumes it depends on consumption and output, where the role of output is to capture the disutility of work. This change does not affect the results. The second, more substantive change, is that we allow explicitly for stochastic variation in the share of output going to the government. Finally, the steady-state values in our model are not independent of monetary policy (as they are in Woodford) because the level of the nominal interest rate in the steady-state affects equilibrium output and employment.

We make use of the following notation:

Notation	
$\bar{X}$	Steady-state value
X_t^*	Efficient level
X_t^f	Flex-price equilibrium level
$\tilde{X}_t$	$X_t - \bar{X}$
$\hat{X}_t$	$\log X_t - \log \bar{X}$

Given this notation,

$$\frac{X_t}{\bar{X}} \approx 1 + \log\left(\frac{X_t}{\bar{X}}\right) + \frac{1}{2}\left[\log\left(\frac{X_t}{\bar{X}}\right)\right]^2 = 1 + \hat{X}_t + \frac{1}{2}\hat{X}_t^2.$$

Because one can always write $\tilde{X}_t = \bar{X}\left(\frac{X_t}{\bar{X}} - 1\right)$, it follows that $\tilde{X}_t \approx \bar{X}\left(\hat{X}_t + \frac{1}{2}\hat{X}_t^2\right)$.

Utility is assumed to be separable in consumption and leisure and takes the form

$$\mathrm{E}_t \sum_{i=0}^{\infty} \beta^i \left[\frac{\xi_{t+i} C_{t+i}^{1-\sigma}}{1-\sigma} - \chi \frac{N_{t+i}^{1+\eta}}{1+\eta}\right]. \tag{4}$$

We begin by approximating the utility of consumption. The second order Taylor expansion for $U(C_t, \xi_t)$ is

$$U(C_t,\xi_t) \approx U(\bar{C},1)+U_c(\bar{C},1)\tilde{C}_t+\frac{1}{2}U_{cc}(\bar{C},1)\tilde{C}_t^2+U_{\ \xi}(\bar{C},1)\tilde{\xi}_t+\frac{1}{2}U_{\xi,\xi}\tilde{\xi}_t^2+U_{c,\xi}\tilde{\xi}_t\tilde{C}_t.$$

We assume the government purchases individual goods in the same proportions as households and that aggregate government purchases are proportional to output; $G_t = (1-\gamma_t)Y_t$, where γ_t is stochastic and bounded between zero and one. The aggregate resource constraint then takes the form $Y_t = C_t + G_t = C_t + (1-\gamma_t)Y_t$, or

$$C_t = \gamma_t Y_t. \tag{5}$$

This implies that $\tilde{C}_t \approx \bar{\gamma}\bar{Y}\left[\hat{\gamma}_t + \hat{Y}_t + \frac{1}{2}\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right]$. Given the utility function specification (4), the utility of consumption becomes

$$\begin{aligned} U(C_t,\xi_t) \quad \approx \quad & U(\bar{C},1) + U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[\hat{\gamma}_t + \hat{Y}_t + \frac{1}{2}\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right] \\ & -\frac{1}{2}\sigma U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[\hat{\gamma}_t + \hat{Y}_t + \frac{1}{2}\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right]^2 \\ & +U_{\ \xi}(\bar{C},1)\hat{\xi}_t + \frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2 + U_c(\bar{C})\bar{\gamma}\bar{Y}\hat{\xi}_t\left[\hat{\gamma}_t + \hat{Y}_t + \frac{1}{2}\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right]. \end{aligned}$$

Ignoring terms of order X^i for $i > 2$,

$$\begin{aligned} U(C_t,\xi_t) \quad \approx \quad & U(\bar{C},1) + U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[(1+\hat{\xi}_t)\left(\hat{\gamma}_t + \hat{Y}_t\right) + \frac{1}{2}(1-\sigma)\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right] \\ & +U_\xi(\bar{C},1)\hat{\xi}_t + \frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2. \end{aligned} \tag{6}$$

The next step is to obtain an approximation for the disutility of work. The second order Taylor expansion for $V(N_t)$ is

$$V(N_t) \approx V(\bar{N}) + V_N(\bar{N})\tilde{N}_t + \frac{1}{2}V_{NN}(\bar{N})\tilde{N}_t^2 \tag{7}$$

where aggregate employment is

$$\tilde{N}_t = \int_0^1 \tilde{n}_t(i)di.$$

For employment at firm i,

$$\tilde{n}_t(i) \approx \bar{n}\left[\hat{n}_t(i) + \frac{1}{2}\hat{n}_t(i)^2\right].$$

Each firm has a production technology given by $y_t(i) = A_t n_t(i)$. Hence,

$$\hat{n}_t(i) = \hat{y}_t(i) - \hat{A}_t.$$

We can then write

$$\begin{aligned}\tilde{N}_t &\approx \int_0^1 \tilde{n}_t(i)di = \bar{n}\int_0^1 \left[\hat{n}_t(i) + \frac{1}{2}\hat{n}_t(i)^2\right] di \\ &= \bar{y}\left[\int_0^1 \hat{y}_t(i)di - \hat{A}_t + \frac{1}{2}\int_0^1 \left(\hat{y}_t(i) - \hat{A}_t\right)^2 di\right].\end{aligned}$$

Substituting this into (7), and ignoring terms of order X^2 and higher powers,

$$\begin{aligned}V(N_t) &\approx V(\bar{N}) + V_N(\bar{N})\bar{y}\left[\int_0^1 \hat{y}_t(i)di - \hat{A}_t + \frac{1}{2}\int_0^1 \left(\hat{y}_t(i) - \hat{A}_t\right)^2 di\right] \\ &\quad + \frac{1}{2}V_{NN}(\bar{N})\bar{y}^2\left[\int_0^1 \hat{y}_t(i)di - \hat{A}_t\right]^2. \qquad (8)\end{aligned}$$

Given the demand functions facing each individual firm, the aggregate output variable Y_t is defined as

$$Y_t = \int_0^1 y_t(i)^{\frac{\theta-1}{\theta}} di.$$

This implies

$$\hat{Y}_t \approx \int_0^1 \hat{y}_t(i)di + \frac{1}{2}\left(\frac{\theta-1}{\theta}\right) var_i \hat{y}_t(i).$$

Hence,

$$\left[\int_0^1 \hat{y}_t(i)di\right]^2 = \left[\hat{Y}_t - \frac{1}{2}\left(\frac{\theta-1}{\theta}\right) var_i \hat{y}_t(i)\right]^2 \approx \hat{Y}_t^2.$$

Note also that

$$\int_0^1 \hat{y}_t(i)^2 di = \left[\int_0^1 \hat{y}_t(i)di\right]^2 + var_i \hat{y}_t(i).$$

Therefore,

$$\int_0^1 \hat{y}_t(i)^2 di \approx \hat{Y}_t^2 + var_i \hat{y}_t(i).$$

In addition,

$$\hat{A}_t \int_0^1 \hat{y}_t(i)di \approx \hat{A}_t\hat{Y}_t - \frac{1}{2}\hat{A}_t\left(\frac{\theta-1}{\theta}\right) var_i \hat{y}_t(i) \approx \hat{A}_t\hat{Y}_t.$$

Using these results, equation (8) becomes

$$\begin{aligned}V(N_t) &\approx V(\bar{N}) + V_N(\bar{N})\bar{y}\left[\hat{Y}_t - \hat{A}_t - \frac{1}{2}\left(\frac{\theta-1}{\theta}\right) var_i \hat{y}_t(i)\right] \\ &\quad + V_N(\bar{N})\bar{y}\left[\frac{1}{2}\left(\hat{Y}_t^2 + var_i \hat{y}_t(i)\right) - \hat{A}_t\hat{Y}_t + \frac{1}{2}\hat{A}_t^2 di\right] \\ &\quad + \frac{1}{2}V_{NN}(\bar{N})\bar{y}^2\left(\hat{Y}_t - \hat{A}_t\right)^2. \qquad (9)\end{aligned}$$

Combining terms, and using the utility function (4),

$$V(N_t) \approx V(\bar{N}) + V_N(\bar{N})\bar{y}\left[\hat{Y}_t - \hat{A}_t + \frac{1}{2}\left(\frac{1}{\theta}\right) var_i \hat{y}_t(i) + \frac{1}{2}(1+\eta)\left(\hat{Y}_t - \hat{A}_t\right)^2\right]. \tag{10}$$

Combining equations (6) and (10),

$$\begin{aligned} U(C_t,\xi_t) - V(N_t) &= U(\bar{C},1) - V(\bar{N}) \\ &+ U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[(1+\hat{\xi}_t)\left(\hat{\gamma}_t + \hat{Y}_t\right) + \frac{1}{2}(1-\sigma)\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right] \\ &+ U_{\ \xi}(\bar{C},1)\hat{\xi}_t + \frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2 \\ &- V_N(\bar{N})\left[\bar{y}\hat{Y}_t - \hat{A}_t + \frac{1}{2}\left(\frac{1}{\theta}\right) var_i \hat{y}_t(i)\right. \\ &\left. + \frac{1}{2}(1+\eta)\left(\hat{Y}_t - \hat{A}_t\right)^2\right] \end{aligned} \tag{11}$$

Before simplifying this expression, note that the steady-state labor market equilibrium condition becomes $\bar{V}_N/\bar{U}_c = \bar{w} = 1/\Phi\bar{R}$. If we define Θ such that

$$1 - \Theta \equiv \frac{1}{\bar{\gamma}\Phi\bar{R}},$$

then $V_N(\bar{N})\bar{y}$ can be written as $U_c(\bar{C})\bar{\gamma}\bar{Y}(1-\Theta)$. We will assume Θ is small so that terms such as $\left(\bar{\gamma}\Phi\bar{R}\right)^{-1}\hat{Y}_t^2 = (1-\Theta)\hat{Y}_t^2$ become simply $\hat{Y}_t^2$.[1] In this case, we can now write equation (**??**) as

$$\begin{aligned} U(C_t,\xi_t) - V(N_t) &\approx U(\bar{C},1) - V(\bar{N}) + U_c(\bar{C})\bar{\gamma}\bar{Y}\left[(1+\hat{\xi}_t)\left(\hat{\gamma}_t + \hat{Y}_t\right) + \frac{1}{2}(1-\sigma)\left(\hat{\gamma}_t + \hat{Y}_t\right)^2\right] \\ &- U_c(\bar{C})\bar{\gamma}\bar{Y}(1-\Theta)\left[\hat{Y}_t - \hat{A}_t + \frac{1}{2}(1+\eta)\left(\hat{Y}_t - \hat{A}_t\right)^2\right] \\ &- \frac{1}{2}U_c(\bar{C})\bar{\gamma}\bar{Y}(1-\Theta)\left(\frac{1}{\theta}\right) var_i \hat{y}_t(i) + U_{\ \xi}(\bar{C},1)\hat{\xi}_t + \frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2. \end{aligned}$$

Collecting terms,

$$\begin{aligned} U(C_t,\xi_t) - V(N_t) &\approx U(\bar{C},1) - V(\bar{N}) + U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[\Theta\hat{Y}_t + \hat{\xi}_t\hat{Y}_t\right] \\ &+ U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[\frac{1}{2}(1-\sigma)\left(\hat{\gamma}_t + \hat{Y}_t\right)^2 - \frac{1}{2}(1+\eta)\left(\hat{Y}_t - \hat{A}_t\right)^2\right] \\ &- \frac{1}{2}U_c(\bar{C},1)\bar{Y}\left(\frac{1}{\theta}\right) var_i \hat{y}_t(i) \\ &+ U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[(1+\hat{\xi}_t)\hat{\gamma}_t + (1-\Theta)\hat{A}_t\right] + U_{\ \xi}(\bar{C},1)\hat{\xi}_t + \frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2. \end{aligned}$$

[1] This is stronger than the corresponding assumption made by Woodford (1999). He assumes $(\Phi)^{-1}$ is small. The presence of $\bar{\gamma}$ and a positive steady-state nominal interest rate increase the average distortions in the economy relative to the case in which the monopoly distortion is the only source of inefficiency.

Define

$$\hat{Z}_t \equiv \left[\frac{(1+\eta)\hat{A}_t + \hat{\xi}_t + (1-\sigma)\hat{\gamma}_t}{\sigma+\eta}\right]$$

and

$$z^* = \frac{\Theta}{\sigma+\eta}.$$

Then the utility approximation can be written as

$$\begin{aligned} U(\gamma_t Y_t, \xi_t) - V(N_t) \quad \approx \quad & U(\bar{Y},1) - V(\bar{N}) - \frac{1}{2}(\sigma+\eta)\, U_c(\bar{Y})\bar{Y}\left(\hat{Y}_t - \hat{Z}_t - z^*\right)^2 \\ & -\frac{1}{2}U_c(\bar{Y})\bar{Y}\left(\frac{1}{\theta}\right) var_i \hat{y}_t(i) + t.i.s.p.. \end{aligned} \tag{12}$$

where

$$\begin{aligned} t.i.s.p. \quad = \quad & U_c(\bar{C},1)\bar{\gamma}\bar{Y}\left[(1+\hat{\xi}_t)\hat{\gamma}_t + (1-\Theta)\hat{A}_t\right] + U_{\ \xi}(\bar{C},1)\hat{\xi}_t \\ & +\frac{1}{2}U_{\xi,\xi}\hat{\xi}_t^2 - \frac{1}{2}U_c(\bar{C},1)\bar{Y}c\hat{Z}_t^2 \end{aligned}$$

are terms independent of stabilization policy.

Recalling that

$$\hat{Y}_t^f = \left(\frac{1+\eta}{\sigma+\eta}\right)\hat{A}_t - \left(\frac{1}{\sigma+\eta}\right)\left(\hat{R}_t^f - \hat{\xi}_t\right) - \left(\frac{\sigma}{\sigma+\eta}\right)\hat{\gamma}_t,$$

$\hat{Z}_t$ can be written as

$$\hat{Z}_t = \hat{Y}_t^f + \left(\frac{1}{\sigma+\eta}\right)\left(\hat{R}_t^f + \hat{\gamma}_t\right).$$

With the assumed utility function,

$$\log y_t(i) = \log Y_t - \theta\left(\log p_t(i) - \log P_t\right)$$

so

$$var_i \log y_t(i) = \theta^2 var_i \log p_t(i)$$

The price adjustment mechanism involves a randomly chosen fraction $1-\omega$ of all firms optimally adjusting price each period. Define $\bar{P}_t \equiv \mathrm{E}_t \log p_t(i)$ and $\Delta_t \equiv var_i \log p_t(i)$. Then Woodford (2000) shows that

$$\Delta_t \approx \omega\Delta_{t-1} + \left(\frac{\omega}{1-\omega}\right)\pi_t^2,$$

where $1-\omega$ is the fraction of firms that reset their price each period. If Δ_{-1} is the initial degree of price dispersion, then

$$\sum_{t=0}^{\infty}\beta^t\Delta_t = \left[\frac{\omega}{(1-\omega)(1-\omega\beta)}\right]\sum_{t=0}^{\infty}\beta^t\pi_t^2 + t.i.p.,$$

where $t.i.p.$ denotes terms independent of monetary policy.

Combining this with (12), the present discounted value of the utility of the representative household can be approximated by

$$\sum_{t=0}^{\infty} \beta^t \mathrm{U}_t \approx \bar{\mathrm{U}} - \Omega \sum_{t=0}^{\infty} \beta^t L_t$$

where

$$L_t = \pi_t^2 + \lambda \left(\hat{Y}_t - \hat{Z}_t - z^* \right)^2,$$

$$\Omega = \frac{1}{2} U_c \bar{Y} \left[\frac{\omega}{(1-\omega)(1-\omega\beta)} \right] \theta,$$

and

$$\lambda = \left[\frac{(1-\omega)(1-\omega\beta)}{\omega} \right] \left(\frac{\sigma+\eta}{\theta} \right) = \frac{\kappa\,(\sigma+\eta)}{\theta}.$$

The parameter κ is the coefficient on real marginal costs in the inflation adjustment equation.

2.2 The demand shock in eq. 26

In equation (26) of the text, a demand shock u_t appears. Using the results from section 1.3 above, and replacing Y^f with Y^*,

$$\hat{Y}_t - \hat{Y}_t^* = \mathrm{E}_t \left(\hat{Y}_{t+1} - \hat{Y}_{t+1}^* \right) - \left(\frac{1}{\sigma} \right) (r_t - r_t^*),$$

where

$$\hat{r}_t^* = \sigma \left(\mathrm{E}_t \hat{Y}_{t+1}^* - \hat{Y}_t^* \right) - \left(\mathrm{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t \right) + \sigma \left(\mathrm{E}_t \hat{\gamma}_{t+1} - \hat{\gamma}_t \right).$$

From the definition of $\hat{Y}_t^*$ given in equation (24),

$$\mathrm{E}_t \hat{Y}_{t+1}^* - \hat{Y}_t^* = \left(\frac{1}{\sigma+\eta} \right) \left[(1+\eta)(\mathrm{E}_t \hat{A}_{t+1} - \hat{A}_t) - \sigma(\mathrm{E}_t \hat{\gamma}_{t+1} - \hat{\gamma}_t) + (\mathrm{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t) \right]$$

Hence, the Euler condition can be written as

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma} \right) (\hat{R}_t - \mathrm{E}_t \pi_{t+1}) + u_t \tag{13}$$

where

$$\begin{aligned} u_t &\equiv \left(\frac{1}{\sigma} \right) r_t^* = \left(\mathrm{E}_t \hat{Y}_{t+1}^* - \hat{Y}_t^* \right) - \left(\frac{1}{\sigma} \right) \left(\mathrm{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t \right) + \left(\mathrm{E}_t \hat{\gamma}_{t+1} - \hat{\gamma}_t \right) \\ &= \left(\frac{1}{\sigma+\eta} \right) \left[(1+\eta)(\mathrm{E}_t \hat{A}_{t+1} - \hat{A}_t) - \left(\frac{\eta}{\sigma} \right) (\mathrm{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t) + \frac{\eta}{1+\eta} (\mathrm{E}_t \hat{\gamma}_{t+1} - \hat{\gamma}_t) \right], \end{aligned}$$

which is equation (29) of the text.

3 Optimal monetary policy

In order to contrast results with and without the cost channel, it will be useful to introduce the index variable δ defined as

$$\delta = \begin{cases} 0 \text{ if there is no cost channel} \\ 1 \text{ if there is a cost channel} \end{cases}$$

and to write the inflation adjustment equation as

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa(\sigma + \eta) x_t + \delta \kappa \hat{R}_t, \tag{14}$$

3.1 Discretion: derivation of equation 30

The problem of the central banker is to choose a path for $\hat{R}_t$, and the implied paths for x_t and π_t, to maximize

$$U_t \equiv -\frac{1}{2} \mathrm{E}_t \left\{ \sum_{i=0}^{\infty} \left[\pi_{t+i}^2 + \lambda \left(x_{t+i} - \frac{1}{\sigma + \eta} \hat{\gamma}_t \right)^2 \right] \right\}$$

subject to

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma} \right) (\hat{R}_t - \mathrm{E}_t \pi_{t+1}) + u_t$$

and

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa(\sigma + \eta) x_t + \delta \kappa \hat{R}_t,$$

where

$$u_t \equiv \left(\frac{1 + \eta}{\sigma + \eta} \right) \left[(\mathrm{E}_t \hat{A}_{t+1} - \hat{A}_t) - \left(\frac{\eta}{\sigma} \right) (\mathrm{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t) + \frac{\eta}{1 + \eta} (\mathrm{E}_t \hat{\gamma}_{t+1} - \hat{\gamma}_t) \right].$$

Let χ_t and ψ_t denote the Lagrangian multipliers associated with each of these constraints at time t. Under optimal discretion, the first order conditions for the central bank's problem are:

$$\begin{aligned} \frac{\partial U_t}{\partial x_t} &= -\lambda \left(x_t - \frac{1}{\sigma + \eta} \hat{\gamma}_t \right) - \kappa(\sigma + \eta) \psi_t + \chi_t = 0; \\ \frac{\partial U_t}{\partial \pi_t} &= -\pi_t + \psi_t = 0; \\ \frac{\partial U_t}{\partial \hat{R}_t} &= -\delta \kappa \psi_t + \left(\frac{1}{\sigma} \right) \chi_t = 0. \end{aligned}$$

From the last of these, $\chi_t = \sigma \delta \kappa \psi_t$ so that $\chi_t = 0$ in the absence of a cost channel ($\delta = 0$). From the second first order condition, $\psi_t = \pi_t$. Using these results in the first of the first order conditions yields

$$\pi_t = -\left(\frac{\lambda}{\kappa \left[\sigma(1 - \delta) + \eta \right]} \right) \left[x_t - \left(\frac{1}{\sigma + \eta} \right) \hat{\gamma}_t \right], \tag{15}$$

which is equation (30) of the text for $\delta = 1$.

The equilibrium behavior of inflation is found by substituting equation (15) into the constraints (13) and (14), and solving the resulting two-equation system.

3.2 Commitment: derivation of first order conditions under commitment

The fully optimal commitment policy involves a choice of current and future values of inflation, the output gap, and the nominal interest rate to maximize

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{E}_t\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2+\lambda\left[x_t-\left(\frac{1}{\sigma+\eta}\right)\hat{\gamma}_t\right]^2\right.\\
&+\chi_t\left[x_t-\mathrm{E}_t x_{t+1}+\frac{1}{\sigma}(\hat{R}_t-\mathrm{E}_t\pi_{t+1})-g_t\right]\\
&\left.+\psi_t\left[\pi_t-\beta\mathrm{E}_t\pi_{t+1}-\kappa(\sigma+\eta)x_t-\kappa\delta\hat{R}_t\right]\right\},
\end{aligned}
$$

where χ and ψ are Lagrangian multipliers. The basic problem of time inconsistency is illustrated by contrasting the first order conditions for time t and for future periods $t+i$ for $i>0$. At time t, the central bank sets π_t, x_t, and $\hat{R}_t$ such that

$$\pi_t+\psi_t=0 \tag{16}$$

$$\lambda\left[x_t-\left(\frac{1}{\sigma+\eta}\right)\hat{\gamma}_t\right]+\chi_t-\kappa(\sigma+\eta)\psi_t=0 \tag{17}$$

$$\left(\frac{1}{\sigma}\right)\chi_t-\delta\kappa\psi_t=0, \tag{18}$$

while for $t+i>t$,

$$\pi_{t+i}+\left(\psi_{t+i}-\psi_{t+i-1}\right)-\chi_{t+i-1}\left(\frac{1}{\sigma}\right)=0 \tag{19}$$

$$\lambda\left[x_{t+i}-\left(\frac{1}{\sigma+\eta}\right)\hat{\gamma}_{t+i}\right]+\chi_{t+i}-\beta^{-1}\chi_{t+i-1}-\kappa(\sigma+\eta)\psi_{t+i}=0 \tag{20}$$

$$\left(\frac{1}{\sigma}\right)\chi_{t+i}-\delta\kappa\psi_{t+i}=0. \tag{21}$$

If $\delta=0$ (no cost channel), $\chi_{t+i}=0$ for all i and these first order conditions reduce to the case considered by Woodford (2000), Clarida, Galí, and Gertler (1999), or McCallum and Nelson (2000). When $\delta=1$, the nature of the time inconsistency inherent in this problem shows up in the comparisons of (16) to (19) and (17) to (20). These first order conditions for time t can be rewritten as

$$\pi_t=-\left(\frac{\lambda}{\kappa\eta}\right)\left[x_t-\left(\frac{1}{\sigma+\eta}\right)\hat{\gamma}_t\right]$$

and for $t+i$, $i>0$, as

$$\pi_{t+i}=-\psi_{t+i}+(1+\kappa)\,\psi_{t+i-1} \tag{22}$$

$$\lambda\left[x_{t+i}-\left(\frac{1}{\sigma+\eta}\right)\hat{\gamma}_{t+i}\right]=\beta^{-1}\sigma\kappa\psi_{t+i-1}+\kappa\eta\psi_{t+i} \tag{23}$$

Svensson and Woodford (1999) have described a policy that implements equations (22) and (23) for all t as the timeless perspective precommitment policy.

3.3 Simulations

To carry out the simulations, we write the model in state space form:

$$\begin{bmatrix}\hat{\gamma}_{t+1}\\ \hat{\xi}_{t+1}\\ \hat{a}_{t+1}\\ \hat{Y}_t^*\\ \hat{Y}_t^e\\ \tilde{x}_t\\ E_t x_{t+1}\\ E_t\pi_{t+1}\end{bmatrix}=A\begin{bmatrix}\hat{\gamma}_{t+1}\\ \hat{\xi}_{t+1}\\ \hat{a}_{t+1}\\ \hat{Y}_t^*\\ \hat{Y}_t^e\\ \tilde{x}_t\\ E_t x_{t+1}\\ E_t\pi_{t+1}\end{bmatrix}+\begin{bmatrix}0\\0\\0\\0\\0\\0\\ \frac{\beta+\kappa}{\sigma\beta}\\ \frac{1}{\beta}\end{bmatrix}R_t+\begin{bmatrix}u_{\gamma t+1}\\ u_{\xi_{t+1}}\\ u_{at+1}\\ 0\\0\\0\\0\\0\end{bmatrix},$$

where

$$A=\begin{bmatrix}\rho_\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \rho_\xi & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \rho_a & 0 & 0 & 0 & 0 & 0\\ -\frac{\sigma}{\sigma+\eta} & \frac{1}{\sigma+\eta} & \frac{1+\eta}{\sigma+\eta} & 0 & 0 & 0 & 0 & 0\\ \frac{1-\sigma}{\sigma+\eta} & \frac{1}{\sigma+\eta} & \frac{1+\eta}{\sigma+\eta} & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{\sigma+\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \frac{\eta(1-\rho_\gamma)}{\sigma+\eta} & \frac{-\eta(1-\rho_\xi)}{\sigma(\sigma+\eta)} & \frac{(1+\eta)(1-\rho_a)}{\sigma+\eta} & 0 & 0 & 0 & 1+\frac{\kappa(\sigma+\eta)}{\sigma\beta} & -\frac{1}{\sigma\beta}\\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\kappa(\sigma+\eta)}{\beta} & \frac{1}{\beta}\end{bmatrix}.$$

The equilibrium under optimal discretion and commitment are obtained using the programs of Paul Söderlind, available at http://home.tiscalinet.ch/paulsoderlind/.