How credible are inflation announcements at the ELB?*

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Abstract

Cochrane (2013) has emphazied the multiplicity of equilibrium at the effective lower bound (ELB) on nominal interest rates, with the economy's behavior at the ELB dependent on the inflation rate the central bank promises to delivery when the ELB is exited. If the probability of reverting to the ELB in the future is zero, any such promise is not sustainable as, absent the ability to commit, optimal discretion dominates delivering on the promise once the ELB episode has ended. However, if the economy may return to the ELB in the future, I show that that are welfare improving inflation promises that are sustainable even in the absence of a commitment technology.

1 Introduction

The current era of very low interest rates have raised troubling questions for central banks. If policy is constrained by an effective lower bound (ELB) for short-term interest rates, a binding ELB can be very costly in an environment of discretionary policy making, as shown for example by Adam and Billi (2007). In contrast, if the central bank is assumed able to commit itself to future policies, the costs of the ELB are small. For example, this is the conclusion of the work by Jung, Teranishi, and Watanabe (2005), Adam and Billi

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(2006) and Nakov (2008).[1] A central bank able to commit to future actions is not unduly constrained when its current policy rate is at its lower bound; making promises about the future path of the policy rate is sufficient to allow policymakers to influence economic activity effectively. If commitment is the appropriate way to understand the monetary policy environment, then the ELB does not call for any reform of inflation targeting or for raising the average inflation target.

Finding policy regimes that can limit the adverse effects of the ELB is important, as episodes of very low interest rates cannot, as they once were, be viewed as extremely rare events. Figure 1 shows histograms of U.S. short term interest rates. The top panel is based on the monthly effective federal funds rate from January 1960 to July 2016, while the lower panel is for the 3-month Treasury bill rate since 1934. Both show that a large fraction of months have seen rates below 25 basis points. For the shorter sample based on the funds rate, 13% of months since January 1960 have seen the funds rate at or below 25 basis points. For the longer period, the 3-month T-bill rate fell below 25 basis points in 17% of all months.[2] Recent estimates of the economy's natural rate of interest suggest it has declined, for example, see Holston, Laubach, Williams, Holston, and Williams (2016), implying nominal interest rates will average lower, making the probability of hitting the ELB higher, unless target inflation rates rise.

The possibility of recurring episodes at the ELB has important consequences for the credibility of policy promises. Most of the literature that has focused on the monetary policy consequences of the ELB has treated the credibility of the central bank as either complete, as in commitment equilibria, or totally absent, as in analyses of discretion. In the one case, future promises are fully believed and subsequently delivered on. In the latter case, the public places no weight on any promises the central bank might make. But even in the absence of a formal commitment mechanism, the possibility of a future ELB event may create incentives for a policy maker, acting with discretion, to fulfill past

[1] Reifschneider (2016) demonstrates the effectiveness of credible forward guidance (together with balance sheet policies) using the FRB/US model. Levin, López-Salido, Nelson, and Yun (2010) argue that forward guidance may be less effective in the face of large and persistent shocks that drive the economy to the ELB.

[2] This histogram is misleading in the sense that, to take the top panel, all the months at or below 25 basis points occurred consecutively between December 2008 and December 2015.

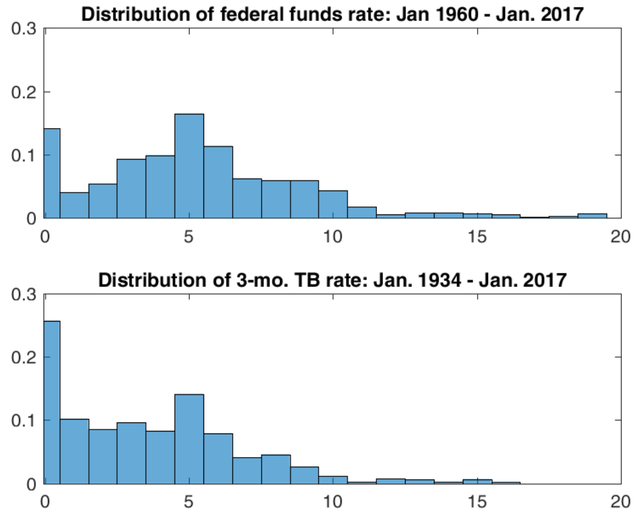

Figure 1: Histogram of U.S. interest rates. Upper panel: federal funds rate. Lower panel: 3-month T-Bill rate.

promises. Doing so brings a future benefit of credibility should the ELB again bind. In fact, Nakata (2014) has shown that the fully optimal Ramsey policy can be sustained if there is only a slight probability the ELB will occur in the future.

Despite Nakata's results, the literature has focused on discretionary environments in nonrecurring ELB episodes in which the central bank reverts to the time-consistent solution or to a simple Taylor rule as soon as the ELB no longer binds. This assumption allows a unique equilibrium at the ELB binds to be selected by uniquely determining the inflation rate once the ELB constraint is removed. In the analysis of discretion in *Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Rebelo (2011), Werning (2011), Braun, Körber, and Waki (2012), Kiley (2016), McKay, Nakamura, and Steinsson (2016b),* and Carlstrom, Fuerst, and Paustian (2012), for example, this terminal inflation rate equals zero. Cochrane (2013), in contrast, has argued the ELB periods are consistent with multiple equilibrium and that the assumption that inflation reverts to zero once the ELB constraint no longer binds is an equilibrium selection device. He shows that one can index equilibria by the inflation rate the central bank promises for the exit period.

In this paper, I investigate whether such promises of future inflation are credible in the absence of a commitment mechanism. A promise of future inflation is defined as sustainable if, at each point in time, the present discounted value of losses is less that is achieved by defecting to the optimal discretionary policy. I show that the multiplicity of equilibrium highlighted by Cochrane (2013) can be refined if the central bank's promise of future inflation is restricted to be sustainable. When the ELB will never again bind, the pure discretionary, time-consistent equilibrium with a zero inflation rate is the unique, sustainable equilibrium. When future episodes at the ELB may occur, some, but not all, exit inflation rates are sustainable under discretion, and so there remain multiple equilibria, but requiring exit inflation to be sustainable reduces the set of possible equilibria. Within the set of set of sustainable exit inflation rates, equilibria can be ranked by their welfare consequences. Under the assumption that the central bank maximizes welfare, a unique equilibrium is selected. The welfare maximizing exit inflation rate may not be sustainable, so discretion may involve the promise of a positive inflation rate upon exiting the ELB, but this rate may be less than the welfare maximizing but unsustainable rate.

Pure discretionary and optimal commitment are extreme alternatives. One implies a complete absence of credibility to fulfill promises; the other involves complete credibility. If future promises are credible even in a discretionary environment, the sharp distinction between discretion and commitment is blurred and credibility is no longer an all or nothing property of policy actions. Two literatures have developed approaches that allow for partial credibility. The first follows the stochastic planning problem analyzed by Roberds (1987), and includes the work by Schaumburg and Tambalotti (2007), Debortoli and Nunes (2010), Bodenstein, Hebden, and Nunes (2012), and Debortoli, Maih, and Nunes (2014). The second approach, and the one followed here, builds on notion of sustainable plans developed by Chari and Kehoe (1990) and Stokey (1991) and employed by Ireland (1997), Kurozumi (2008), Loisel (2008), Kurozumi (2012), and Nakata (2014).[3] That is, I assume the absence of any commitment technology. A past promise might be honored, but only if doing so is the best strategy for the policymaker at the time the promise needs to be honored.[4] Kurozumi (2008) has investigated whether the optimal

[3]In the presence of endogenous state variables, current policy choices can affect the incentives faced by future policymakers, thereby generating a channel through which the policymaker can effectively influence expectations about future policy. For example, Jeanne and Svensson (2007) have investigated how generating a large increase in the government's nominal debt can create an incentive for future inflation. Thus, a government's concerns about its balance sheet can provide a mechanism for current policy to influence future policy choices. This channel is absent in the present paper which employs a basic new Keynesian model in which there are no endogenous state variables. See also Eggertsson (2006).

[4]The sustainable plans literature builds on Abreu (1988). See also Levine, McAdam, and Pearlman

commitment policy in the basic new Keynesian model is sustainable under discretion. He shows that the optimal sustainable policy falls between that of optimal discretion and optimal commitment, but it converges over time to the optimal commitment policy if the policymaker's discount rate is not too large. Kurozumi (2012) shows that a regime of flexible inflation targeting is sustainable, but only if the central banker places more weight – but not too much weight – on inflation stability than is reflected in social welfare. That is, the central banker must be a Rogoff (1985) conservative, but not too conservative, while Loisel (2008) finds that a finite-period punishment can sustain a reputational equilibria that can overcome both an inflation bias and a stabilization bias in a new Keynesian model. These papers do not focus on issues related to the effective lower bound on nomina interest rates, the primary concern of the present paper.

The framework I use is similar to that employed by Nakata (2014), whose paper is closely related to the approach I adopt but whose focus differs somewhat. Nakata (2014) focuses on the sustainability of policy at the ELB and shows that the optimal Ramsey policy can be sustained if future episodes at the ELB can occur. However, as Svensson (2003) has noted, commitment to the optimal Ramsey rule is impractical, making it important to investigate the sustainability of simpler forms of forward guidance, such as the promise to deliver inflation once an ELB period ends. Walsh (2017) also uses a framework that allows for recurring episodes at the ELB but there the focus is on forward guidance about the future path of the nominal interest rate; here, it is on the sustainability of announcements about future inflation.

The rest of the paper is organized as follows. Section 2 modifies the basic framework of Eggertsson and Woodford (2003) to allow for a positive probability of recurring ELB episode and derives the optimal time-consistent policy under discretion. In section 3, forward guidance is interpreted as a promise to deliver a specific inflation rate when the ELB period ends, and the sustainability of the optimal inflation announcement is investigated in section 4. Conclusions are summarized in section 5.

2 The basic model, discretion, and sustainable policies

In the interests of tractability, I work with the simple Markov structure of Eggertsson and Woodford (2003). This model is briefly reviewed before extending it to allow for

(2008).

recurring episodes at the ELB.[5]

Assume the objective of policy is to minimize

$$L_t = \frac{1}{2}\mathrm{E}_t \sum_{j=0}^{\infty} \beta^j \left(\pi_{t+j}^2 + \lambda x_{t+j}^2\right). \tag{1}$$

A sustainable policy is defined as follows. Let L_j^o be the present value of losses when the economy is in state j under an arbitrary policy o. Let L_j^d denote the present value of losses in state j under the optimal time-consistent discretionary policy. In the present context, by optimal discretionary policy I mean the policy that, in each period, minimizes the policy maker's loss function, taking expectations and future policy as given. The policy o may involve promises made in the past about policy actions in the current state. The policy o is sustainable if $L_j^o < L_j^d$ for each j. That is, continuing to implement policy o, including any promises made in the past, constitutes a sustainable plan if the present value of losses obtained by implementing the policy is, in every state, less than that obtained by reverting to the policy d. A sustainable policy is time-consistent; the policymaker has no incentive to switch from the policy and adopt the discretionary policy.[6]

Consider a simple new Keynesian model, given by

$$x_t = \mathrm{E}_t x_{t+1} - \left(\frac{1}{\sigma}\right)(i_t - \mathrm{E}_t\pi_{t+1} - r_t) \tag{2}$$

$$\pi_t = \beta \mathrm{E}_t \pi_{t+1} + \kappa x_t \tag{3}$$

$$i_t \geq 0, \tag{4}$$

[5] A number of authors (Jung, Teranishi, and Watanabe (2005), Adam and Billi (2006), Adam and Billi (2007), Nakov (2008), Levin, López-Salido, Nelson, and Yun (2010), Billi (2015)) have examined stochastic equilibria in new Keynesian models subject to occasionally binding lower bounds on the nominal interest rate. In these models, the economy can pass into, out of, and back into periods during which the lower bound constraint is binding. However, this literature has not investigated specific examples of forward guidance. Work on forward guidance in stochastic models or on assessing the empirical effects of such policies include Campbell, Evans, Fisher, and Justiniano (2012) and Campbell (2016).

[6] The concept of a sustainable policy plans was first introduced by Chari and Kehoe (1990). Stokey (1991) defines a pair of strategies (for the government and private sector) that is compatible with a competitive equilibium in the private sector, given the government's strategy, and for which the government has no incentive to alter its strategy as a *credible policy*. See Nakata (2014) for a formal treatment of sustainability in the context of the Markov structure I employ.

together with a specification of monetary policy, where x_t is the output gap, π_t the inflation rate, i_t is the nominal interest rate, and r_t is an exogenous stochastic process.[7] For convenience the ELB on the nominal interest rate is taken to be zero. Any contingent sequence of inflation, the output gap, and the nominal interest rate that satisfies (2) - (4) for every $t \geq 0$ is sustainable if for each $t \geq 0$ the present discounted value of losses is less than the present value of losses under the optimal, time-consistent discretionary policy. Thus, policies for which the current period's loss exceeds that obtained under the discretionary policy may still be sustainable if future losses under the policy are less than those under discretion.

Following Eggertsson and Woodford (2003), assume there are two states: in one, the ELB is binding; in the other, it is not. The shock r_t in (2) follows a two-state Markov process. If $r_t = r_z < 0$, then $r_{t+1} = r_z$ with probability q, $0 \leq q < 1$, and $r_{t+1} = \beta^{-1} - 1 \equiv \rho > 0$ with probability $1 - q$. With the exception of Nakata (2014), the standard assumption is that if $r_t = \rho$, then $r_{t+j} = \rho$ for all $j \geq 0$; that is, the economy never returns to the ELB once it has left it. Instead, assume if $r_t = \rho$, then $r_{t+1} = \rho$ with probability s, $0 < s \leq 1$, and $r_{t+1} = r_z < 0$ with probability $1 - s$. Thus, $1 - s$ is the probability of reverting to the ELB. The earlier literature building on the analytical structure of Eggertsson and Woodford (2003) assumed $s = 1$, as does the literature that treats the ELB as binding for a fixed number of periods after which it never binds again (see, for example, Werning (2011), Cochrane (2013), Kiley (2014)).

Let π_z and x_z denote equilibrium inflation and the output gap when the ELB binds, and let π_k and x_k denote inflation and the output gap after k periods away from the ELB. From (2), (3) and the assumed process for r_t,

$$x_z = (qx_z + (1 - q)x_1) + \left(\frac{1}{\sigma}\right)[q\pi_z + (1 - q)\pi_1 + r_z]$$

and

$$\pi_z = \beta[q\pi_z + (1 - q)\pi_1] + \kappa x_z.$$

[7] The underlying nonlinear model that leads to the reduced form equations employed here is so well known that providing details on it seems unnecessary. See, for example, chapter 8 of Walsh (2010); chapter 11 of the forthcoming fourth edition provides an extended discussion of the ELB. Braun, Körber, and Waki (2012) discuss how, at least for some issues, the log linearized version used here may give misleading answers to some questions. Some of the properties of the model that they emphasize as problematic are absent in a consumption only version of the model such as the one I use. McKay, Nakamura, and Steinsson (2016a) argue that more plausible results on the power of forward guidance are obtained using a discounted Euler equation; see section **??**.

In the first period away from the ELB, (2) and (3) imply

$$x_1 = [sx_2 + (1-s)\,x_z] - \left(\frac{1}{\sigma}\right)[i_1 - s\pi_2 - (1-s)\,\pi_z - \rho]$$

and

$$\pi_1 = \beta\,[s\pi_2 + (1-s)\,\pi_z] + \kappa x_1.$$

The standard analysis, which assumes $s = 1$ and there is never a reversion to the ELB, implies these last two equations become

$$x_k = x_{k+1} - \left(\frac{1}{\sigma}\right)(i_k - \pi_{k+1} - \rho)$$

and

$$\pi_k = \beta\pi_{k+1} + \kappa x_k,$$

for all $k \geq 1$. In this case, the optimal policy under discretion implies $\pi_k = x_k = 0$ and $i_k = \rho$ for all $k \geq 1$. This yields the standard solutions at the ELB:

$$x_z = \Delta\,(1-\beta q)\,r_z \tag{5}$$

and

$$\pi_z = \left(\frac{\kappa}{1-\beta q}\right) x_z = \Delta\kappa r_z, \tag{6}$$

where

$$\Delta \equiv \frac{1}{\sigma(1-q)(1-\beta q) - q\kappa}.$$

Employing Eggertsson and Woodford's calibration ($\beta = 0.99$, $\sigma = 2$, $\kappa = 0.02$, and $q = 0.9$) yields $\Delta = 263$.[8] Assume that $r_z = -2\%$ (expressed at an annual percentage rate), the equilibrium output gap and inflation rate at the ELB are $x_z = -0.1434$ and $\pi_z = -0.0263$ (-14.34% and -10.53% respectively, when inflation is expressed at an annual rate). When agents expect the economy to return to a zero inflation rate and output gap once the ELB episode ends, with the probability of reverting to the ELB also equal to zero, then the binding ELB generates a large deflation and negative output gap.

[8] These values are also used by McKay, Nakamura, and Steinsson (2016b). The large value of Δ implies the negative value of r_z has a large effect on x_z and π_z. Eggertsson (2011) limits attention to cases in which the denominator of Δ is positive; this is his condition C1 (p. 70). Braun, Körber, and Waki (2012) discuss the equilibrium when the denominator of Δ is negative.

As is well known, outcomes at the ELB can be improved if the central bank can commit to a positive inflation rate on exiting from an ELB episode. However, when $s = 1$, $\pi_1 = 0$ is the only rate that is consistent with optimal policy in the absence of a commitment technology. Any equilibrium in which $\pi_1 \neq 0$ and $x_1 \neq 0$ is dominated by the zero inflation and output gap obtained under optimal discretion. Thus, only $\pi_1 = 0$ and therefore $\pi_k = x_k = 0$ for all $k > 0$ is sustainable.

But if $s < 1$, deviations from zero inflation – promising $\pi_1 > 0$ – may be sustainable. given that a policy will be defined as sustainable if it yields a lower loss in present value terms than could be achieved by reverting to the time-consistent discretionary policy, the next section defines equilibrium under optimal discretion.

2.1 Optimal discretion

Consider the policy problem when the economy is not at the ELB. Under pure discretion, the policymaker takes private sector expectations and future policy as given and faces a static decision problem each period that involves minimizing

$$l_t = \frac{1}{2}\left(\pi_t^2 + \lambda x_t^2\right), \tag{7}$$

subject to (2), (3), and (4). Optimal policy is characterized by a targeting rule of the form[9]

$$\kappa\pi_t + \lambda x_t = 0. \tag{8}$$

Using superscript d to denote the equilibrium under discretion, inflation π_n^d and the output gap x_n^d when the ELB is not binding solve

$$\pi_n^d = \beta\left[s\pi_n^d + (1-s)\,\pi_z^d\right] + \kappa x_n^d, \tag{9}$$

and

$$\kappa\pi_n^d + \lambda x_n^d = 0, \tag{10}$$

[9] In most of the literature using this model, policy after the ELB episode ends is characterized by a simple instrument rule rather than by optimal discretion. In the present context, $\pi_n = x_n = 0$ is also the locally unique stationary equilibrium if the nominal rate is given by $i_n = \rho + \phi\pi_n$ once the ELB constrain no longer binds, with $\phi > 1$. The choice of ϕ, as long as it exceeds 1, plays no role in affecting equilibrium at the ELB or away from the ELB when the ELB episode is a one-off event.

where expected inflation is equal to $s\pi_n^d + (1-s)\,\pi_z^d$. Given the non-negative constraint on i_t, it must also hold from (2) that

$$i_n^d = \rho + \left[s\pi_n^d + (1-s)\,\pi_z^d\right] + \sigma\,(1-s)\left(x_z^d - x_n^d\right) \geq 0, \tag{11}$$

where this last equation is obtained by solving the Euler condition (2) in state n.

When $s = 1$, $\pi_n^d = x_n^d = 0$ constitutes an equilibrium under discretion when the ELB is nonbinding. When $s < 1$, it is no longer feasible to achieve a zero inflation rate and output gap, as neither expected inflation, $s\pi_n^d + (1-s)\,\pi_z^d$, nor the expected output gap, $sx_n^d + (1-s)\,x_z^d$, equal zero. As long as some probability is assigned to that possibility the economy will return to the ELB, expected inflation and the expected output gap when not at the ELB will depend on x_z^d and π_z^d.

At the ELB, equilibrium is characterized by

$$\pi_z^d = \beta\left[q\pi_z^d + (1-q)\,\pi_n^d\right] + \kappa x_z^d \tag{12}$$

$$x_z^d = \left[qx_z^d + (1-q)\,x_n^d\right] + \left(\frac{1}{\sigma}\right)\left[q\pi_z^d + (1-q)\,\pi_n^d + r_z\right]. \tag{13}$$

Equations (9) - (13) can be solved jointly to obtain equilibrium inflation, the output gap, and the nominal rate in states z and n.[10]

2.2 Calibration and results under discretion

To evaluate equilibria as a function of the announced exit-period inflation rate, a calibrated version of the model is employed. The baseline calibration is given in Table 1, which is based on the values employed by Eggertsson and Woodford (2003) and used more recently by Nakata (2014) and McKay, Nakamura, and Steinsson (2016b). The loss function (**??**) is interpreted as derived from a second-order approximation of the welfare of the representative household around the economy's efficient equilibrium; Woodford (2003) showed that in this case $\lambda = \kappa/\theta$, where θ is the price elasticity of demand faced by individual firms. Using Woodford's value of $\theta = 7.88$ implies $\lambda = 0.003$. If inflation is

[10] Walsh (2017) shows that optimal discretion involves a positive inflation rate and a positive output gap when $s < 1$, but this may not be consistent with $i_n^d \geq 0$. This situation can arise if s is small (for example, less than 0.9750 for $q = 0.9$). However, for all s and q values considered here, the ELB constaint does not bind in state n.

expressed at an annual rate, then $\lambda^a = 16\lambda = 0.048$.

Table 1: Benchmark Values

Parameter	Values (quarterly rates)
β	0.99
σ	2
κ	0.02
r_z	-0.005
ρ	$\beta^{-1} - 1 = 0.01$
λ	0.003

The final two parameters of the model are s and q. To discipline the calibration of the two transition probabilities when $s < 1$, I employ the evidence based on figure 1. Define

$$\Phi \equiv \begin{bmatrix} s & 1-q \\ 1-s & q \end{bmatrix}.$$

The steady-state fractions of time spend away from the ELB and at the ELB are given by the diagonal elements of $\lim_{T\to\infty} \Phi^T$. I match these fractions to either the 1960-2016 sample frequencies(88% of the time away from the ELB, 12% of the time at the ELB) or the longer 1934-2016 sample period (83% and 17% in the two states respectively). For $q = 0.9$, the standard value of q used by Woodford (2003), Nakata (2014), McKay, Nakamura, and Steinsson (2016b) and Walsh (2017), $s = 0.9858$ implies a steady-state frequency at the ELB of 12%, matching the sample frequency for 1960-2017, while $s = 0.9792$ implies a steady-state frequency of 17%, matching the sample frequency for 1934-2017. These two values of s, together with $q = 0.9$, will be employed in the baseline calibration. To also assess the effects of a smaller continuation probability at the ELB, calibrations based on $q = 0.85$ will also be used. The associated values of s for the two samples are given in Table 2.[11]

[11] As discussed in Eggertsson (2011), the economy experiences what he describes as deflationary back hole if q rises above 0.9. As $q \to 1$, it enters what Braun, Körber, and Waki (2012) characterize as a type 2 equilibrium. As discussed in the next section, I restrict attention to vaues of $q \leq 0.90$ to be consistent with equilibria in which $x_z < 0$, $\pi_z < 0$, and $i_n > 0$ under optimal discretion.

Table 2: Benchmark Transition Probabilities

	s	
	$q = 0.85$	$q = 0.90$
1960-2016	0.9788	0.9858
1934-2016	0.9687	0.9792

If $r_z = -2\%$ (expressed at an annual percentage rate), using the parameter values given in Table 1 and setting $q = 0.90$ and $s = 1$ as in Eggertsson and Woodford (2003), the equilibrium output gap and inflation rate at the ELB are $x_z = -0.1434$ and $\pi_z = -0.0263$ (-14.34% and -10.53% respectively, when inflation is expressed at an annual rate) for $s = 1$. This translates into a consumption-equivalent loss of $\mu_z = 5.04\%$.[12] Outcomes under optimal discretion for the four combinations of q and s in Table 2, as well as for $s = 1$, are given in Table 3. Also reported are the present discounted values of losses at the ELB and away from the ELB. A decline in s implies that returns to the ELB are more frequent, so losses increase as s falls. This effect on loss is driven by the effects on equilibrium when the ELB constraint is not binding. With returns to the ELB more likely, expectations in state n place greater weight on outcomes at the ELB. The deterioration in expected inflation reduces actual inflation and the optimal policy response under discretion is to offset partially the effect of lower expected future inflation through a positive output gap, consistent with (8).

A fall in q reduces the expected duration of ELB episodes, and this significantly lowers losses. When q is reduced from 0.90 to 0.85, L_z^d and L_n^d decline by 98%.

[12] Following Billi (2011), losses are expressed in terms of their steady-state consumption equivalence given by

$$\mu_z = (1-\beta)\left[\frac{\omega\theta\,(1+\eta\theta))}{(1-\omega)(1-\omega\beta)}\right] L_z. \tag{14}$$

Thus, a loss of L_z is equivalent to a $100\mu_z$ percent reduction in steady-state consumption.

Table 3: Outcomes under optimal discretion

q	s	π_z^d	x_z^d	π_n^d	x_n^d	L_z^d	L_n^d
0.90	1.0000	−10.5263	−14.3421	−0.0000	0.0000	5.0389	0.0000
0.90	0.9858	−9.4458	−11.8258	−0.8437	1.4061	8.8318	5.3134
0.90	0.9792	−9.0675	−10.9449	−1.1390	1.8984	9.9629	6.9250
0.85	1.0000	−1.3093	−2.5941	−0.0000	−0.0000	0.0585	0.0000
0.85	0.9788	−1.3520	−2.3581	−0.1727	0.2878	0.1773	0.1251
0.85	0.9687	−1.3695	−2.2614	−0.2434	0.4057	0.2328	0.1834

Outcomes with inflation announcements will be sustainable only if they yield losses when the economy exits the ELB that are less then what is achieved under optimal discretion. For given q and s, any policy that yields a loss in present value terms that exceeds L_n^d will not be sustainable as the policymaker has an incentive to defect to the pure discretion equilibrium.

3 Announcing future inflation

Even when $s = 1$, Cochrane (2013) shows that the equilibrium given by (5), (6), $x_k = \pi_k = 0$ for $k \geq 1$, $i_k = 0$ for $k < 1$ and $i_k = \rho$ for $k \geq 1$ is not the only possible equilibria. He shows that the same interest rate path, with the nominal rate at zero while the ELB binds and then equal to ρ when it doesn't, is consistent with multiple equilibria indexed by the value of π_1, the exit period inflation rate. The standard results are obtained when it is assumed that $\pi_1 = 0$. In the next section, it is shown that a unique, stationary equilibrium when $s \leq 1$ can be defined as a function of the inflation rate the central bank announces for the exit period. Regardless of the inflation rate the central bank announces, the path of the nominal interest rate is assumed to be the same: while at the ELB, $i = 0$ and when not at the ELB, $i = \rho > 0$. By announcing an inflation rate for the exit period, the policy maker is selecting a particular equilibrium.[13] Given this result,

[13] Equivalently, consider an instrument rule of the form

$$i_t = \max\left[0,\, \rho + \phi\left(\pi_t - \pi_t^a\right)\right],$$

where π_t^a is the announced path for inflation when the economy remains away from the ELB and $\phi > 1$. In equilibrium, $\pi_t = \pi_t^a$ when the economy is not at the ELB, so $i_t = \rho$.

two questions are addressed. What is the optimal inflation rate to announce? And is this optimal rate sustainable?

3.1 Equilibrium with an arbitrary exit inflation rate

Denote the exit period by $k = 1$ and the announced inflation rate by π_1^a. Given π_1^a, I construct an equilibrium for π_z and x_z, as well as for x_k for $k \geq 1$ and for π_k for $k > 1$. The resulting paths for inflation and the output gap will be denoted by a superscript a.

When Cochrane solved for the equilibrium associated with an announced inflation rate, he assumed the economy never reverted to the ELB. In that case, the equilibrium for $k \geq 1$ is independent of the equilibrium at the ELB; one can obtain x_1^a as the unique value consistent with a stationary equilibrium and then solve the model forward for the paths of inflation and the output gap in the post-ELB periods. Given π_1^a and x_1^a as terminal conditions, one can also solve backward for the equilibrium during the ELB period. This separation, in which the post-ELB equilibrium is independent of the equilibrium during the ELB, is no longer possible when $s < 1$. Knowledge of π_z^a and x_z^a is required to determine x_1^a, because expectations for all $k \geq 1$ put some weight on reverting to the ELB. Hence, the equilibrium after the exit period depends on π_z^a and x_z^a as well as on π_1^a and x_1^a.

When away from the ELB, equilibrium inflation and the output gap satisfy

$$\begin{bmatrix} \pi_{k+1}^a \\ x_{k+1}^a \end{bmatrix} = P \begin{bmatrix} \pi_k^a \\ x_k^a \end{bmatrix} + Q \begin{bmatrix} \pi_z^a \\ x_z^a \end{bmatrix}, \ k \geq 1, \tag{15}$$

given π_1^a, while when the ELB is binding,

$$\begin{bmatrix} \pi_z^a \\ x_z^a \end{bmatrix} = S \begin{bmatrix} \pi_1^a \\ x_1^a \end{bmatrix} + Tr_z. \tag{16}$$

The matrices appearing in these two systems of equations are given in the appendix, which also provides the details on how (15) and (16) are solved. If $s = 1$, then $Q = 0$, and (15) for the post-exit equilibrium can be solved independently of the equilibrium when the ELB binds.[14] If $s < 1$, then $Q \neq 0$, and it is necessary to solve (15) and (16)

[14] This corresponds to the case considered by Cochrane. Cochrane assumed the ELB period lasted for a fixed number of periods rather than that the model of stochastic duration employed here. However, the points made apply in either case.

jointly. And this means the equilibrium when away from the ELB is affected not just by the expected time until another ELB episode, as determined by s, but also by the expected duration of ELB episodes, as determined by q.

In addition to solving for the equilibrium paths of inflation and the output gap while the economy remains away from the ELB, the sustainability of the equilibrium for each $k > 0$ must be assessed. Valuation equations must account for the fact that inflation and the output gap are not constant once the exit period is over. Let l_z^a, and l_k^a, $k \geq 1$ denote the period losses at the ELB, in the exit period, and after $k \geq 1$ periods during which the ELB constraint has been nonbinding. Then

$$L_z^a = l_z^a + \beta \left[q L_z^a + (1-q) L_1^a \right]$$

$$L_k^a = l_k^a + \beta \left| s L_{k+1}^a + (1-s) L_z^a \right|, \, k \geq 1.$$

Given the assumption of stationarity,

$$\lim_{k \to \infty} L_k^a = \frac{1}{1 - \beta s} \left[\lim_{k \to \infty} l_k^a + \beta (1-s) L_z^a \right].$$

3.2 Results for announced inflation

The gain in terms of the reduction in the present value of losses at the ELB from making an inflation announcement is measured by $L_z^d - L_z^a$, and this gain as a function of s and π_1^a for $q = 0.90$ is shown by the black surface in figure 2. For all values of s shown, there is a range of inflation announcements that reduce the PDV of losses at the ELB relative to optimal discretion and so produce a positive gain. The lowest announced inflation rate that yields a gain increases as s decreases. A decrease in s implies a higher probability of reverting to the ELB. This reduces expected inflation and the output gap when not at the ELB, thereby worsening outcomes when not at the ELB as well as when at the ELB. To offset the decline in welfare induced by a fall in s, a higher inflation rate must be announced for the exit period if outcomes are to be improved over discretion. Thus, the lowest announced inflation rate that yields a gain increases as s falls.

Are inflation announcements sustainable? To answer this questions requires an examination of the temptation to revert to the discretionary equilibrium rather than deliver the promised inflation rate in the exit period. The temptation to defect to discretion is present when $L_1^a - L_n^d > 0$. This temptation for $q = 0.9$ is shown by the black surface

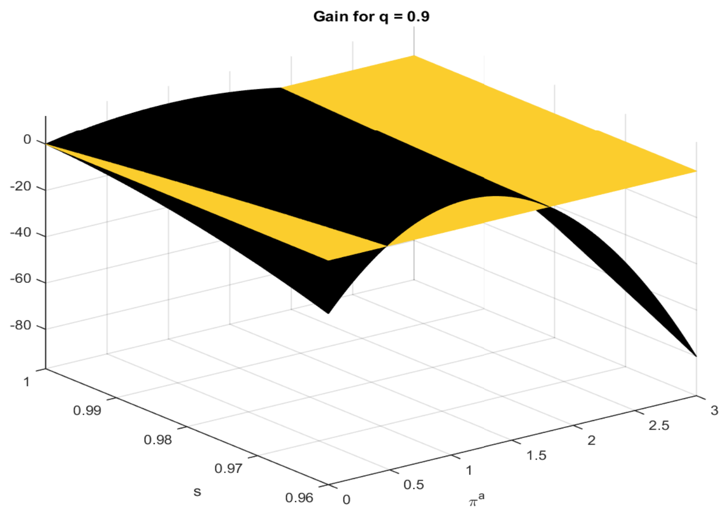

Figure 2: Gain for $q = 0.9$ as a function of s and π_1^a.

in figure 3. An announced inflation rate π_1^a is sustainable if it yields a positive welfare gain at the ELB and the temptation to defect is negative. Comparing figures 2 and 3 shows the announced inflation rates for which temptation is negative are a subset of the rates for which the gain is positive. Thus, some but not all announced rates that would improve welfare are sustainable.

Figure 4 illustrates this by plotting L_1^a and L_n^d as a function of π_1^a for $q = 0.90$ and associated values of s that match the 1960-2017 (solid) and 1934-2017 (dashed) frequencies at the ELB. For $q = 0.90$ and $s = 0.9858$, sustainable inflation announcements are ones for which the solid blue line showing L_1^a falls below the black solid line showing L_n^d. Announced rates that are too low or too high are not sustainable. The dashed lines show L_1^a and L_n^d for $s = 0.9792$, the value consistent with 1934-2017 frequency of ELB quarters. With a smaller s, the expected duration away from the ELB falls. Losses rise under discretion and the set of sustainable inflation announcements shrinks.[15]

The value of q has a significant impact on the present value of losses and on the set of sustainable inflation announcements. Figure 5 shows results for $q = 0.85$ and for the

[15] For $s = 1$, no non-zero inflation announcement is sustainable as $L_n^d = 0$.

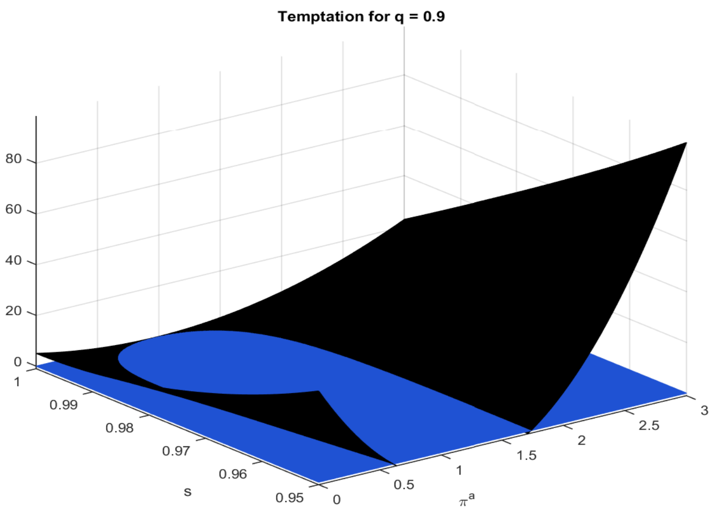

Figure 3: Temptation for $q = 0.90$ as a function of s and π_1^a.

two values of s consistent with the observed number of quarters at the ELB. With ELB episodes of shorter expected duration, the losses under discretion are much less than when $q = 0.90$ (compare the scale of the vertical axis in figures 4 and 5. And as q declines, fewer inflation announcements are sustainable. For sufficiently small q (not shown), no inflation announcement is sustainable.

4 Optimal inflation announcements

The previous section considered equilibria indexed by the inflation rate promised once the ELB episode ends and showed that some inflation announcements were unsustainable as they yielded a larger loss (in present value terms) when it came time to implement the promised inflation rate then could be achieved by reverting to the time-consistent optimal discretionary policy. This means that some announcements are ruled out as possible equilibria at the ELB is sustainability is a necessary condition for a candidate equilibria. However, as figures 4 and 5 also showed, multiple sustainable equilibria remain, each indexed by the inflation rate promised for when the economy exits the ELB.

Different choices for π_1^a led to different losses at the ELB, and figure 6 shows L_z^a as a function of π_1^a for the different combinations of s and q from Table 2. The dotted lines for

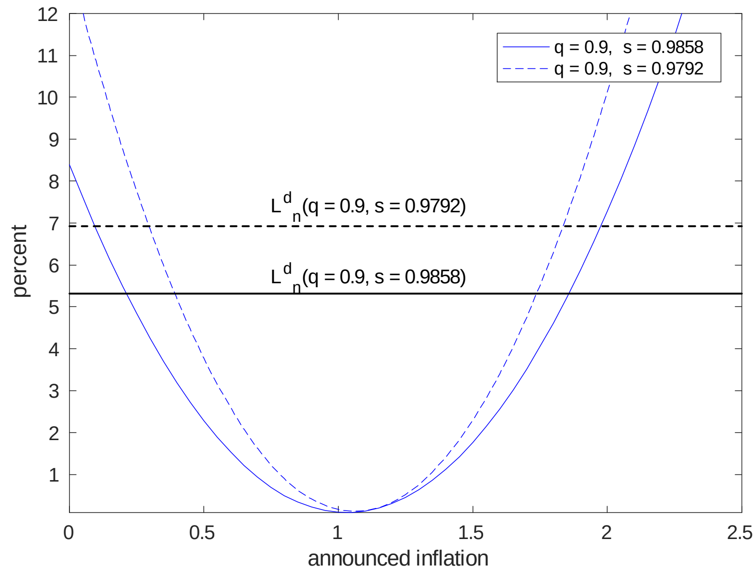

Figure 4: PDV of loss in exit period ($k = 1$) for $q = 0.90$ and values of s from Table 2.

$q = 0.90$ (blue) and $q = 0.85$ (black) for $s = 1$ achieve essentially a zero loss at the ELB if the inflation rate that minimizes loss L_z^a is chosen. However, no announcement involving a positive announcement is sustainable when $s = 1$, since in this standard case, $L_n^d = 0$. For $s < 1$, is increasing in q; a rise in the expected persistence of an ELB episode (a rise in q) worsens outcomes at the ELB and makes it optimal to announce a higher exit inflation rate. A fall in s represents an increase in the probability of reverting to the ELB which worsens outcomes when away from the ELB; this in turn makes outcomes at the ELB worse and increases the optimal exit inflation rate to announce. These conclusions hold over the range of q and s considered; figure 7 shows the optimal π_1^a as a function os q and s. The value of π_1^a at which L_z^a reaches a minimum decreases with s and increases with q. In all cases, the optimal inflation rate to promise for the exit period is quite small.

Table 4 shows the optimal inflation rate to announce and the associated equilibrium outcomes at the ELB and when the economy remains away from the ELB. For example, when $q = 0.90$ and $s = 0.9858$, values consistent with the observed fraction of quarters at the ELB in the 1960-2016 period (see Table 1), the optimal inflation rate to announcement

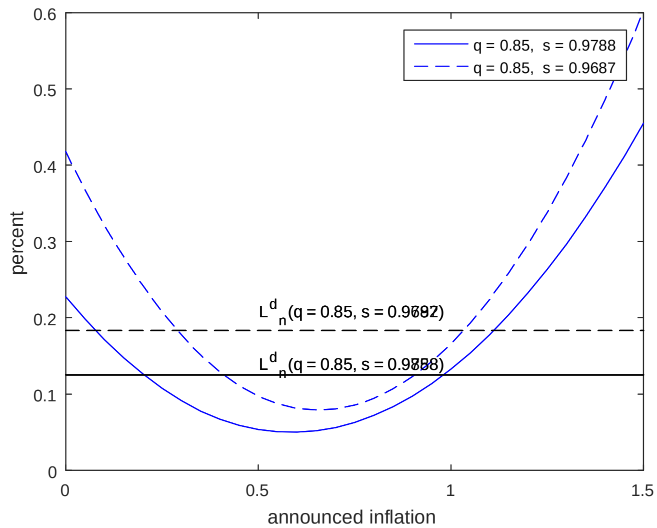

Figure 5: PDV of loss in exit period ($k = 1$) for $q = 0.90$ and values of s from Table 2.

for the exit period is 1.025%. This boosts inflation at the ELB from −9.446% under discretion (see Table 3) to 0.152%. The output gap at the ELB rises from −11.823% under discretion to −1.062%. If the economy remains away from the ELB, DON'T π_n^a and x_n^a HAVE TO BE OF SAME SIGN??

Table 4: Outcomes under optimal announced inflation*

q	s	π_1^a	x_1^a	π_z^a	x_z^a	π_n^a	x_n^a
0.90	1.0000	0.9563	1.1953	−0.0076	−1.1937	−0.0000	0.0000
0.90	0.9858	1.0250	1.1394	0.1516	−1.0619	0.1213	0.0098
0.90	0.9792	1.0563	1.1185	0.2427	−0.9764	0.1606	−0.0011
0.85	1.0000	0.9563	0.5938	−0.1036	−1.0869	0.0000	−0.0000
0.85	0.9788	0.6000	0.5810	0.0810	−0.9532	0.1712	0.0451
0.85	0.9687	0.6500	0.5717	0.1515	−0.9065	0.2470	0.0679

* Inflation at annual rates; output gap in percent.

Is the optimal inflation rate to announce sustainable? For the optimal announcement that minimizes L_z^a to be sustainable, it must be that (a) the gain $L_z^d - L_z^a$ is positive,

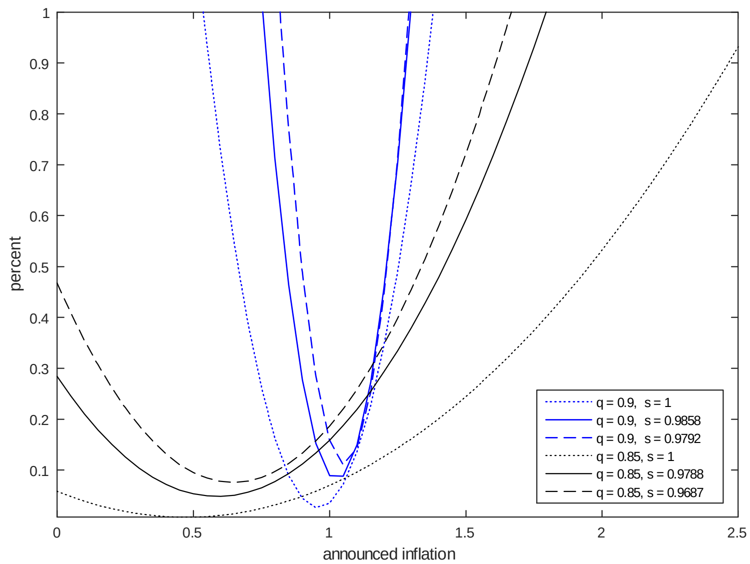

Figure 6: PDV of loss at the ELB for $q = 0.90$ (blue lines) and $q = 0.85$ (black lines).

and (b) the temptation to defect $L_1^a - L_n^d$ be negative. Table 5 reports the present value of the loss function for the same combinations of q and s as shown in Table 4 and for a lower q (0.81). Rows in bold denote sustainable equilibria.

Table 5: Losses under optimal π_1^a

q	s	π_1^a	L_z^a	L_1^a	L_n^a	L_n^d
0.90	1.0000	0.9563	0.0261	0.0256	0.0015	0.0000
0.90	**0.9858**	**1.0250**	**0.0808**	**0.0854**	**0.0549**	**5.3134**
0.90	**0.9792**	**1.0563**	**0.1106**	**0.1170**	**0.0844**	**6.9250**
0.85	1.0000	0.9563	0.0079	0.0063	0.0004	0.0000
0.85	**0.9788**	**0.6000**	**0.0484**	**0.0501**	**0.0398**	**0.1251**
0.85	**0.9687**	**0.6500**	**0.0761**	**0.0793**	**0.0681**	**0.1834**
0.81	1.0000	0.3000	0.0036	0.0025	0.0001	0.0000
0.81	**0.9731**	**0.4250**	**0.0388**	**0.0399**	**0.0356**	**0.0402**
0.81	0.9604	0.5000	0.0655	0.0677	0.0628	0.0603

Table 5 shows the equilibrium outcomes for different states under the optimal inflation announcement and discretion for the values of q and s from Table 2.

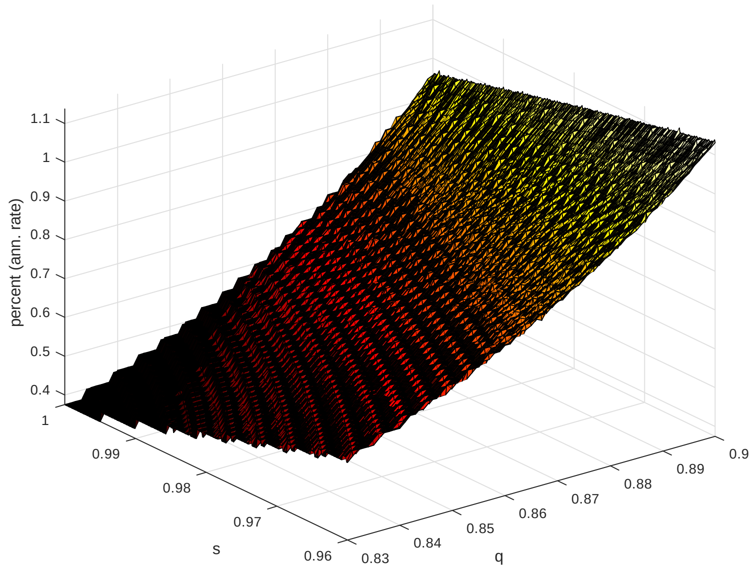

Figure 7: Optimal inflation announcement

Table 6: Outcomes under optimal announcement and discretion

Inflation						
q	s	π_1^a	π_z^a	π_n^a	π_z^d	π_n^d
0.90	0.9858	0.1516	1.0250	0.1606	-9.4458	-0.8437
0.90	0.9792	0.2427	1.0563	0.1606	-9.0675	-1.1390
0.85	0.9788	0.0810	0.6000	0.1712	-1.3520	-0.1727
0.85	0.9687	0.1515	0.6500	0.2470	-1.3695	-0.2434
Output gap						
q	s	x_1^a	x_z^a	x_n^a	x_z^d	x_n^d
0.90	0.9858	-1.0619	1.1394	0.0098	-11.8258	1.4061
0.90	0.9792	-0.9764	1.1185	-0.0011	-10.9449	1.8984
0.85	0.9788	-3.8130	2.3241	0.1803	-9.4325	1.1512
0.85	0.9687	-3.6259	2.2866	0.2713	-9.0457	1.6229

The previous figures and tables have focused on the values of q and s from Table 2. To get a more understanding of the sustainability of the optimal inflation announcement across a range of values for these two transition probabilities, Figure **??** divides the area defined by $s \in [0.96\ 1]$ and $q \in [0.75\ 0.90]$ into two regions based on the sustainability

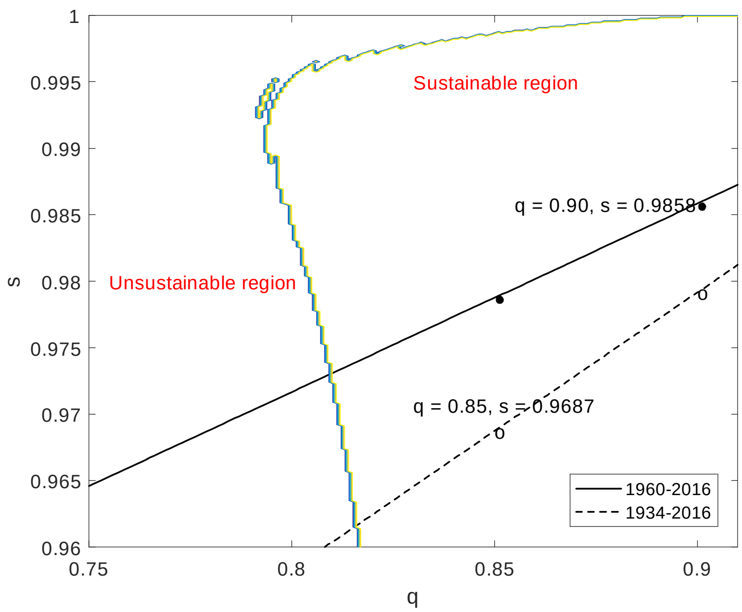

Figure 8: Region of sustainable optimal inflation announcements.

of the optimal inflation announcement. The four baseline calibrations for s and q from Table 2 are indicated in the figure. For $s = 1$, no announcement is sustainable. As s falls below 1, the optimal inflation announcement policy becomes sustainable, but only if the expected duration of ELB episodes is sufficiently long (i.e, for sufficiently high values of q). As the expected duration of spells at the ELB falls, the maximum s for which the optimal announcement is sustainable also falls. For example, if s falls from 1 to just 0.9999, the optimal announcement is sustainable if $q = 0.90$. With $q = 0.90$, the equilibrium under discretion at the ELB is so bad (see Table 3), that even a tiny probability of revisiting the ELB provides sufficient incentive for the policy maker to fulfill promises made at the ELB. If $q = 0.85$, however, s must be less than 0.9987 for the optimal inflation announcement to be sustainable. Outcomes generate much smaller losses at the ELB when $q = 0.85$, that the incentive to defect from the promised inflation in the exit period is to large to sustain the promise unless the probability of returning to the ELB is much larger (i.e., s smaller).

The figure also shows that, for some values of q, and announcement can be unsustainable if s is very large or if s is small. For example, if $q = 0.81$, the optimal announcement is sustainable for s such that $0.9724 < s < 0.9967$. Ceteris paribus, a fall in s makes ELB episodes more frequent and worsens outcomes while at the ELB. This increases the optimal inflation rate to announce for the exit period, which in turn, increases equilibrium inflation both at the ELB and for periods beyond the exit period. Declines in s worsens outcomes away from the ELB under discretion, but the deterioration during the exit period under the inflation announcement policy is even greater, leading the optimal announcement policy to be unsustainable.

5 Summary and conclusions

Recent research has emphasized the adverse consequences for the economy when the central bank's policy instrument is constrained by an effective lower bound on the short-term nominal interest rate and policy is implemented in a time-consistent, discretionary manner. These adverse effects stand in contrast to the situation in which the central bank is able to implement the optimal but time-inconsistent commitment policy. Under the presumption that discretion is the more realistic assumption about policy, proposals for reforming inflation targeting policy frameworks have emphasized changes that either make it less likely the ELB will be encountered or that establish alternative regimes, such as price-level targeting, that can cause expectations to move in a manner that promotes stabilization and mimics a commitment policy regime.

Proposed reforms presume that the ELB will be encountered again in the future, yet much of the simple analytics of the ELB have been studied in models that assume an episode at the ELB is a one-off event, never encounter again.[16] If this is the case, then any promises about future policy – a promise about post-ELB inflation for example – lack credibility. Once the economy is out of the ELB period, there is no incentive for the policymaker to implement the policies that were promised in the past.

But if the economy may revert to the ELB, then promises made during an ELB episode may be credible even in the absence of a commitment mechanism. If the promised policy actions improve outcomes when at the ELB, then it may be rational for the central bank to fully implement those promises because, while doing so generates a cost, it also brings

[16] As noted earlier, the exception is Nakata (2014). See also Walsh (2017).

an expected future benefit. Future promises may be sustainable.

I modify the basic model of Eggertsson and Woodford (2003) to allow for both a constant probability of exiting the ELB and a constant probability of returning to the ELB. Unlike the standard analysis, the economy does not achieve zero inflation and a zero output gap once it exits the ELB. With a positive probability of reverting to the ELB, expected inflation and the output gap are no longer zero as in the Eggertsson-Woodford analysis. Within this framework, I investigate the effects of promising a specific inflation rate on exiting the ELB, with the nominal rate held to zero during the ELB and set equal the natural rate when not at the ELB. Here the results depended on the calibration. For transition probabilities betweeen states that are consistent with the fraction of quarters U.S. interest rates have been at or below 25 basis points, the optimal inflation announcement is sustainable as long as the expected duration of ELB episodes is not too short. For commonly used value of the probability of exiting the ELB, the optimmal inflation announcement is sustainable if the probability of reverting to the ELB is even as small as 0.01% per quarter.

The results obtained here were derived using a very stylized model. The basic model does, however, generalizes the framework that has been employed widely in analyzes of the ELB. There are many directions in which the basic model could be extended to determine how robust the reported findings are. The key implication, a result consistent with the findings of Nakata (2014), is likely to be robust: if future episodes at the ELB are likely, then promises made during the ELB period may be credible despite the absence of any mechanism to ensure commitment.

6 Appendix

This appendix outlines the method used to solve for the equilibrium with an exogenous interest rate path and an announced rate of inflation for the exit period. To solve the equilibrium involves several steps. Letting $k = 1$ denote the first period in which the ELB constraint is no longer binding,

$$\pi_k^a = \beta s \pi_{k+1}^a + \beta(1-s)\pi_z^a + \kappa x_k^a$$

and

$$
\begin{aligned}
x_k^a &= \left[s x_{k+1}^a + (1-s)x_z^a\right] - \left(\frac{1}{\sigma}\right)\left[i_k - s\pi_{k+1}^a - (1-s)\pi_z^a - \rho\right] \\
&= \left[s x_{k+1}^a + (1-s)x_z^a\right] + \left(\frac{1}{\sigma}\right)\left[s\pi_{k+1}^a + (1-s)\pi_z^a\right],
\end{aligned}
$$

where terms such as π_{k+1}^a and x_{k+1}^a denote equilibrium values along the path that remains out of the ELB and $i_k = \rho$ has been used.[17]

Rewrite these two equations as

$$
\begin{bmatrix} \pi_{k+1}^a \\ x_{k+1}^a \end{bmatrix} = P \begin{bmatrix} \pi_k^a \\ x_k^a \end{bmatrix} + Q \begin{bmatrix} \pi_z^a \\ x_z^a \end{bmatrix}, \; k \geq 1, \tag{17}
$$

where

$$
P = \begin{bmatrix} \beta s & 0 \\ s & \sigma s \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\kappa \\ 0 & \sigma \end{bmatrix}
$$

and

$$
Q = \begin{bmatrix} \beta s & 0 \\ s & \sigma s \end{bmatrix}^{-1} \begin{bmatrix} -\beta(1-s) & 0 \\ -(1-s) & -\sigma(1-s) \end{bmatrix}.
$$

The dependence of the post-ELB inflation rate and output gap on equilibrium at the ELB is illustrated clearly in (17). If $s = 1$, $Q = 0$ and the post-exit equilibrium is independent of the pre-exit equilibrium.

Define $\bar{\pi}$ and $\bar{x}$ as the steady-state, stationary equilibrium values of inflation and the output gap if the economy were to remain away from the ELB. Then from (17),

$$
\begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} = P \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} + Q \begin{bmatrix} \pi_z^a \\ x_z^a \end{bmatrix}. \tag{18}
$$

Subtracting this from (17) results in

$$
\begin{bmatrix} \hat{\pi}_{k+1}^a \\ \hat{x}_{k+1}^a \end{bmatrix} = P \begin{bmatrix} \hat{\pi}_k^a \\ \hat{x}_k^a \end{bmatrix}, \tag{19}
$$

where $\hat{\pi} = \pi^a - \bar{\pi}$ and $\hat{x} = x^a - \bar{x}$ are now expressed in deviation form. The matrix P has

[17] That is, if the economy has remained away from the ELB for $j+1$ periods, then inflation is equal to π_{k+j}^a. In the following period, inflation will equal π_{+j+1}^a with probability s and π_z^a with probability $1-s$.

one eigenvalue outside the unit circle and one inside, so multiple equilibria are feasible; a locally unique, stationary equilibrium is selected by the policymaker's announcement of the inflation rate in the exit period.

Note that (18) consists of the following two equations:

$$(1-\beta s)\,\bar{\pi} = \beta(1-s)\pi_z + \kappa\bar{x}$$

$$(1-s)\bar{x} = (1-s)x_z + \left(\frac{1}{\sigma}\right)[s\bar{\pi} + (1-s)\pi_z.]$$

Solving these two equations yields

$$\bar{x} = \frac{(1-s)\left[\sigma\left(1-\beta s\right)x_z + \pi_z\right]}{\left[\sigma(1-s)\left(1-\beta s\right) - s\kappa\right]},$$

$$\bar{\pi} = \left[\frac{\beta(1-s)\pi_z + \kappa\bar{x}}{1-\beta s}\right].$$

The denominator in the expression for $\bar{x}$ is negative for $s > 0.909$, which is less than the benchmark values for s given in Table 2. Since the numerator is also negative, $\bar{x} > 0$ but it is decreasing in π_z – i.e., since increasing π^a increases π_z and x_z, it decreases $\bar{x}$.

The matrix P in (19) can be written as $P = VDV^{-1}$, where D is a diagonal matrix with elements equal to the eigenvalues of P, and V consists of the eigenvectors of P. P has one eigenvalue greater than one and one less than one. Assume D is ordered such that the largest eigenvalue is ordered first. Premultiplying (19) by V^{-1} yields

$$V^{-1}\begin{bmatrix} \hat{\pi}^a_{k+1} \\ \hat{x}^a_{k+1} \end{bmatrix} = DV^{-1}\begin{bmatrix} \hat{\pi}^a_{k} \\ \hat{x}^a_{k} \end{bmatrix}$$

which can be written as

$$\begin{bmatrix} z^a_{1,k+1} \\ z^a_{2,k+1} \end{bmatrix} = D\begin{bmatrix} z^a_{1,k} \\ z^a_{2,k} \end{bmatrix} \tag{20}$$

where

$$\begin{bmatrix} z^a_{1,k} \\ z^a_{2,k} \end{bmatrix} \equiv V^{-1}\begin{bmatrix} \hat{\pi}^a_{k} \\ \hat{x}^a_{k} \end{bmatrix}$$

for $k \geq 1$.

The system in (20) consists of two equations:

$$z_{1,k+1} = \lambda_1 z_{1,k}\colon \ \lambda_1 \geq 1;$$

$$z_{2,k+1} = \lambda_1 z_{2,k}\colon \ \lambda_2 < 1.$$

Since $\lambda_1 \geq 1$, the first of these equations is nonstationary. Hence, if attention is restricted to stationary equilibria, we require that

$$z_{1,k} = 0.$$

Let

$$V^{-1} \equiv \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}.$$

Then $z_{1,k} = 0$ if and only if

$$v_{11}\hat{\pi}_k^a + v_{12}\hat{x}_k^a = 0,$$

or

$$\hat{x}_k^a = -\left(\frac{v_{11}}{v_{12}}\right)\hat{\pi}_k^a. \tag{21}$$

This determines $\hat{x}_1^a$ once $\hat{\pi}_1^a$ is fixed (but fixing $\hat{\pi}_1^a$ requires knowing the announced inflation rate π_1^a and $\bar{\pi}$, the mean post-exit inflation rate which depends on π_z^a and x_z^a).

Given $\hat{\pi}_k$ and $\hat{x}_k$, $z_{2,k}$ is given by

$$z_{2,k} = v_{21}\hat{\pi}_k^a + v_{22}\hat{x}_k^a.$$

Since $z_{2,k+1} = \lambda_2 z_{2,k}$, knowledge of $z_{2,k}$ determines $z_{2,k+1}$. Future values of π_{k+j}^a and x_{k+j}^a for $j > 1$, if the economy remains away from the ELB are obtained by jointly solving

$$0 = v_{11}\hat{\pi}_{k+j}^a + v_{12}\hat{x}_{k+j}^a$$

and

$$z_{2,k+j} = v_{21}\hat{\pi}_{k+j}^a + v_{22}\hat{x}_{k+j}^a = \lambda_2 z_{2,k+j-1} = \lambda_2\left(v_{21}\hat{\pi}_{k+j-1}^a + v_{22}\hat{x}_{k+j-1}^a\right).$$

for the $k+j$ equilibrium, and so on. More compactly, once we have $\hat{\pi}_1^a$ and $\hat{x}_1^a$, then future values of inflation and the output in the non-ELB equilibrium can be obtained from

$$\begin{bmatrix} v_{11}\hat{\pi}^a_{k+1} + v_{12}\hat{x}^a_{k+1} \\ v_{21}\hat{\pi}^a_{k+1} + v_{22}\hat{x}^a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2\left(v_{21}\hat{\pi}^a_k + v_{22}\hat{x}^a_k\right) \end{bmatrix},$$

or

$$\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \hat{\pi}^a_{k+1} \\ \hat{x}^a_{k+1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \begin{bmatrix} \hat{\pi}^a_k \\ \hat{x}^a_k \end{bmatrix}$$

as $v_{11}\hat{\pi}^a_k + v_{12}\hat{x}^a_k = 0$. Pre-multiplying both sides by V (recalling that the matrix of the $v'_{ij}s$ is V^{-1} yields

$$\begin{aligned} \begin{bmatrix} \hat{\pi}^a_{k+1} \\ \hat{x}^a_{k+1} \end{bmatrix} &= V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1} \begin{bmatrix} \hat{\pi}^a_k \\ \hat{x}^a_k \end{bmatrix} \\ &= VDV^{-1} \begin{bmatrix} \hat{\pi}^a_k \\ \hat{x}^a_k \end{bmatrix} = P \begin{bmatrix} \hat{\pi}^a_k \\ \hat{x}^a_k \end{bmatrix}, \end{aligned} \tag{22}$$

as required.

It remains to determine $\bar{\pi}$ and $\bar{x}$. Recall we started with an announced inflation rate π^a_1. We need $\bar{\pi}$ to map π^a_1 into $\hat{\pi}^\alpha_1$, and we need $\bar{x}$ to map $\hat{x}^a_1$ into x^a_1. Once x^a_1 is known, it, together with π^a_1, allow π^a_z and x^a_z to be obtained. At the ELB, π^a_x and x^a_z must satisfy

$$(1-\beta q)\,\pi_z = \beta(1-q)\pi^a_1 + \kappa x_z$$

and

$$\sigma\,(1-q)\,x_z = \sigma(1-q)x^a_1 + \left[q\pi_z + (1-q)\pi^a_1 + r_z\right],$$

or

$$\begin{bmatrix} \pi^a_z \\ x^a_z \end{bmatrix} = S \begin{bmatrix} \pi^a_1 \\ x^a_1 \end{bmatrix} + Tr_z \tag{23}$$

where

$$S = \begin{bmatrix} 1-\beta q & -\kappa \\ -q & \sigma\,(1-q) \end{bmatrix}^{-1} \begin{bmatrix} \beta(1-q) & 0 \\ (1-q) & \sigma\,(1-q) \end{bmatrix},$$

and

$$T = \begin{bmatrix} 1-\beta q & -\kappa \\ -q & \sigma\,(1-q) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From (21),

$$\begin{bmatrix} \hat{\pi}_1^a \\ \hat{x}_1^a \end{bmatrix} \equiv \begin{bmatrix} \pi_1^a \\ x_1^a \end{bmatrix} - \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{v_{11}}{v_{12}} \end{bmatrix} \hat{\pi}_1^a,$$

or

$$\begin{aligned} \begin{bmatrix} \pi_1^a \\ x_1^a \end{bmatrix} &= \begin{bmatrix} 1 \\ -\frac{v_{11}}{v_{12}} \end{bmatrix} \hat{\pi}_1^a + \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -\frac{v_{11}}{v_{12}} \end{bmatrix} \pi_1^a + \begin{bmatrix} 0 & 0 \\ \frac{v_{11}}{v_{12}} & 1 \end{bmatrix} \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix}. \end{aligned}$$

This allows us to rewrite (23) as

$$\begin{bmatrix} \pi_z^a \\ x_z^a \end{bmatrix} = S \begin{bmatrix} 1 \\ -\frac{v_{11}}{v_{12}} \end{bmatrix} \pi_1^a + S \begin{bmatrix} 0 & 0 \\ \frac{v_{11}}{v_{12}} & 1 \end{bmatrix} \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} + Tr_z.$$

Using this in (18) yields

$$\begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} = P_2 \begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} + P_3 \pi_1^a + QTr_z,$$

where

$$P_2 \equiv P + QS \begin{bmatrix} 0 & 0 \\ \frac{v_{11}}{v_{12}} & 1 \end{bmatrix},$$

and

$$P_3 \equiv QS \begin{bmatrix} 1 \\ -\frac{v_{11}}{v_{12}} \end{bmatrix}.$$

Since r_z is exogenous and π_1^a is a policy choice, we can then solve for $\bar{\pi}$ and $\bar{x}$ as

$$\begin{bmatrix} \bar{\pi} \\ \bar{x} \end{bmatrix} = (I_2 - P_2)^{-1} \left(P_3 \pi_1^a + QTr_z \right), \tag{24}$$

where I_2 is the 2×2 identity matrix. From $\bar{\pi}$, $\hat{\pi}_1^a = \pi_1^a - \bar{\pi}$ while from (21),

$$x_1^a = \bar{x} - \left(\frac{v_{11}}{v_{12}} \right) \left(\pi_1^a - \bar{\pi} \right). \tag{25}$$

This completes the derivation of the solution. Given the announced inflation rate π_1^a

for the exit period, (24) can be solved for the post-exit average inflation and output gap. Equation (25) then pins down x_1^a consistent with a stationary equilibrium. Equilibrium when the ELB is binding is given by (23), while (22) can be solved for future inflation rates and output gaps should the economy remain away from the ELB.

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