**Econ 105, Spring 2008**
**C. E. Walsh**

## Problem set 2: Answers

1. Suppose the central bank's objective is to minimize

$$V = \frac{1}{2}\lambda(u - ku_n)^2 + \frac{1}{2}\pi^2,$$

where $u$ is the unemployment rate and $u_n$ is the natural rate of unemployment, with $k < 1$. Assume the economy is described by

$$u = u_n - a(\pi - \pi^e).$$

For simplicity, assume the central bank can control inflation directly.

(a) What is the equilibrium rate of inflation under discretion? *Substitute the equation for the unemployment rate into the objective function V to yield*

$$V = \frac{1}{2}\lambda\left[u_n - a(\pi - \pi^e) - ku_n\right]^2 + \frac{1}{2}\pi^2.$$

*Under discretion, the central bank picks $\pi$ to minimize this, taking $\pi^e$ as given. The first order condition is*

$$-a\lambda\left[u_n - a(\pi - \pi^e) - ku_n\right] + \pi = 0.$$

*Solving for $\pi$,*

$$\left(1 + a^2\lambda\right)\pi - a^2\lambda\pi^e - a\lambda(1-k)u_n = 0 \Rightarrow \pi = \frac{a^2\lambda\pi^e + a\lambda(1-k)u_n}{(1 + a^2\lambda)}.$$

*Since private sector expectations will be correct (there is no uncertainty here), it must hold in equilibrium that $\pi^e = \pi$. Using this result, we have that the equilibrium inflation rate under discretion $\pi^d$ is*

$$\pi^d = \frac{a^2\lambda\pi^d + a\lambda(1-k)u_n}{(1 + a^2\lambda)} \Rightarrow \pi^d = a\lambda(1-k)u_n.$$

(b) How does a fall in $u_n$ affect the equilibrium rate of inflation? Explain. *With $k < 1$, the results in part (a) show that average inflation is positive. A fall in $u_n$ reduces the gap between $u_n$ and the target unemployment rate of $ku_n$. This lowers the incentive to try to expand the economy to reduce unemployment, so the equilibrium inflation rate under discretion declines:*

$$\frac{\partial\pi^d}{\partial u_n} = a\lambda(1-k) > 0.$$

(c) During the 1970s, the natural rate of unemployment in the US rose due to the entry of the baby boomers into the labor force. *Using your results from part (b), could this rise in $u_n$ explain the rise in inflation in the 1970s? Yes, or at least, potentially. A rise in $u_n$ according to the results from part (b) would increase the equilibrium rate of i*nflation under discretion.

2. Suppose the economy is subject to productivity shocks $e_t$, so that

$$u = u_n - a(\pi - \pi^e) + e_t.$$

Assume $e$ has expected value zero and is serially uncorrelated. Assume the central bank's loss function is the same as in question 1. Also assume that private agents must form expectations before observing the realization of $e_t$, but the central bank can set policy after observing $e_t$.

(a) What is the equilibrium rate of inflation under discretion? *The only difference from part (a) of the previous problem arises from the shock e. Carrying out the same steps as before, one finds that the central bank's first order condition implies*

$$\pi = \frac{a^2\lambda\pi^e + a\lambda(1-k)u_n + a\lambda e}{(1+a^2\lambda)}. \tag{1}$$

*Since private agents do not know e when they form their expectations, their best guess is simply that e* $= 0$*, so their best guess is that the central bank will pick* $\pi$ *to satisfy*

$$\pi = \frac{a^2\lambda\pi^e + a\lambda(1-k)u_n}{(1+a^2\lambda)}.$$

*So the public will expect*

$$\pi^e = \frac{a^2\lambda\pi^e + a\lambda(1-k)u_n}{(1+a^2\lambda)} \Rightarrow \pi^e = a\lambda(1-k)u_n.$$

*Using this in (1),*

$$\pi = \frac{a^2\lambda\left[a\lambda(1-k)u_n\right] + a\lambda(1-k)u_n + a\lambda e}{(1+a^2\lambda)} \Rightarrow \pi = a\lambda(1-k)u_n + \left(\frac{a\lambda}{1+a^2\lambda}\right)e,$$

*so the equilibrium inflation rate under discretion is*

$$\pi^d = a\lambda(1-k)u_n + \left(\frac{a\lambda}{1+a^2\lambda}\right)e.$$

(b) What is the equilibrium unemployment rate under discretion? *From the equation for the unemployment rate,*

$$\begin{aligned}
u &= u_n - a(\pi - \pi^e) + e \\
&= u_n - a\left[a\lambda(1-k)u_n + \left(\frac{a\lambda}{1+a^2\lambda}\right)e - a\lambda(1-k)u_n\right] + e \\
&= u_n + \left(\frac{1}{1+a^2\lambda}\right)e
\end{aligned}$$

(c) What is the equilibrium rate of inflation under commitment? *Under commitment, the central bank can announce it will set inflation according to*

$$\pi_t = \bar{\pi} + be,$$

*where $\bar{\pi}$ is the average inflation rate and $b$ measures how the central bank adjusts inflation in response to $e$. Since the central bank can commit, private agents will expect an inflation rate of $\bar{\pi}$ (since their best guess for $e$ is that it will be zero). Thus, unemployment will be given by*

$$u = u_n - a\left(\bar{\pi} + be - \bar{\pi}\right) + e = u_n + (1 - ab)e.$$

Hence, the loss function of the central bank will be

$$V = \frac{1}{2}\lambda\left[(1-k)u_n + (1-ab)e\right]^2 + \frac{1}{2}\left(\bar{\pi} + be\right)^2.$$

*Since $\bar{\pi}$ only apears in the last term, we can minimize it by setting $\bar{\pi} = 0$ – so on average inflation will be zero. Now pick $b$ to minimize the expected value of*

$$V = \frac{1}{2}\lambda\left[(1-k)u_n + (1-ab)e\right]^2 + \frac{1}{2}b^2e^2.$$

*The first order condition is*

$$E\left\{-ea\lambda\left[(1-k)u_n + (1-ab)e\right] + be^2\right\}.$$

*Recall that the expected value of $e$ is zero and so $Ee^2 = \sigma_e^2$ is the variance of $e$. Hence, the first order condition becomes*

$$\begin{aligned} E\left\{-ea\lambda(1-k)u_n - a\lambda(1-ab)e^2 + be^2\right\} &= -Eae\lambda(1-k)u_n - a\lambda(1-ab)Ee^2 + bEe^2 \\ &= 0 - a\lambda(1-ab)\sigma_e^2 + b\sigma_e^2 = 0. \end{aligned}$$

*This means that*

$$-a\lambda(1-ab)\sigma_e^2 + b\sigma_e^2 = 0 \text{ or } b = \frac{a\lambda}{1+a^2\lambda}.$$

*Thus, under commitment, the equilibrium inflation rate $\pi^c$ is*

$$\pi^c = \left(\frac{a\lambda}{1+a^2\lambda}\right)e$$

*and expected inflation is zero.*

(d) What is the equilibrium unemployment rate under commitment?

$$\begin{aligned} u &= u_n - a\left(\pi - \pi^e\right) + e \\ &= u_n - a\left[\left(\frac{a\lambda}{1+a^2\lambda}\right)e\right] + e \\ &= u_n + \left(\frac{1}{1+a^2\lambda}\right)e \end{aligned}$$

(e) How do the reactions of inflation and the unemployment rate to a productivity shock compare under discretion and commitment? *They are the same.*

3. Suppose social loss is given by

$$V^s = \frac{1}{2}\lambda^s (y - y^*)^2 + \frac{1}{2}\pi^2,$$

and

$$y = \bar{y} + a(\pi - \pi^e) + e.$$

Assume $y^* = \bar{y} + k$, where $k > 0$. Policy is conducted by a central bank whose preferences are to minimize

$$V^{cb} = \frac{1}{2}\lambda^{cb} (y - y^*)^2 + \frac{1}{2}\pi^2.$$

Policy is conducted under discretion and private agents must form expectations before observing $e$. Let $e$ have a mean of zero and variance of $\sigma_e^2$.

(a) Derive the equilibrium for output and inflation. *This is just like problem 2 except that things are expressed in terms of output $y$ rather than unemployment $u$. Substitute the supply equation into the central bank's loss function to obtain*

$$V^{cb} = \frac{1}{2}\lambda^{cb} \left[\bar{y} + a(\pi - \pi^e) + e - y^*\right]^2 + \frac{1}{2}\pi^2.$$

*The first order condition is*

$$a\lambda^{cb} \left[\bar{y} + a(\pi - \pi^e) + e - y^*\right] + \pi = 0 \Rightarrow \pi = \frac{a^2\lambda^{cb}\pi^e + a\lambda^{cb}(y^* - \bar{y}) - a\lambda^{cb}e}{1 + a^2\lambda^{cb}}.$$

*Since the private sector does not observe $e$, their best guess is that $e = 0$, so they expect*

$$\pi^e = \frac{a^2\lambda^{cb}\pi^e + a\lambda^{cb}(y^* - \bar{y})}{1 + a^2\lambda^{cb}} \Rightarrow \pi^e = a\lambda^{cb}(y^* - \bar{y}).$$

*Using this in the first order condition, equilibrium inflation under discretion is*

$$\pi^d = \frac{a^2\lambda^{cb}a\lambda^{cb}(y^* - \bar{y}) + a\lambda^{cb}(y^* - \bar{y}) - a\lambda^{cb}e}{1 + a^2\lambda^{cb}} = a\lambda^{cb}(y^* - \bar{y}) - \left(\frac{a\lambda^{cb}}{1 + a^2\lambda^{cb}}\right) e$$

*and output is*

$$\begin{aligned} y &= \bar{y} + a(\pi - \pi^e) + e = \bar{y} - a\left(\frac{a\lambda^{cb}}{1 + a^2\lambda^{cb}}\right) e + e \\ &= \bar{y} + \left(\frac{1}{1 + a^2\lambda^{cb}}\right) e \end{aligned}$$

(b) Find the expected value of $V^s$. Let $k \equiv y^* - \bar{y}$. Social loss is

$$\begin{aligned}\frac{1}{2}\lambda^s(y-y^*)^2+\frac{1}{2}\pi^2 &= \frac{1}{2}\lambda^s\left[-k+\left(\frac{1}{1+a^2\lambda^{cb}}\right)e\right]^2 \\ &+\frac{1}{2}\left[a\lambda^{cb}k-\left(\frac{a\lambda^{cb}}{1+a^2\lambda^{cb}}\right)e\right]^2\end{aligned}$$

The expected value of $V^s$ is

$$\begin{aligned}EV^s &= \frac{1}{2}E\lambda^s\left[-k+\left(\frac{1}{1+a^2\lambda^{cb}}\right)e\right]^2 \\ &\quad+\frac{1}{2}E\left[a\lambda^{cb}k-\left(\frac{a\lambda^{cb}}{1+a^2\lambda^{cb}}\right)e\right]^2 \\ &= \frac{1}{2}\lambda^s k^2+\frac{1}{2}\lambda^s\left(\frac{1}{1+a^2\lambda^{cb}}\right)^2\sigma_e^2+\frac{1}{2}a^2\left(\lambda^{cb}\right)^2k^2+\frac{1}{2}\left(\frac{a\lambda^{cb}}{1+a^2\lambda^{cb}}\right)^2\sigma_e^2 \\ &= \frac{1}{2}\left(\lambda^s+a^2\lambda^{cb}\right)k^2+\frac{1}{2}\left[\lambda^s\left(\frac{1}{1+a^2\lambda^{cb}}\right)^2+\left(\frac{a\lambda^{cb}}{1+a^2\lambda^{cb}}\right)^2\right]\sigma_e^2 \\ &= \frac{1}{2}\left(\lambda^s+a^2\lambda^{cb}\right)k^2+\frac{1}{2}\left(\frac{1}{1+a^2\lambda^{cb}}\right)^2\left[\lambda^s+a^2(\lambda^{cb})^2\right]\sigma_e^2.\end{aligned}$$

*The first terms is increasing in* $\lambda^{cb}$, so appointing a central banker with a lower $\lambda^{cb}$ would decrease expected social loss. The second term, however, is increasing in $\lambda^{cb}$ – to see this, note that

$$\begin{aligned}\frac{\partial\left(\frac{1}{1+a^2\lambda^{cb}}\right)^2\left[\lambda^s+a^2(\lambda^{cb})^2\right]}{\partial\lambda^{cb}} &= \frac{2a^2(\lambda^{cb})\left(1+a^2\lambda^{cb}\right)^2-2\left(1+a^2\lambda^{cb}\right)a^2\left[\lambda^s+a^2(\lambda^{cb})^2\right]}{\left(1+a^2\lambda^{cb}\right)^4} \\ &= \frac{2a^2(\lambda^{cb})\left(1+a^2\lambda^{cb}\right)-2a^2\left[\lambda^s+a^2(\lambda^{cb})^2\right]}{\left(1+a^2\lambda^{cb}\right)^3} \\ &= \frac{2a^2(\lambda^{cb})+2a^4\left(\lambda^{cb}\right)^2-2a^2\lambda^s-2a^4(\lambda^{cb})^2}{\left(1+a^2\lambda^{cb}\right)^3} \\ &= \left[\frac{2a^2}{\left(1+a^2\lambda^{cb}\right)^3}\right](\lambda^{cb}-\lambda^s)\end{aligned}$$

which has the sign of $(\lambda^{cb}-\lambda^s)$. So this terms is reduced if $\lambda^{cb}<\lambda^s$.

(c) Explain why society might achieve a lower value of expected loss if $\lambda^{cb} < \lambda^s$ than if $\lambda^{cb} = \lambda^s$. *A lower* $\lambda^{cb}$ *reduces the average inflation bias under discretion.*

(d) Does an increase in $\sigma_e^2$ increase or decrease the "optimal degree of central bank conservatism"? (Discuss – no derivations necessary.) *A larger* $\sigma_e^2$ *means the economy is hit by larger shocks – this makes stabilization policy more important and raises the costs of having a conservative central bank who is less concerned than society as a whole with stabilizing the real economy.*