$\mathscr{L}$-invariants and Shimura curves

Samit Dasgupta
Matthew Greenberg

October 20, 2009

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Abstract

In [8], the second author described how one may extract Darmon-style $\mathscr{L}$-invariants from modular forms on Shimura curves that are special at p. In this paper, we show that these $\mathscr{L}$-invariants are preserved by the Jacquet-Langlands correspondence. As a consequence, we prove the outstanding period conjecture of [8] in the case where the base field is $\mathbb{Q}$. As a further application of our methods, we use integrals of Hida families to describe the Abel-Jacobi maps and Stark-Heegner points appearing in [8].

1 Introduction

Let N and p be relatively prime positive integers with p prime and let $f \in S_2(\Gamma_0(Np))^{p\text{-new}}$ be a Hecke eigenform. In their study of p-adic L-functions associated to modular forms, Mazur, Tate and Teitelbaum [11] introduce a p-adic invariant of f which they call its $\mathscr{L}$-*invariant*. Let $L_p(f,s)$ be the p-adic L-function of f. If the U_p-eigenvalue a_p of f is equal to 1, as opposed to -1, then the interpolation property of $L_p(f,s)$ implies that it has an "exceptional zero" at $s=1$. In this case, they conjecture in [11] that there is a p-adic number $\mathscr{L}^{\mathrm{MTT}}(f)$ such that

$$L'_p(f,\chi,1) = \mathscr{L}^{\mathrm{MTT}}(f)\frac{p^{\mathrm{cond}(\chi)}}{\tau(\overline{\chi})}\frac{L(f,\overline{\chi},1)}{\Omega_f^{\chi(-1)}}, \tag{1}$$

for all finite-order characters χ of $\mathbb{Z}_p^\times$ of conductor $\operatorname{cond}\chi = p^m$. Here, $\tau(\overline{\chi})$ is the Gaussian sum associated to χ and $\Omega_f^{\chi(-1)}$ is the real or imaginary period of f depending on the parity of χ. All of this makes sense as $L(f,\overline{\chi},1)/\Omega_f^\pm$ is algebraic. The uniqueness of $\mathscr{L}^{\mathrm{MTT}}(f)$ follows from a nonvanishing result of Rohrlich [12] which asserts the existence of a finite order character χ of p-power conductor such that $L(f,\overline{\chi},1) \neq 0$.

This conjecture was proved by Greenberg and Stevens in the influential paper [9]. Since f is p-ordinary, i.e., $a_p(f)$ is a p-adic unit, f lives in a p-adic analytic family of eigenforms by the work of Hida. More precisely, there is a p-adic disk U containing 2 and p-adic analytic functions $\boldsymbol{a}_n(\boldsymbol{f})$ on U, $n \geq 2$, such that

1. $\boldsymbol{f}(k) := q + \sum_{n\geq 2} \boldsymbol{a}_n(\boldsymbol{f},k)q^n$ is the q expansion of an element of $S_k(\Gamma_0(Np))$ for all positive integers $k \geq 2$ with $k \in U$.

2. $\boldsymbol{f}(2) = f$.

Moreover, up to shrinking U around 2, $\boldsymbol{f}$ is completely determined by f. Note that $1 - \boldsymbol{a}_p(\boldsymbol{f},k)^2$ vanishes at $k=2$ as $a_p(f) = \pm 1$. Thus, it is natural to consider the derivative of this quantity. Greenberg and Stevens show that

$$\mathscr{L}^{\mathrm{MTT}}(f) = \frac{d}{dk}\Big|_{k=2}\big(1 - \boldsymbol{a}_p(\boldsymbol{f},k)^2\big) =: \mathscr{L}^{\mathrm{GS}}(f). \tag{2}$$

Observe also that (2) extends the definition of the $\mathscr{L}$-invariant from the case $a_p(f) = 1$ originally considered in [11] to the case $a_p(f) = \pm 1$.

Mazur, Tate, and Teitelbaum further conjecture in [11] that the factor $\mathscr{L}^{\mathrm{MTT}}(f)$ is of local type, i.e., depends only on the two-dimensional p-adic representation $\sigma_p(f)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated to f. Greenberg and Stevens prove this in [9] by showing that $\mathscr{L}^{\mathrm{GS}}(f)$ may be described in terms of the deformation theory of $\sigma_p(f)$.

Since the $\mathscr{L}$-invariant is a local-at-p invariant of f, it is natural to attempt to extract the $\mathscr{L}$-invariant of f from its Jacquet-Langlands lift g to another indefinite quaternion algebra B split at p, i.e., with $B_p \cong M_2(\mathbb{Q}_p)$, since the corresponding automorphic representations will have the same local components at p. (The case of definite quaternion algebras was resolved by Bertolini, Darmon and Iovita [2].) Following Darmon [6], the second author [8] proposed a conjectural method for doing this, as follows. Let $M^0(X)$ be the space of $\mathbb{C}_p$-valued measures on

$$X := \mathbb{P}^1(\mathbb{Q}_p)$$

with total measure zero. A Mayer-Vietoris argument, together with multiplicity-one, shows that for each choice of sign ± at infinity,

$$\dim_{\mathbb{C}_p} H^1(\Theta, M^0(X))^{g,\pm} = 1,$$

where Θ is a p-arithmetic subgroup of $B^\times$ of level

$$N^+ := N/\operatorname{disc} B. \tag{3}$$

Let $\varphi_g^\pm$ be a nonzero element of $H^1(\Theta, M^0(X))^{g,\pm}$.

For each $\mathscr{L} \in \mathbb{Q}_p$, there is a unique branch $\log_{\mathscr{L}}$ of the p-adic logarithm such that

$$\log_{\mathscr{L}} p = \mathscr{L}.$$

Let

$$\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$$

be the p-adic upper half-plane. Associated to each branch of the p-adic logarithm, there is a $\mathrm{PGL}_2(\mathbb{Q}_p)$-invariant integration pairing

$$\langle \cdot\,, \cdot \rangle_{\mathscr{L}} : M^0(X) \times \operatorname{Div}^0 \mathcal{H}_p \longrightarrow \mathbb{C}_p$$

defined by

$$\langle \mu, \{\tau'\} - \{\tau\} \rangle_{\mathscr{L}} = \int_X \log_{\mathscr{L}} \left(\frac{x - \tau'}{x - \tau} \right) \mu(x),$$

which, in turn, induces a pairing

$$H^1(\Theta, M^0(X)) \times H_1(\Theta, \operatorname{Div}^0 \mathcal{H}_p) \longrightarrow \mathbb{C}_p.$$

Let

$$\partial : H_2(\Theta, \mathbb{Z}) \longrightarrow H_1(\Theta, \operatorname{Div}^0 \mathcal{H}_p)$$

be the boundary map in the long exact sequence in Θ-cohomology associated to the short exact sequence defining $\operatorname{Div}^0 \mathcal{H}_p$.

Proposition 1 (cf. [8, Prop. 30]). *There are unique Darmon-style $\mathscr{L}$-invariants $\mathscr{L}^D(\varphi_g^\pm)$ such that*

$$\left\langle \varphi_g^\pm, \partial H_2(\Theta, \mathbb{Z}) \right\rangle_{\mathscr{L}^D(\varphi_g^\pm)} = \{0\}.$$

The goal of this paper is to relate these $\mathscr{L}$-invariants $\mathscr{L}^{\mathrm{D}}(\varphi_g^\pm)$ arising from the cohomology of Shimura curves to those whose origins lie in the arithmetic of classical modular curves. Our main result is:

Theorem 2. *$\mathscr{L}^{D}(\varphi_g^{\pm}) = \mathscr{L}^{GS}(f)$.*

From Theorem 2, we may deduce Conjecture 2 of [8] in the case where the base field is $\mathbb{Q}$; see §8 for details. The proof of Theorem 2 falls into two steps. Applying a result of Hida's theory, we may deform the Jacquet-Langlands lift g of f into a cohomological Hida family $\Phi_g^{\pm}$. Let $\boldsymbol{a}_p = \boldsymbol{a}_p(k)$ be the eigenvalue of U_p acting on $\Phi_g^{\pm}$. Group cohomological calculations inspired by those in the first author's thesis [7] show that

$$\mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm}) = \frac{d}{dk}\Big|_{k=2}\big(1 - \boldsymbol{a}_p(\boldsymbol{g}, k)^2\big) =: \mathscr{L}^{\mathrm{GS}}(g).$$

It remains to show that $\mathscr{L}^{\mathrm{GS}}(g) = \mathscr{L}^{\mathrm{GS}}(f)$. This follows from Theorem 8, a result asserting a compatibility between the Jacquet-Langlands correspondence with the formation of Hida families. This proposition, which is a weak analogue of results of Chenevier [5] for definite quaternion algebras, may be of independent interest.

In the last section of this paper, we apply our computations to the theory of *Stark-Heegner points*. Let $E/\mathbb{Q}$ be an elliptic curve and suppose that $\mathcal{O}$ is a real quadratic order with fraction field K such that $(\operatorname{disc}\mathcal{O}, N) = 1$. Assume further that the sign in the functional equation of $L(E/K, s)$ is -1. Then for each character $\chi : \mathrm{Cl}_{\mathcal{O}}^{+} \to \mathbb{C}^{\times}$ of the narrow ideal class group of $\mathcal{O}$, the sign in the functional equation of $L(E/K, \chi, s)$ is also -1. Thus, the conjecture of Birch and Swinnerton-Dyer leads one to expect that

$$\operatorname{rank} E(H_{\mathcal{O}}) = \operatorname{ord}_{s=1} L(E/H_{\mathcal{O}}, s) = \operatorname{ord}_{s=1} \prod_{\chi:\mathrm{Cl}_{\mathcal{O}}^{+}\to\mathbb{C}^{\times}} L(E/K, \chi, s) \geq |\,\mathrm{Cl}_{\mathcal{O}}^{+}\,|, \tag{4}$$

where $H_{\mathcal{O}}$ is the narrow ring class field associated to the order $\mathcal{O}$. In [8], the second author presented a p-adic analytic construction of local *Stark-Heegner points* on E. The local definition of these points is contingent upon Conjecture 2 of [8] over the base field $\mathbb{Q}$, which is implied by Theorem 2. The analytically defined Stark–Heegner points are conjectured to be defined over the field $H_{\mathcal{O}}$, and are expected to generate a finite index subgroup of $E(H_{\mathcal{O}})$ when the inequality in (4) is an equality. The construction of [8] generalized a construction of Darmon [6] applicable when there is a unique prime p dividing of the conductor of $E/\mathbb{Q}$ that is inert in K.

The strongest theoretical evidence presented to date for the conjectures of [6] is the main result of [4] which proves the rationality of certain linear combinations of Stark-Heegner points. A key tool in the proof of this result is a description of the formal group logarithms of Stark-Heegner points in terms of periods of Hida families. In §9, we prove such a formula for the Stark-Heegner points of [8]. We intend to pursue the analogue of the rationality result of [4] in future work.

2 Modular forms on quaternion algebras and the cohomology of Shimura curves

Let f be as in the introduction and suppose, also as in the introduction, that f (or, more precisely, the automorphic representation π_f associated to f) admits a Jacquet-Langlands lift to $B^\times$, where B is an indefinite quaternion $\mathbb{Q}$-algebra split at p. Let N^- be the discriminant of B. Then f is necessarily N^--new, and the tame level N of f admits the factorization

$$N = pN^-N^+, \quad p \nmid N^-N^+, \quad (N^-, N^+) = 1.$$

Let R be an Eichler order in B of level N^+. Since B is split at p, we may choose an embedding

$$\iota_p : B \to M_2(\mathbb{Q}_p).$$

We may make this selection so that $\iota_p(R) \subset M_2(\mathbb{Z}_p)$ and define

$$R_0 = \left\{ \alpha \in R : \iota_p(\alpha) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p} \right\}. \tag{5}$$

Thus, R_0 is an Eichler order in B of level pN^+. Set

$$\Gamma = R_+^\times/\{\pm 1\}, \quad \Gamma_0 = R_{0,+}^\times/\{\pm 1\},$$

where the subscript $+$ indicates elements with positive reduced norm.

Since B is split at the infinite place of $\mathbb{Q}$, we may choose an embedding

$$\iota_\infty : B \longrightarrow M_2(\mathbb{R}). \tag{6}$$

The groups Γ and Γ_0 may be viewed as discrete groups of transformations of the complex upper half-plane $\mathcal{H}$ by identifying them with subgroups of $\mathrm{PGL}_2(\mathbb{R})$ via ι_∞. The quotients

$$Y(\mathbb{C}) := \Gamma \backslash \mathcal{H} \quad Y_0(\mathbb{C}) := \Gamma_0 \backslash \mathcal{H}$$

are Riemann surfaces, compact exactly when $N^- \neq 1$. Let $\mathcal{H}^*$ be the extended complex upper half-plane and define

$$X(\mathbb{C}) = \begin{cases} Y(\mathbb{C}) & \text{if } N^- \neq 1, \\ \Gamma \backslash \mathcal{H}^* & \text{if } N^- = 1. \end{cases}$$

Define $X_0(\mathbb{C})$ analogously. The Riemann surfaces $X(\mathbb{C})$ and $X_0(\mathbb{C})$ are compact and may be identified with the loci of complex points of *Shimura curves* X and X_0 that admit canonical models over $\mathbb{Q}$. Of course, these are just the classical modular curves in the case $N^- = 1$. The space $S_2(\Gamma)$ (resp. $\overline{S_2(\Gamma)}$) of holomorphic (resp. antiholomorphic), weight two cusp forms of level Γ is, by definition, the space of functions $f : \mathcal{H} \to \mathbb{C}$ such that $f(z)dz$ is the pullback of a holomorphic (resp. antiholomorphic) differential 1-form on X. The spaces $S_2(\Gamma_0)$ and $S_2(\Gamma)$ are defined analogously. These spaces admit the action of a commutative algebra of Hecke operators, all commuting with complex conjugation.

Theorem 3 (Jacquet-Langlands correspondence)**.** *There are Hecke-module isomorphisms*

$$S_2(\Gamma_0(N))^{N^-\text{-}new} \cong S_2(\Gamma), \quad S_2(\Gamma_0(Np))^{N^-\text{-}new} \cong S_2(\Gamma_0).$$

Therefore, there is a one-dimensional subspace of $S_2(\Gamma_0)$, independent of the choice of isomorphism in the Jacquet-Langlands correspondence, on which the Hecke operators act via the eigenvalues of f. Let g be a nonzero element of this space. We call g a *Jacquet-Langlands lift of f*.

We are also interested in cohomological avatars of g. We have canonical isomorphisms of Betti and group cohomology

$$H^*(\Gamma, E) = H^*(X(\mathbb{C}), E), \quad H^*(\Gamma_0, E) = H^*(X_0(\mathbb{C}), E)$$

for any characteristic zero field E. By the de Rham theorem and the Hodge decomposition,

$$H^1(\Gamma_0, \mathbb{C}) = H^1(X_0(\mathbb{C}), \mathbb{C}) = H^{1,0}(X_0(\mathbb{C}), \mathbb{C}) \oplus H^{0,1}(X_0(\mathbb{C}), \mathbb{C}) = S_2(\Gamma_0) \oplus \overline{S_2(\Gamma_0)}.$$

Therefore, if E is any field containing the Hecke eigenvalues of g,

$$\dim_E H^1(\Gamma_0, E)^g = 2,$$

where the superscript g indicates Hecke eigenspace corresponding to the system of Hecke eigenvalues of g. Conjugation by an element of $R_0^\times$ of reduced norm -1 induces an automorphism of $H^1(\Gamma_0, E)$ under which the subspace $H^1(\Gamma_0, E)^g$ is stable. (This action corresponds to complex conjugation of cusp forms.) Therefore, $H^1(\Gamma_0, E)^g$ decomposes into one-dimensional $\pm$-eigenspaces for this action:

$$H^1(\Gamma_0, E)^g = H^1(\Gamma_0, E)^{g,+} \oplus H^1(\Gamma_0, E)^{g,-}.$$

Let $g^\pm$ be a nonzero element of $H^1(\Gamma_0, E)^{g,\pm}$.

3 Hecke operators and group cohomology

In anticipation of the delicate group cohomological calculations to follow, we carefully set up notation for describing the action of Hecke operators on various cohomology groups. Let $G \subset K$ be an inclusion of groups, x an element of K, M a G-module, and N an xGx^{-1}-module. Suppose that $\xi : M \to N$ is a group homomorphism such that

$$\xi(gm) = xgx^{-1}\xi(m). \tag{7}$$

for all $g \in G$ and $m \in M$. Then ξ induces a homomorphism

$$\xi_* : H^*(G, M) \longrightarrow H^*(xGx^{-1}, N) \tag{8}$$

as follows: Let $F_\bullet \to \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by free K-modules. Note that F_r is also a free G-module and a free xGx^{-1}-module. In what follows, we will often take $F_r = \mathbb{Z}[K^{r+1}]$. Formally, ξ induces a map of cochain complexes relative to this resolution,

$$\xi_* : \mathrm{Hom}_G(F_r, M) \longrightarrow \mathrm{Hom}_{xGx^{-1}}(F_r, N), \quad \xi_*(\varphi)(f_r) = \xi(\varphi(x^{-1}f_r)),$$

which induces (8).

Let $w_p \in R_0$ be an element of reduced norm p that generates the normalizer of Γ_0 in $R[1/p]_+^\times$ and define

$$\widetilde{\Theta} = R[1/p]_+^\times \Big/ \mathbb{Z}[1/p]^\times.$$

The groups Γ_0, Γ and

$$\Gamma' := w_p \Gamma w_p^{-1}$$

are all subgroups of $\widetilde{\Theta}$. Using the above formalism with $G = \Gamma_0$ or Γ, $K = \widetilde{\Theta}$ and $x = w_p$ yields *Atkin-Lehner maps*

$$W_p : H^r(\Gamma_0, M) \longrightarrow H^r(\Gamma_0, N), \quad W_p : H^r(\Gamma, M) \longrightarrow H^r(\Gamma', N). \tag{9}$$

We note that these maps are isomorphisms, as applying the same formalism with w_p^{-1} instead of w_p yields inverse homomorphisms W_p^{-1}.

We choose an element $\Pi \in R_0$ of reduced norm p such that

$$\iota_p(\Pi)\mathfrak{I} \in \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \mathfrak{I}, \tag{10}$$

where $\mathfrak{I}$ is the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ defined by

$$\mathfrak{I} = \left\{ \alpha \in \mathrm{GL}_2(\mathbb{Z}_p) : \alpha \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod p \right\}.$$

Set $\Pi' = w_p^{-1} \Pi w_p$. We may form double coset decompositions

$$\Gamma_0 \cdot \Pi \cdot \Gamma_0 = \bigcup_{i=0}^{p-1} \gamma_a \Gamma_0, \qquad \Gamma_0 \cdot \Pi' \cdot \Gamma_0 = \bigcup_{i=0}^{p-1} \gamma_a' \Gamma_0, \tag{11}$$

where $\gamma_i' = w_p^{-1} \gamma_i w_p$. Let Σ and Σ' be the subsemigroups of $\widetilde{\Theta}$ generated by Γ_0 together with Π and Π' and let M and M' be Σ- and Σ'-modules, respectively. Let $F_\bullet \to \mathbb{Z}$ be a resolution of $\mathbb{Z}$ by free $\widetilde{\Theta}$-modules and define an endomorphism U_p of the cochain complex $\mathrm{Hom}_{\Gamma_0}(F_\bullet, M)$ by

$$(U_p \varphi)(f_r) = \sum_{i=0}^{p-1} \gamma_i \varphi(\gamma_i^{-1} f_r), \quad f_r \in F_r. \tag{12}$$

It is routine to check that U_p does not depend on our choice of coset representatives and descends to a well defined endomorphism U_p of $H^*(\Gamma_0, M)$. By replacing γ_i by γ_i', we analogously define an endomorphism U_p' of $H^*(\Gamma_0, M')$. Similarly, let U_p' be the Hecke operator associated to the double coset $\Gamma_0 \Pi' \Gamma_0$. It is easy to check that

$$U_p = W_p^{-1} \circ U_p' \circ W_p. \tag{13}$$

4 p-adic measures, Hida families, and Greenberg-Stevens $\mathscr{L}$-invariants

Let Y be a compact p-adic manifold and let A be a subring of $\mathbb{C}_p$. Write $C^\infty(Y) = C^\infty(Y, A)$ for the group of locally-constant, A-valued functions on Y, equipped with the sup-norm. An *E-valued measure on* Y is a bounded E-linear functional on $C^\infty(Y, A)$. We write $M(Y) = M(Y, A)$ for the space of such measures. $M(Y)$ can be identified with the space of finitely additive, A-valued functions on the set of compact-open subsets of Y whose values are bounded.

Let

$$\mathbb{X} = (\mathbb{Z}_p^2)' := \mathbb{Z}_p^2 - p(\mathbb{Z}_p^2)$$
$$\mathbb{X}_\infty = \mathbb{Z}_p^\times \times p\mathbb{Z}_p \subset \mathbb{X}.$$

The spaces $M(\mathbb{X})$ and $M(\mathbb{X}_\infty)$ are naturally modules for the Iwasawa algebra

$$\Lambda := \mathbb{Z}_p[\![1 + p\mathbb{Z}_p]\!],$$

where group-like elements act via the natural diagonal action of $1 + p\mathbb{Z}_p$ on $\mathbb{X}$:

$$([\ell]\mu)(h(x, y)) := \mu(h(\ell x, \ell y)), \qquad \ell \in 1 + p\mathbb{Z}_p.$$

Let

$$\varepsilon : \Lambda \longrightarrow \mathbb{Z}_p \tag{14}$$

be the augmentation map defined by $[\ell] \mapsto 1$ and let I_ε be the kernel of ε. Letting γ be a topological generator of $1 + p\mathbb{Z}_p$, it follows that I_ε is generated by

$$\varpi := [\gamma] - 1.$$

The group Γ acts on $\mathbb{X}$ via the embedding ι_p and $\mathbb{X}_\infty$ is stable under Γ_0. Therefore, we may consider the cohomology groups $H^*(\Gamma, M(\mathbb{X}))$ and $H^*(\Gamma_0, M(\mathbb{X}_\infty))$. These cohomology groups are canonically isomorphic:

Lemma 4. *The map $H^*(\Gamma, M(\mathbb{X})) \to H^*(\Gamma_0, M(\mathbb{X}_\infty))$ induced by the Γ_0-equivariant inclusion $\mathbb{X}_\infty \hookrightarrow \mathbb{X}$ is an isomorphism.*

Proof. The $p + 1$ translates of $\mathbb{X}_\infty$ by Γ cover $\mathbb{X}$. From this, it follows that

$$M(\mathbb{X}) = \text{Co-Ind}_{\Gamma_0}^{\Gamma} M(\mathbb{X}_\infty).$$

The lemma now follows from Shapiro's lemma. □

We set

$$\mathbb{W} := H^1(\Gamma_0, M(\mathbb{X}_\infty)) = H^1(\Gamma, M(\mathbb{X}))$$

One may check that $\Pi\mathbb{X}_\infty \subset \mathbb{X}_\infty$, so the semigroup Σ of §3 acts on $M(\mathbb{X}_\infty)$. Therefore, the formalism of §3 endows $\mathbb{W}$ with an action of the U_p operator. By Lemma 4 and transfer of structure, U_p also acts on $\mathbb{W}$. The group $\mathbb{W}$ enjoys an action of:

- Hecke operators T_ℓ, for primes $\ell \nmid Np = N^-N^+p$ and U_ℓ, for primes $\ell \mid pN^+$;
- Atkin-Lehner involutions W_ℓ for $\ell \mid N^-$;

Let $\mathbb{T}$ be the commutative Λ-subalgebra of $\mathrm{End}_\Lambda \mathbb{W}$ generated by these operators. Let $\rho : M(\mathbb{X}_\infty) \longrightarrow \mathbb{Z}_p$ be the total measure map. It induces a corresponding map

$$\rho : \mathbb{W} \longrightarrow H^1(\Gamma_0, \mathbb{Z}_p). \tag{15}$$

Viewing $\mathbb{Z}_p$ as a Λ-algebra via ε, we see that the map ρ is the composite of the maps

$$M(\mathbb{X}_\infty) \longrightarrow M(\mathbb{X}_\infty) \otimes_\Lambda \mathbb{Z}_p \xrightarrow{\sim} M(X_\infty) \longrightarrow \mathbb{Z}_p,$$

where the rightmost arrow in the above sequence is given by $\mu \mapsto \mu(X_\infty)$. The map ρ respects the decomposition into $\pm$-eigenspaces:

$$\rho : \mathbb{W}^\pm \longrightarrow H^1(\Gamma_0, \mathbb{Z}_p)^\pm.$$

Let $e = \lim_{n\to\infty} U_p^{n!}$ denote Hida's ordinary idempotent and, for any $\mathbb{T}$-module M, let $M^o = eM$. In particular, $\mathbb{T}^o = e\mathbb{T}$ is Hida's ordinary Hecke algebra.

Theorem 5 (Hida's control theorem). *There is an exact sequence*

$$0 \longrightarrow \varpi \mathbb{W}^{\pm,o} \longrightarrow \mathbb{W}^{\pm,o} \xrightarrow{\rho} H^1(\Gamma_0, \mathbb{Z}_p)^{\pm,o} \longrightarrow 0 \tag{16}$$

The kernel of the Λ-algebra homomorphism $\mathbb{T}^o \to \mathbb{Z}_p$ given by sending a Hecke operator to its eigenvalue on g is a prime ideal $\mathfrak{p} \subset \mathbb{T}^o$ lying above the augmentation ideal $I_\varepsilon \subset \Lambda$. The following fundamental result is due to Hida in the case $N^- = 1$ (see [9]), and was extended in [1] to the case $N^- \neq 1$.

Theorem 6. *There is a unique minimal prime $\mathfrak{P} \subset \mathfrak{p}$, and the quotient $R := \mathbb{T}^o/\mathfrak{P}$ is a finite flat extension of Λ unramified above I_ε.*

Let R be as in the theorem and let $R_\mathfrak{p}$ be the localization of R at $\mathfrak{p}$. Let E be the field of fractions of the integral closure of $\mathbb{Z}_p$ in R. It is a finite extension of $\mathbb{Q}_p$. Write $(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$ for the subspace of $(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^\pm$ on which $\mathbb{T}$ acts via the canonical map $\mathbb{T} \to R_\mathfrak{p}$. Note that $(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \subset (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,o} = \mathbb{W}^{\pm,o} \otimes R_\mathfrak{p}$ and that $H^1(\Gamma_0, \mathbb{Z}_p) \otimes_\Lambda R_\mathfrak{p} = H^1(\Gamma_0, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathcal{O}_E = H^1(\Gamma_0, \mathcal{O}_E)$.

Corollary 7 (see [1, §3.6]). *The sequence*

$$0 \longrightarrow \varpi(\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \longrightarrow (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \longrightarrow H^1(\Gamma_0, \mathcal{O}_E)^{\pm,g} \to 0$$

is exact and

$$\mathrm{rank}_{R_\mathfrak{p}} (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} = 1.$$

Fixing an embedding of E into $\overline{\mathbb{Q}}_p$, we may view $g^{\pm}$ as an element of $H^1(\Gamma_0, \mathcal{O}_E)^{g,\pm}$. Corollary 7, we may choose a lift

$$\Phi_g^{\pm} \in (\mathbb{W} \otimes_{\Lambda} R_{\mathfrak{p}})^{\pm,g} \tag{17}$$

of $g^{\pm} \in H^1(\Gamma_0, \mathcal{O}_E)^{\pm,g}$. The element $\Phi_g^{\pm}$ is well defined up to multiplication by an element of $1+\varpi R_{\mathfrak{p}}$. We call $\Phi_g^{\pm}$ a *Hida family through* $g^{\pm}$. We denote its U_p-eigenvalue by $\boldsymbol{a}_p(\Phi_g^{\pm}) \in R_{\mathfrak{p}}$.

By abuse of notation we write ε for the map $R_{\mathfrak{p}} \to \mathcal{O}_E$ given by reduction modulo ϖ. This abuse is justified as this map extends the augmentation $\varepsilon : \Lambda \to \mathbb{Z}_p$. As

$$\varepsilon(\boldsymbol{a}_p(\Phi_g^{\pm})) = a_p(g^{\pm}) = a_p(g) = a_p(f) = \pm 1,$$

we see that $1 - \boldsymbol{a}_p(\Phi_g^{\pm})^2$ lies in $\varpi R_{\mathfrak{p}}$. There is a "derivative map"

$$d_{\varepsilon} : \varpi R_{\mathfrak{p}}/(\varpi R_{\mathfrak{p}})^2 \longrightarrow \mathcal{O}_E$$

that extends the p-adic logarithm map $\log : I_{\varepsilon}/I_{\varepsilon}^2 \to \mathbb{Z}_p$ given by

$$[\ell] - 1 \mapsto \log(\ell). \tag{18}$$

Note that since $\ell \in \mathbb{Z}_p^{\times}$, we need not specify a branch of the p-adic logarithm. We define the *Greenberg-Stevens $\mathscr{L}$-invariant of g* by

$$\mathscr{L}^{\mathrm{GS}}(\Phi_g^{\pm}) = d_{\varepsilon}\big(1 - \boldsymbol{a}_p(\Phi_g^{\pm})^2\big).$$

Theorem 8. *We have the equality of Greenberg-Stevens $\mathscr{L}$-invariants $\mathscr{L}^{GS}(\Phi_g^{\pm}) = \mathscr{L}^{GS}(f)$.*

Proof. For $0 < r \leq 1$, let $\mathcal{A}_r$ be the subring of $\overline{\mathbb{Q}}_p[\![x]\!]$ consisting of those powers series which converge on the closed disk centered at 0 with radius r. Evidently, if $r < s$, then there is a canonical inclusion $\mathcal{A}_s \subset \mathcal{A}_r$. Therefore, we may set $\mathcal{A} = \bigcup_r \mathcal{A}_r$. Define $i : \Lambda \to \mathcal{A}_1$ by sending a group-like element $[\ell]$, for $\ell \in 1 + p\mathbb{Z}_p$, to the function $k \mapsto \ell^{k-2}$. Since R is unramified over I_{ε} and $\mathcal{A}$ is Henselian, there is a unique extension of i to a Λ-algebra homomorphism $i : R_{\mathfrak{p}} \to \mathcal{A}$. Let R' be a finitely generated R-subalgebra of $R_{\mathfrak{p}}$ such that $\Phi_g^{\pm} \in (\mathbb{W} \otimes_{\Lambda} R')^{g,\pm}$. Then there is some r_0 such that $i(R')$ is contained in $\mathcal{A}_{r_0}$.

Let $P_{k-2}(\overline{\mathbb{Q}}_p)$ be the space of homogeneous polynomials of degree $k-2$ in indeterminates x and y and let $V_{k-2}(\overline{\mathbb{Q}}_p)$ be its $\overline{\mathbb{Q}}_p$-linear dual. Define a "specialization to weight k map"

$$\rho_k : M(\mathbb{X}_{\infty}) \longrightarrow V_{k-2}(\overline{\mathbb{Q}}_p)$$

by the rule

$$\rho_k(\Phi)(P) = \int_{\mathbb{X}_{\infty}} P(x,y)\Phi(x,y).$$

This map being Γ_0-equivariant, it induces a homomorphism

$$\rho_k : H^1(\Gamma_0, M(\mathbb{X}_{\infty})) \longrightarrow H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}}_p)).$$

The maps ρ defined above coincide with ρ_2 in this generalized notation.

If $|k-2|_p \leq r$, we may extend ρ_k to a map

$$\rho_k : H^1(\Gamma_0, M(\mathbb{X}_\infty)) \otimes_\Lambda \mathcal{A}_r \longrightarrow H^1(\Gamma_0, V_{k-2}(\overline{\mathbb{Q}}_p))$$

by setting

$$\rho_k\left(\sum_i \varphi_i \otimes a_i\right) = \sum_i a_i(k)\rho_k(\varphi_i).$$

One may verify formally that ρ_k is Hecke-equivariant.

Let $\boldsymbol{a}_\ell$ be the image in $\mathcal{A}_{r_0}$ of the eigenvalue of T_ℓ, W_ℓ or U_ℓ acting on $\Phi_g^\pm$ in the cases $\ell \nmid Np$, $\ell \mid N^-$, and $\ell \mid N^+p$, respectively. Set $\boldsymbol{a}_1 = 1$ and define $\boldsymbol{a}_n$ in terms of the $\boldsymbol{a}_\ell$ with $\ell \mid n$ by the usual Euler product expansion.

We may shrink r_0 if necessary to ensure that $\rho_k(\Phi_g^\pm)$ is a nonzero element of $H^1(\Gamma_0, V_{k-2})$ for all $k \geq 2$ with $|k-2|_p \leq r_0$. It is an eigenvector for the ℓ-th Hecke operator with eigenvalue $\boldsymbol{a}_\ell(k)$. Thus, $\{\boldsymbol{a}_\ell(k)\}$ is a system of Hecke eigenvalues occuring in $H^1(\Gamma_0, V_{k-2})$. Therefore, $\{\boldsymbol{a}_\ell(k)\} \subset \overline{\mathbb{Q}}$. By the Eichler-Shimura isomorphism, this system of Hecke eigenvalues also occurs in $S_k(\Gamma_0)$. By the Jacquet-Langlands correspondence, it occurs in $S_k(\Gamma_0(pN))$ as well. Thus, if we set

$$\boldsymbol{h}(q) := \sum_{n \geq 1} \boldsymbol{a}_n q^n \in \mathcal{A}_{r_0}[\![q]\!],$$

then $\boldsymbol{h}(k) = \sum \boldsymbol{a}_n(k)q^n$ is in fact the q-expansion of a classical cusp form of weight k on $\Gamma_0(Np)$. Furthermore, it is clear that $\boldsymbol{h}(2) = f$. Therefore, by the uniqueness of the Hida family through f, it follows that $\boldsymbol{a}_n(k) = \boldsymbol{a}_n(\boldsymbol{f}_k)$ for $k \geq 2$ with $|k-2|_p \leq r_0$. Theorem 8 follows. □

Finally, we record a result that will be important later. Set

$$\mathbb{W}^0 = H^1(\Gamma, M^0(\mathbb{X})).$$

Lemma 9. *The canonical map*

$$(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \to (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g} \tag{19}$$

is an isomorphism.

Proof. The map $\rho : M(\mathbb{X}) \to \mathbb{Z}_p$ gives rise to the short exact sequence

$$0 \longrightarrow M^0(\mathbb{X}) \longrightarrow M(\mathbb{X}) \xrightarrow{\rho} \mathbb{Z}_p \longrightarrow 0.$$

Since R is Λ-flat, we may tenor the associated long exact sequence in cohomology with $R_\mathfrak{p}$ to obtain

$$\cdots \longrightarrow H^0(\Gamma, E) \longrightarrow \mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p} \longrightarrow \mathbb{W} \otimes_\Lambda R_\mathfrak{p} \longrightarrow H^1(\Gamma, E) \cdots .$$

Since the maps in this sequence are Hecke-equivariant, and $H^0(\Gamma, E)$ is Eisenstein, the map (19) is injective. Similarly, if $\Phi \in (\mathbb{W} \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$, then its image in $H^1(\Gamma, E)$ must be zero. The reason for this is that since g is p-new of level Γ_0, the system of Hecke eigenvalues of g does not occur in $H^1(\Gamma, E)$. Therefore Φ is the image of an element $\widetilde{\Phi} \in \mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p}$. Let ℓ be any prime such that the eigenvalue $a_\ell(g)$ of the Hecke operator T_ℓ is not equal to $\ell + 1$. Let $\boldsymbol{a}_\ell(\Phi)$ denote the T_ℓ eigenvalue of Φ, i.e. the image of T_ℓ in $R_\mathfrak{p}$. We claim that

$$\widetilde{\Phi}' := \frac{T_\ell - (\ell+1)}{\boldsymbol{a}_\ell(\Phi) - (\ell+1)} \widetilde{\Phi} \tag{20}$$

is a lift of Φ to $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$. First note that the division in (20) is allowed in the localization, since the image of $\boldsymbol{a}_\ell(\Phi) - (\ell+1)$ in E is $a_\ell(g) - (\ell+1) \neq 0$. Next, it is clear that $\widetilde{\Phi}'$ maps to Φ under (19) since Φ has T_ℓ eigenvalue $\boldsymbol{a}_\ell(\Phi)$. Finally, let $\lambda \in \mathbb{T}^o$, and let $\boldsymbol{a}_\lambda(\Phi)$ be the corresponding eigenvalue of Φ. Then $(\lambda - \boldsymbol{a}_\lambda(\Phi))\widetilde{\Phi}$ maps to 0 in $\mathbb{W} \otimes_\Lambda R_\mathfrak{p}$ and hence arises from $H^0(\Gamma, E)$. Since this module is Eisenstein, it is killed by $T_\ell - (\ell+1)$, and it follows that $(\lambda - \boldsymbol{a}_\lambda(\Phi))\widetilde{\Phi}' = 0$. This shows that $\widetilde{\Phi}'$ lies in $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$, and concludes the proof of the lemma. ☐

Using Lemma 9, we may view $\Phi_g^\pm$ an element of $(\mathbb{W}^0 \otimes_\Lambda R_\mathfrak{p})^{\pm,g}$.

5 Some commutative diagrams

In this section, we establish some commutative diagrams involving the operators U_p, U_p' and W_p acting on the group cohomology of various spaces of p-adic measures. In fact, these diagrams are so natural, that they commute on the level of cochains; this fact will be used heavily in the calculations of §7. We describe cohomology classes in terms of homogeneous cochains relative to the complex of projective $\widetilde{\Theta}$-modules

$$F_r := \mathbb{Z}[\widetilde{\Theta}^{r+1}]. \tag{21}$$

Thus, if G is a subgroup of $\widetilde{\Theta}$, our group of M-valued r-cochains is

$$C^r(G, M) := \mathrm{Hom}_G(F_r, M). \tag{22}$$

Coboundary maps $d : C^r(G, M) \to C^{r+1}(G, M)$ are defined by the usual formula

$$d\varphi(g_0, \ldots, g_{r+1}) = \sum_{i=0}^{r+1} (-1)^i \varphi(g_0, \ldots, \widehat{g}_i, \ldots, g_{r+1}).$$

We write $Z^r(G, M) = \mathrm{Ker}(d : C^r(G, M) \to C^{r+1}(G, M))$ and $B^r(G, M) = \mathrm{Image}(d : C^{r-1}(G, M) \to C^r(G, M))$, and have

$$H^r(G, M) = Z^r(G, M)/B^r(G, M).$$

Defining

$$\mathbb{X}_p = \mathbb{Z}_p \times \mathbb{Z}_p^\times = w_p^{-1}\mathbb{X}_\infty,$$

we obtain mutually inverse Atkin-Lehner maps as in (9) with $M = M(\mathbb{X}_\infty)$ and $N = M(\mathbb{X}_p)$.

Proposition 10. *The following diagrams commute:*

1.
$$\begin{tikzcd}
& C^r(\Gamma, M(\mathbb{X})) \arrow[dl, "\rho_{\mathbb{X}_\infty}"'] \arrow[dr, "\rho_{\mathbb{X}_p}"] & \\
C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p^{-1}"'] & C^r(\Gamma_0, M(\mathbb{X}_p))
\end{tikzcd}$$

2.
$$\begin{tikzcd}
& C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dl, "\rho'_{p\mathbb{X}_p}"'] \arrow[dr, "\rho'_{\mathbb{X}_\infty}"] & \\
C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[r, "U'_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[r, "W_p^{-1}"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty))
\end{tikzcd}$$

Here the maps ρ are the natural restriction maps.

Proof. Let $\varphi \in Z^r(\Gamma, M(\mathbb{X}))$. Let $g \in \widetilde{\Theta}^{r+1}$ and let h be a locally analytic function on $\mathbb{X}_p$. In the following, we will write $j_!$ for the extension-by-zero of a function j on $\mathbb{X}_\infty$ to a function on $\mathbb{X}$. We compute:

$$\begin{aligned}
(W_p^{-1}U_p\rho_{\mathbb{X}_\infty}\varphi)(g)(h) &= (U_p\rho_{\mathbb{X}_\infty}\varphi)(w_pg)(h|w_p^{-1}) \\
&= \sum_{0\le i\le p-1} (\rho_{\mathbb{X}_\infty}\varphi)(\delta_i^{-1}w_pg)(h|w_p^{-1}\delta_i) \\
&= \sum_{0\le i\le p-1} \varphi(\delta_i^{-1}w_pg)((h|w_p^{-1}\delta_i)_!) \\
&= \sum_{0\le i\le p-1} \varphi(g)((h|w_p^{-1}\delta_i)_!|\delta_i^{-1}w_p) \\
&= \sum_{0\le i\le p-1} \varphi(g)(h_!\mathbf{1}_{\pi^{-1}(i+p\mathbb{Z}_p)}) \\
&= (\rho_{\mathbb{X}_p}\varphi)(g)(h).
\end{aligned}$$

Key in the above calculation is that $w_p^{-1}\delta_i$ belongs to Γ and that

$$w_p^{-1}\delta_i(\mathbb{X}_\infty) = \gamma_i w_p^{-1}(\mathbb{X}_\infty) = \gamma_i(\mathbb{X}_p) = \pi^{-1}(i + p\mathbb{Z}_p).$$

Part 2 of the proposition follows from applying the operator W_p to part 1. □

Next, we will be interested in understanding the map

$$W_pU_p : H^r(\Gamma, M(\mathbb{X})) \to H^r(\Gamma', M(w_p\mathbb{X}))$$

with respect to the decomposition $w_p\mathbb{X} = \mathbb{X}_\infty \sqcup p\mathbb{X}_p$.

Proposition 11. *The following diagram commutes:*

$$\begin{tikzcd}
C^r(\Gamma, M(\mathbb{X})) \arrow[r, "\rho_{\mathbb{X}_\infty}"] \arrow[d, "W_pU_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[d, "U_p^2"] \\
C^r(\Gamma', M(w_p\mathbb{X})) \arrow[r, "\rho'_{\mathbb{X}_\infty}"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty))
\end{tikzcd}$$

Proof. The result follows from the following commutative diagram and equation (13). Note that the commutativity of the triangle on the right is the statement of part 2 of Proposition 10.

$$\begin{tikzcd}
C^r(\Gamma, M(\mathbb{X})) \arrow[r, "U_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma, M(\mathbb{X})) \arrow[r, "W_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dd, "\rho'_{p\mathbb{X}_p}"'] \arrow[dr, "\rho'_{\mathbb{X}_\infty}"] & \\
 & & & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \\
C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[ur, "W_p^{-1}U'_p"'] &
\end{tikzcd}$$

$\square$

Proposition 12. *The following diagram commutes:*

$$\begin{tikzcd}
H^r(\Gamma, M(\mathbb{X})) \arrow[r, "\rho_{\mathbb{X}_p}"] \arrow[d, "W_pU_p"'] & H^r(\Gamma_0, M(\mathbb{X}_p)) \arrow[d, "p_*"] \\
H^r(\Gamma', M(w_p\mathbb{X})) \arrow[r, "\rho'_{p\mathbb{X}_p}"'] & H^r(\Gamma_0, M(p\mathbb{X}_p))
\end{tikzcd}$$

Here the map $p_* : H^r(\Gamma_0, M(\mathbb{X}_p)) \to H^r(\Gamma_0, M(p\mathbb{X}_p))$ *is induced by* $p_*h(x,y) = h(px,py)$ *for a locally analytic function* h *on* $p\mathbb{X}_p$.

Proof. The result follows from the following commutative diagram.

$$\begin{tikzcd}
C^r(\Gamma, M(\mathbb{X})) \arrow[r, "U_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] \arrow[drrr, "\rho_{\mathbb{X}_p}"] & C^r(\Gamma, M(\mathbb{X})) \arrow[r, "W_p"] \arrow[dd, "\rho_{\mathbb{X}_\infty}"'] & C^r(\Gamma', M(w_p\mathbb{X})) \arrow[dd, "\rho'_{p\mathbb{X}_p}"'] & \\
 & & & C^r(\Gamma_0, M(\mathbb{X}_p)) \\
C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "U_p"'] & C^r(\Gamma_0, M(\mathbb{X}_\infty)) \arrow[r, "W_p"'] & C^r(\Gamma_0, M(p\mathbb{X}_p)) \arrow[ur, "W_p^{-2}=p_*^{-1}"'] &
\end{tikzcd}$$

The commutativity of the diagonal map $\rho_{\mathbb{X}_p}$ with the arrows that lie below it follows from part 1 of Proposition 10. The fact that $W_p^2 = p_*$ follows from the fact that $w_p^2 \in p\Gamma_0$ and hence induces the same map on Γ_0-cohomology as multiplication by p. $\square$

6 p-arithmetic cohomology classes and Darmon $\mathscr{L}$-invariants

Let

$$\Theta = \ker\big(\operatorname{ord}_p \circ \operatorname{nrd} : \widetilde{\Theta} \longrightarrow \mathbb{Z}/2\mathbb{Z}\big).$$

Thus, Θ is a normal subgroup of $\widetilde{\Theta}$ of index two and $\widetilde{\Theta}/\Theta$ is generated by the image of w_p. By analyzing its action on the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, the group Θ can be expressed as an amalgamation (free product) [8]:

$$\Theta \cong \Gamma *_{\Gamma_0} \Gamma'.$$

Associated to such an amalgamation and a Θ-module M, there is an associated Mayer-Vietoris sequence:

$$\cdots \longrightarrow H^{r-1}(\Gamma_0, M) \xrightarrow{\delta} H^r(\Theta, M) \overset{(\mathrm{res}^{\Theta}_{\Gamma}, \mathrm{res}^{\Theta}_{\Gamma'})}{\longrightarrow}$$

$$H^r(\Gamma, M) \oplus H^r(\Gamma', M) \overset{(\mathrm{res}^{\Gamma}_{\Gamma_0} - \mathrm{res}^{\Gamma'}_{\Gamma_0})}{\longrightarrow} H^r(\Gamma_0, M) \longrightarrow \cdots . \quad (23)$$

Recall that we defined $X = \mathbb{P}^1(\mathbb{Q}_p)$. View $\mathbb{Q}_p$ as a subspace of $\mathbb{P}^1(\mathbb{Q}_p)$ via the inclusion $z \mapsto (z:1)$. Thus, $(x:y)$ can be identified with the fraction x/y. Set $\infty = (1:0)$. We view $\mathbb{Z}_p \subset \mathbb{Q}_p$ as a subspace of X and set

$$X_\infty = X - \mathbb{Z}_p = w_p \mathbb{Z}_p.$$

Our first goal in this section is to use (23) in order to construct a cohomology class in $H^1(\Theta, M^0(X))^{\pm}$ associated to $g^{\pm}$. (Such a class is constructed in [8] using different methods.) The map

$$\pi : \mathbb{X} \longrightarrow X, \qquad \pi(x,y) = (x:y)$$

and the induced pushforward of measures $\pi_* : M(\mathbb{X}) \longrightarrow M(X)$ can be described via the following isomorphism, a consequence of the fact that π is a $\mathbb{Z}_p^\times$-fibration:

$$M(X) \cong M(\mathbb{X}) \otimes_\Lambda E, \quad (24)$$

where E is given the structure of Λ-algebra via the augmentation map ϵ defined in (14). Recall that by Lemma 9, we may assume the cohomological Hida family $\Phi_g^{\pm}$ associated to $g^{\pm}$ belongs to $H^1(\Gamma, M^0(\mathbb{X}))$.

Proposition 13. *There is a unique cohomology class $\varphi_g^{\pm} \in H^1(\Theta, M^0(X))$ such that*

$$\mathrm{res}^{\Theta}_{\Gamma} \varphi_g^{\pm} = \pi_* \Phi_g^{\pm}, \quad \mathrm{res}^{\Theta}_{\Gamma'} \varphi_g^{\pm} = \pi_* W_p U_p \Phi_g^{\pm}$$

Proof. The uniqueness follows from (23) as $H^0(\Gamma_0, M^0(X)) = 0$. It remains to show the existence of $\varphi^{\pm}$. To this end, let

$$\varphi^{\pm} = \pi_* \Phi_g^{\pm} \in H^1(\Gamma, M^0(X)), \quad \varphi'^{\pm} = \pi_* W_p U_p \Phi_g^{\pm} \in H^1(\Gamma', M^0(X)).$$

From (23), we must show that $\mathrm{res}^{\Gamma}_{\Gamma_0} \varphi^{\pm} = \mathrm{res}^{\Gamma}_{\Gamma_0} \varphi'^{\pm}$ in $H^1(\Gamma_0, M^0(X))$. Since the kernel of $H^1(\Gamma_0, M^0(X)) \to H^1(\Gamma_0, M(X))$ is Eisenstein, it suffices to prove this equality after viewing $\varphi^{\pm}$ and $\varphi'^{\pm}$ as taking values in $M(X)$. Let

$$\rho_{\mathbb{Z}_p} : H^1(\Gamma, M(X)) \longrightarrow H^1(\Gamma_0, M(\mathbb{Z}_p)) \qquad \rho'_{X_\infty} : H^1(\Gamma', M(X)) \longrightarrow H^1(\Gamma_0, M(\mathbb{Z}_p))$$
$$\rho_{X_\infty} : H^1(\Gamma, M(X)) \longrightarrow H^1(\Gamma_0, M(X_\infty)) \qquad \rho'_{X_\infty} : H^1(\Gamma', M(X)) \longrightarrow H^1(\Gamma_0, M(X_\infty))$$

be the maps induced by the inclusions $\mathbb{Z}_p \hookrightarrow X$ and $X_\infty \hookrightarrow X$ and restriction of groups to Γ_0. From the decomposition

$$H^1(\Gamma_0, M(X))) = H^1(\Gamma_0, M(\mathbb{Z}_p)) \oplus H^1(\Gamma_0, M(X_\infty)),$$

we must show that

$$\rho_{\mathbb{Z}_p}\varphi^{\pm} = \rho_{\mathbb{Z}_p}\varphi'^{\pm}, \quad \rho_{X_\infty}\varphi^{\pm} = \rho'_{X_\infty}\varphi'^{\pm}.$$

By Propositions 11 and 12, the following diagrams commute:

$$\begin{tikzcd}
H^1(\Gamma, M(X)) \arrow[d, "W_pU_p"'] \arrow[dr, "\rho_{\mathbb{Z}_p}"] & \\
H^1(\Gamma', M(X) \arrow[r, "\rho'_{\mathbb{Z}_p}"'] & H^1(\Gamma_0, M(\mathbb{Z}_p))
\end{tikzcd}
\qquad
\begin{tikzcd}
H^1(\Gamma, M(X)) \arrow[r, "\rho_{X_\infty}"] \arrow[d, "W_pU_p"'] & H^1(\Gamma_0, M(X_\infty)) \arrow[d, "U_p^2"] \\
H^1(\Gamma', M(X)) \arrow[r, "\rho'_{X_\infty}"'] & H^1(\Gamma_0, M(X_\infty))
\end{tikzcd}$$

The diagram on the left proves $\rho_{\mathbb{Z}_p}\varphi^{\pm} = \rho_{\mathbb{Z}_p}\varphi'^{\pm}$, one of the desired identities. The one on the right says $\rho'_{X_\infty}\varphi'^{\pm} = U_p^2\rho_{X_\infty}\varphi^{\pm}$. By (24),

$$U_p^2\rho_{X_\infty}\varphi^{\pm} = \epsilon(\boldsymbol{a}_p(\Phi_g^{\pm}))^2\rho_{X_\infty}\varphi^{\pm} = \rho_{X_\infty}\varphi^{\pm},$$

completing the proof. □

For each choice of $\mathscr{L} \in \mathbb{P}^1(E)$, we define an integration map

$$\kappa_{\mathscr{L}} : H^r(\Theta, M^0(X)) \to H^{r+1}(\Theta, E)$$

as follows. Recall that $C(X)$ denotes the space of continuous E-valued functions on X. Choose a base-point $\tau \in \mathcal{H}_p(E) = \mathbb{P}^1(E) - \mathbb{P}^1(\mathbb{Q}_p)$ and define

$$\xi_{\mathscr{L},\tau} \in C^1(\widetilde{\Theta}, C(X)/E)$$

by

$$\xi_{\mathscr{L},\tau}(g_0, g_1) = \begin{cases} \log_{\mathscr{L}}\left(\dfrac{z - g_1\tau}{z - g_0\tau}\right) & \text{if } \mathscr{L} \in E, \\[2ex] \operatorname{ord}_p\left(\dfrac{z - g_1\tau}{z - g_0\tau}\right) & \text{if } \mathscr{L} = \infty. \end{cases}$$

It is easy to see that $d\xi_{\mathscr{L},\tau} = 0$ and that the cohomology class represented by $\xi_{\mathscr{L},\tau}$ does not depend on τ.

Let G be any subgroup of $\widetilde{\Theta}$, let $\varphi \in C^r(G, M^0(X))$, and consider the cup product

$$\xi_{\mathscr{L},\tau} \cup \varphi \in C^{r+1}(G, (C(X)/E) \otimes_E M^0(X)).$$

The $\widetilde{\Theta}$-invariant integration pairing $(C(X)/E) \otimes_E M^0(X) \to E$ induces a map

$$I : C^{r+1}(G, (C(X)/E) \otimes_E M^0(X)) \to C^{r+1}(G, E).$$

Set $\kappa_{\mathscr{L},\tau}(\varphi) = I(\xi_{\mathscr{L},\tau} \cup \varphi) \in C^r(G, E)$, i.e.

$$\kappa_{\mathscr{L},\tau}(\varphi)(g_0, \ldots, g_{r+1}) = \int_X \log_{\mathscr{L}} \left(\frac{z - g_1\tau}{z - g_0\tau} \right) \varphi(g_1, \ldots, g_{r+1}). \tag{25}$$

One may compute directly that

$$d\kappa_{\mathscr{L},\tau}(\varphi) = \kappa_{\mathscr{L},\tau}(d\varphi). \tag{26}$$

Therefore, the correspondence $\varphi \mapsto \kappa_{\mathscr{L},\tau}(\varphi)$ induces a map

$$\kappa_{\mathscr{L}} : H^r(G, M^0(X)) \longrightarrow H^{r+1}(G, E).$$

Define

$$H^1(\Gamma_0, E)_{p\text{-new}} := H^1(\Gamma_0, E) \big/ \operatorname{Image}\left(H^1(\Gamma, E) \oplus H^1(\Gamma', E) \to H^1(\Gamma_0, E)\right)$$

and let

$$\delta : H^1(\Gamma_0, E)_{p\text{-new}} \hookrightarrow H^2(\Theta, E), \tag{27}$$

the injective map induced by the connecting homomorphism in the Mayer-Vietoris sequence (23). For a proof of the following result, see the proof of Lemma 32 in [8].

Proposition 14.

1. *The identity $\kappa_\infty(\varphi_g^\pm) = \delta(g^\pm)$ holds in $H^2(\Theta, E)$.*
2. *There is a unique $\mathscr{L} \in E$, denoted $\mathscr{L}^D(g^\pm)$, such that $\kappa_{\mathscr{L}}(\varphi_g^\pm) = 0$.*

Proof. The first statement is argued in the proof of Lemma 32 of [8]. By Lemmas 32 and 33 of [8], the eigenspace of $H^2(\Theta, E)^\pm$ on which the Hecke operators away from p act via the eigenvalues of g is 1-dimensional and spanned by the $\kappa_\infty(\varphi^\pm) = \delta(g^\pm)$, where δ is as in (27). The class $\delta(g^\pm)$ is nonzero as $g^\pm$ is a nonzero p-new form and δ is injective on these. Since the map κ_0 (the one corresponding to $\mathscr{L} = 0$) is Hecke-equivariant, there is a unique $\mathscr{L}^{\mathrm{D}}(\varphi_g^\pm)$ such that $\kappa_0(\varphi_g^\pm) = \mathscr{L}^{\mathrm{D}}(\varphi_g^\pm)\kappa_\infty(\varphi^\pm(g))$. But the identity $\log_{\mathscr{L}} = \log_p - \mathscr{L}\operatorname{ord}_p$ implies that $\kappa_{\mathscr{L}} = \kappa_0 - \mathscr{L}\kappa_\infty$, and the second statement of the proposition follows. □

The quantity $\mathscr{L}^{\mathrm{D}}(g^\pm)$ is called the *Darmon $\mathscr{L}$-invariant of $g^\pm$*.

7 Equality of the Greenberg-Stevens and Darmon $\mathscr{L}$-invariants

Let $\mathscr{L} \in E$. The goal of this section is to prove the following:

Theorem 15. *We have*

$$\kappa_{\mathscr{L}}(\varphi_g^\pm) = (\mathscr{L}^{GS}(g) - \mathscr{L})\delta(g^\pm)$$

in $H^2(\Theta, E)$. Therefore, $\mathscr{L}^D(g^\pm) = \mathscr{L}^{GS}(g)$.

Since the Riemann surfaces $\Gamma\backslash\mathcal{H}$ and $\Gamma'\backslash\mathcal{H}$ are compact if and only if $N^- \neq 1$, we have

$$H^2(\Gamma, E) \cong \begin{cases} E & \text{if } N^- \neq 1, \\ \{0\} & \text{if } N^- = 1. \end{cases}$$

In either case, this space is Eisenstein for the Hecke operators. Since the restriction maps are Hecke-equivariant,

$$\mathrm{res}^{\Theta}_{\Gamma}\, \kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0, \quad \mathrm{res}^{\Theta}_{\Gamma'}\, \kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0.$$

Fix a base point $\tau \in \mathcal{H}_p(E)$ and a representative $\varphi \in C^1(\Theta, M^0(X))$ for the cohomology class $\varphi_g^{\pm} \in H^1(\Theta, M^0(X))$. Let $\psi \in C^1(\Gamma, E)$ and $\psi' \in C^1(\Gamma', E)$ be 1-cochains such that

$$d\psi = \kappa_{\mathscr{L},\tau}(\varphi), \quad d\psi' = \kappa_{\mathscr{L},\tau}(\varphi).$$

Then $\psi - \psi'$ is a 1-cocycle on $\Gamma_0 = \Gamma \cap \Gamma'$ and

$$\delta([\psi - \psi']) = \kappa_{\mathscr{L}}(\varphi_g^{\pm}), \tag{28}$$

in $H^2(\Theta, E)$, where δ is the map of (27). Through a general cohomological calculation, we will find explicit formulas for ψ and ψ' and show that

$$[\psi - \psi'] = (\mathscr{L}^{\mathrm{GS}}(g) - \mathscr{L})g^{\pm}. \tag{29}$$

Equations (28) and (29) prove Theorem 15.

Let $\varphi \in C^1(\Theta, M^0(X))$ be a cocycle representing the class $\varphi_g^{\pm}$. Let

$$\Phi = \Phi_g^{\pm} \in H^1(\Gamma, M^0(\mathbb{X}))$$

denote the Hida family defined in (17) that lifts $\mathrm{res}^{\Theta}_{\Gamma}[\varphi]$ with respect to the push-forward map $\pi_* : M^0(\mathbb{X}) \to M^0(X)$. Let $\widetilde{\varphi}_0 \in C^1(\Gamma, M^0(\mathbb{X}))$ be a cocycle representing Φ. Then there exists a cochain $m \in Z^0(\Gamma, M(X))$ such that

$$\pi_* \widetilde{\varphi}_0 = \varphi + dm.$$

Since $F_0 = \mathbb{Z}[\widetilde{\Theta}]$ is Θ-projective and thus Γ-projective, we may lift m to cochain $\widetilde{m} \in C^0(\Gamma, M(\mathbb{X}))$. Setting $\widetilde{\varphi} = \widetilde{\varphi}_0 - d\widetilde{m} \in C^1(\Gamma, M^0(\mathbb{X}))$, we obtain a cocycle representing Φ that satisfies

$$\pi_* \widetilde{\varphi} = \varphi. \tag{30}$$

For any $\sigma \in C^r(\Gamma, M^0(\mathbb{X}))$ and $\sigma' \in C^r(\Gamma', M^0(w_p\mathbb{X}))$, define $\lambda_{\mathscr{L}}(\sigma) \in C^r(\Gamma, E)$ and $\lambda'_{\mathscr{L}}(\sigma') \in C^r(\Gamma', E)$ by the formulas

$$\lambda_{\mathscr{L}}(\sigma)(g_0, g_1, \ldots, g_r) = \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, \sigma(g_0, g_1, \ldots, g_r)(x, y). \tag{31}$$

$$\lambda'_{\mathscr{L}}(\sigma')(g_0, g_1, \ldots, g_r) = \int_{w_p\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, \sigma'(g_0, g_1, \ldots, g_r)(x, y)$$

These maps are Γ and Γ'-invariant, respectively, because the values of σ and σ' have total measure zero.

Lemma 16. *For any* $\sigma \in C^r(\Gamma, M^0(\mathbb{X}))$ *and* $\sigma' \in C^r(\Gamma', M^0(w_p\mathbb{X}))$*, we have*

$$d\lambda_{\mathscr{L}}(\sigma) = \kappa_{\mathscr{L}}(\pi_*\sigma) + \lambda_{\mathscr{L}}(d\sigma), \quad d\lambda'_{\mathscr{L}}(\sigma') = \kappa_{\mathscr{L}}(\pi_*\sigma') + \lambda'_{\mathscr{L}}(d\sigma').$$

Proof. Letting $h = (g_0, \ldots, g_{r+1})$ and $h_i = (g_0, \ldots, \widehat{g}_i, \ldots, g_r)$, we have

$$\begin{aligned} d\lambda(\sigma)(h) &= \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_1\tau)y)\, \sigma(h_0)(x,y) + \sum_{i=1}^{r+1}(-1)^i \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, \sigma(h_i)(x,y) \\ &= \int_{\mathbb{X}} \log_{\mathscr{L}}\left(\frac{x - (g_1\tau)y}{x - (g_0\tau)y}\right) \sigma(h_0)(x,y) + \int_{\mathbb{X}} \log_{\mathscr{L}}(x - (g_0\tau)y)\, d\sigma(h)(x,y) \\ &= \int_X \log_{\mathscr{L}}\left(\frac{z - g_1\tau}{z - g_0\tau}\right) \pi_*\sigma(h_0)(z) + \lambda_{\mathscr{L}}(d\sigma)(h) \\ &= \kappa_{\mathscr{L}}(\pi_*\sigma)(h) + \lambda_{\mathscr{L}}(d\sigma)(h), \end{aligned}$$

as desired. The second equality is proved in like manner. ☐

Lemma 16 implies that if we define

$$\psi = \lambda_{\mathscr{L}}(\widetilde{\varphi}) \in C^1(\Gamma, E), \tag{32}$$

then $d\psi = \kappa_{\mathscr{L}}(\varphi)$. Similarly, define

$$\psi' = \lambda'_{\mathscr{L}}(W_pU_p\widetilde{\varphi}) \in C^1(\Gamma', E). \tag{33}$$

Then

$$\begin{aligned} d\psi' &= \kappa_{\mathscr{L}}(\pi_* W_pU_p\widetilde{\varphi}) + d\lambda'_{\mathscr{L}}(dW_pU_p\widetilde{\varphi}) \\ &= \kappa_{\mathscr{L}}(W_pU_p\varphi) + 0 \\ &= \kappa_{\mathscr{L}}(\varphi), \end{aligned}$$

where the last equality is justified by the following lemma:

Lemma 17. *We have the identity of* Θ*-cochains* $W_pU_p\varphi = \varphi$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
& C^r(\Gamma, M(X)) \arrow[dl, "\rho_{X_\infty}"'] \arrow[dr, "\rho_{\mathbb{Z}_p}"] & \\
C^r(\Gamma_0, M(X_\infty)) \arrow[r, "U_p"'] \arrow[d, "\rho^{-1}_{X_\infty}"'] & C^r(\Gamma_0, M(X_\infty)) \arrow[r, "W_p"'] \arrow[d, "\rho^{-1}_{X_\infty}"'] & C^r(\Gamma_0, M(\mathbb{Z}_p)) \arrow[d, "\rho^{-1}_{\mathbb{Z}_p}"] \\
C^r(\Gamma, M(X)) \arrow[r, "U_p"'] & C^r(\Gamma, M(X)) \arrow[r, "W_p"'] & C^r(\Gamma', M(X))
\end{tikzcd}$$

The maps $\rho^{-1}_{X_\infty}$ and $\rho^{-1}_{\mathbb{Z}_p}$ are isomorphisms by Shapiro's lemma. The bottom squares of the diagram commute by definition and the upper triangle commutes as it is the pushforward via π_* of diagram 1 of Proposition 10. The lemma follows. ☐

Having found explicit formulas for ψ and ψ' in (32) and (33), respectively, we now turn towards proving (29). Recall that $\Phi = [\widetilde{\varphi}]$ is a U_p-eigenvector with eigenvalue $\boldsymbol{a}_p(\Phi)$ satisfying $\epsilon(\boldsymbol{a}_p(\Phi)) = \pm 1$. We defined $\mathscr{L}^{\mathrm{GS}}(\Phi) = d_\epsilon(1 - \boldsymbol{a}_p(\Phi)^2)$.

Proposition 18. *The class of the cocycle $\psi - \psi'$ in $H^1(\Gamma_0, E)$ is equal to*

$$(\mathscr{L}^{GS}(\Phi) - \mathscr{L})\rho_*[\varphi],$$

where $\rho_* : H^1(\Theta, M^0(X)) \to H^1(\Gamma_0, M(X_\infty)) \to H^1(\Gamma_0, E)$ is the composition of the canonical restriction map ρ_{X_∞} with the total measure on X_∞ map (as in (15)).

Proof. We use the decompositions

$$\mathbb{X} = \mathbb{X}_\infty \sqcup \mathbb{X}_p, \qquad w_p\mathbb{X} = \mathbb{X}_\infty \sqcup p\mathbb{X}_p$$

to study the integrals defining ψ and ψ'. Writing $h = (g_0, g_1)$, we find:

$$\begin{aligned}(\psi - \psi')(h) = &\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)\widetilde{\varphi}(h) + \int_{\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)\widetilde{\varphi}(h) \\ &- \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h) - \int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h).\end{aligned} \tag{34}$$

Propositions 11 and 12 allow us to rewrite these last two integrals as

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)U_p^2\widetilde{\varphi}(h) \tag{35}$$

and

$$\begin{aligned}\int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)W_pU_p\widetilde{\varphi}(h) &= \int_{p\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)p_*\widetilde{\varphi}(h) \\ &= \int_{\mathbb{X}_p} \log_{\mathscr{L}}(px - (g_0\tau)py)\widetilde{\varphi}(h) \\ &= \int_{\mathbb{X}_p} \log_{\mathscr{L}}(x - (g_0\tau)y)\widetilde{\varphi}(h) + \mathscr{L}\widetilde{\varphi}(h)(\mathbb{X}_p).\end{aligned} \tag{36}$$

Combining (34), (35), and (36), we obtain

$$(\psi - \psi')(h) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)(1 - U_p^2)\widetilde{\varphi}(h) + \mathscr{L}\widetilde{\varphi}(h)(\mathbb{X}_p). \tag{37}$$

We now view $\widetilde{\varphi} \in Z^r(\Gamma_0, M^0(\mathbb{X}_\infty))$ and calculate the class in $H^r(\Gamma_0, E)$ represented by the right side of (37). We have that $\widetilde{\varphi}(h)(\mathbb{X}_p) = \varphi(h)(\mathbb{Z}_p) = -\varphi(h)(X_\infty)$, and hence represents the class $-\rho_*[\varphi]$ in $H^r(\Gamma_0, E)$. Therefore the last term in (37) represents the class $-\mathscr{L}\rho_*[\varphi]$.

Meanwhile, since $(1 - U_p^2)\Phi = \alpha\Phi$ with $\alpha = 1 - \boldsymbol{a}_p(\Phi)^2$, we may write

$$(1 - U_p^2)\widetilde{\varphi} = \alpha\widetilde{\varphi} + d\nu \tag{38}$$

for some $\nu \in C^0(\Gamma_0, M(\mathbb{X}_\infty))$. Pushing forward via π_*, we obtain

$$(1 - U_p^2)\varphi = 0 + \pi_*(d\nu).$$

Since the term on the left is zero, we obtain $d\pi_*(\nu) = 0$. Thus $\pi_*\nu$ represents a class in $H^0(\Gamma_0, M(X_\infty))$.

Lemma 19. *The cohomology group $H^0(\Gamma_0, M(X_\infty))$ is zero.*

Proof. It is easy to see that

$$\mathfrak{I} := \{g \in \mathrm{GL}_2(\mathbb{Z}_p) : g \text{ is upper-triangular modulo } p\}$$

acts transitively on the set of balls in X_∞ of radius p^{-n} for any $n \geq 1$. Since Γ_0 is p-adically dense in $\mathfrak{I}$, Γ_0 acts transitively on this set as well. It follows that if μ is a Γ_0-invariant measure on X_∞, then $\mu(B) = p^{-n+1}\mu(X_\infty)$ for all compact-open balls $B \subset X_\infty$ of radius p^{-n}. Since the values of μ are assumed to be p-adically bounded, it follows that $\mu = 0$. □

By the lemma, we conclude that $\pi_*\nu$ is a coboundary. Arguing above as in the definition of the cocycle $\widetilde{\varphi}$ satisfying (30), we may alter ν by a coboundary to assume that $\pi_*\nu = 0$.

We may now calculate the cohomology class represented by (37). Substituting (38) into (37), the term from $\alpha\widetilde{\varphi}$ yields

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)\alpha\widetilde{\varphi}(h). \tag{39}$$

By Proposition 20 below, the expression in (39) represents the class $\mathscr{L}^{\mathrm{GS}}(\widetilde{\varphi})\rho_*[\varphi]$ in $H^r(\Gamma_0, E)$. It remains to prove that the term arising from $d\nu$ is trivial in cohomology, i.e. that

$$h \mapsto \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)d\nu(h) \tag{40}$$

is a coboundary. Note that the right side of (40) is equal to

$$\int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x)d\nu(h) + \int_{X_\infty} \log_{\mathscr{L}}(1 - (g_0\tau)/z)\pi_*d\nu(h). \tag{41}$$

The last term of (41) is zero since $\pi_*d\nu = 0$. The first term of (41) is equal to the coboundary of the 0-cochain given by

$$g_0 \mapsto \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x)\nu(g_0). \tag{42}$$

We leave to the reader the exercise of using the equation $\pi_*\nu = 0$ to show that the 0-cochain in (42) is Γ_0-invariant. This proves that (40) is a coboundary and completes the proof of the proposition. □

The following proposition, applied with $\alpha = 1 - \boldsymbol{a}_p(\Phi)^2$, was applied above to extract the invariant $\mathscr{L}^{\mathrm{GS}}(\Phi)$ from the cohomology class $[\Phi]$.

Proposition 20. *Let* $\sigma \in Z^r(\Gamma_0, M(\mathbb{X}_\infty))$*, let* $\alpha \in I_\epsilon \subset \Lambda$ *and define*

$$\eta(g_0, \ldots, g_r) = \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)\,\mathrm{d}(\alpha\sigma)(g_0, \ldots, g_r).$$

Then $\eta \in Z^r(\Gamma_0, E)$*, and represents the class*

$$[\eta] = d_\epsilon(\alpha)\rho_*[\sigma] \in H^r(\Gamma_0, E).$$

Proof. Since $\alpha \in I_\epsilon$, we have $\pi_*(\alpha\sigma) = 0$; in particular, $\alpha\sigma$ has total measure 0. It follows from this fact and a routine calculation that η is a cochain. That η is a cocycle follows from the equations $d(\alpha\sigma) = \alpha d\sigma = 0$.

To evaluate the class $[\eta] \in H^r(\Gamma_0, E)$, we consider α of the form $[\ell] - 1$ for $\ell \in 1 + p\mathbb{Z}_p$. Writing $h = (g_0, \ldots, g_r)$, we have

$$\begin{aligned}
\eta(h) &= \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(x - (g_0\tau)y)([\ell]\sigma - \sigma)(h) \\
&= \int_{\mathbb{X}_\infty} (\log_{\mathscr{L}}(\ell x - (g_0\tau)\ell y) - \log_{\mathscr{L}}(x - (g_0\tau)y))\,\sigma(h) \\
&= \int_{\mathbb{X}_\infty} \log_{\mathscr{L}}(\ell)\sigma(h) \\
&= \log(\ell) \cdot \sigma(h)(\mathbb{X}_\infty) \\
&= d_\epsilon([\ell] - 1)\rho_*\sigma(h).
\end{aligned}$$

This proves the result for $\alpha = [\ell] - 1$, and hence gives the result for general $\alpha \in I_\epsilon$ as the ideal I_ϵ is generated over Λ by such elements. □

This concludes the proof of Proposition 18, and since $\rho_*\varphi_g^\pm = g^\pm$, we deduce (29) and hence Theorem 15. Combining with Theorem 8, we also complete the proof of Theorem 2.

8 Multiplicative integrals and period lattices

In this section, we suppose that the Hecke eigenvalues of g belong to $\mathbb{Z}$. In this case, it is shown in [8, §8] that we may take

$$\varphi_g^\pm \in H^1(\Theta, M^0(X, \mathbb{Z}))^{g,\pm}.$$

That is, we may find an element $\varphi_g^\pm \in H^1(\Theta, M^0(X, \mathbb{Z}))^{g,\pm}$ whose image in $H^1(\Theta, M^0(X, E))$ is a basis for $H^1(\Theta, M^0(X, E))^{g,\pm}$. Using this integral cohomology class, we may define multiplicative versions of many of the objects considered in previous sections.

Following Darmon [6], we consider the *multiplicative integration pairing*

$$C(X)^\times/E^\times \times M^0(X, \mathbb{Z}) \longrightarrow E, \quad (f, \mu) \mapsto \times\!\!\!\!\!\!\int_X f\mu \tag{43}$$

defined by

$$\times\!\!\!\!\!\!\int_X f\mu = \lim_{\mathcal{U}\to 0} \prod_{U\in\mathcal{U}} f(z_U)^{\mu(U)}.$$

Here, $\mathcal{U}$ is a finite cover of X by compact-open sets and z_U is an arbitrary point of U. It is clear that for any $\mathscr{L}$,

$$\log_{\mathscr{L}} \times\!\!\!\!\!\!\int f\mu = \int \log_{\mathscr{L}}(f)\mu.$$

The pairing (43) is easily seen to be $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant and thus induces a corresponding pairing

$$\langle\cdot,\cdot\rangle^\times : H_1(\Theta, C(X)^\times/E^\times) \times H^1(\Theta, M^0(X,\mathbb{Z})) \longrightarrow E^\times. \tag{44}$$

Let $\Delta = \mathrm{Div}\,\mathcal{H}_p$ and let $\Delta^0 = \mathrm{Div}^0\,\mathcal{H}_p$. From the long exact sequence associated to the short exact sequence of $\mathrm{GL}_2(\mathbb{Q}_p)$-modules $0 \to \Delta^0 \to \Delta \to \mathbb{Z} \to 0$, we extract a connecting homomorphism

$$\partial : H_2(\Theta,\mathbb{Z}) \longrightarrow H_1(\Theta,\Delta^0).$$

Let $j : \Delta^0 \to C(X)^\times/E^\times$ be the map sending a divisor D to a rational function on X with divisor D. (Note that such a function is only well-defined up to multiplication by a nonzero scalar.) The map j being $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant, it induces a corresponding map

$$j_* : H_1(\Theta,\Delta^0) \longrightarrow H^1(\Theta, C(X)^\times/E^\times).$$

We may also define multiplicative refinements of the cocycles $\kappa_{\mathscr{L},\tau}(\varphi)$ as follows. Let $\tau \in \mathcal{H}_p$, let $\varphi \in C^r(\Theta, M^0(X,\mathbb{Z}))$, and define $\kappa_\tau(\varphi) \in C^{r+1}(\Theta, E^\times)$ by the rule

$$\kappa_\tau(\varphi)(g_0,\ldots,g_{r+1}) = \times\!\!\!\!\!\!\int_X \left(\frac{z-g_1\tau}{z-g_0\tau}\right) \varphi(g_1,\ldots,g_{r+1}) \in E^\times.$$

As with the $\kappa_{\mathscr{L},\tau}$, the homomorphism κ_τ induces a map

$$\kappa : H^r(\Theta, M^0(X,\mathbb{Z})) \longrightarrow H^{r+1}(\Theta, E^\times)$$

which does not depend on τ.

By the universal coefficients theorem, there is a natural surjective map

$$H^{r+1}(\Theta,E^\times) \longrightarrow \mathrm{Hom}(H_{r+1}(\Theta,\mathbb{Z}), E^\times).$$

Lemma 21. *The image of* $\kappa(\varphi_g^\pm)$ *in* $\mathrm{Hom}(H_2(\Theta,\mathbb{Z}),E^\times)$ *is given by*

$$\xi \mapsto \left(\langle j_*\partial\xi, \varphi_g^\pm\rangle^\times\right)^{-1}.$$

Proof. Suppose

$$\xi = \sum_i 1\otimes(\gamma_i,\delta_i,\varepsilon_i) \in Z_2(\Theta,\mathbb{Z}) = \mathbb{Z}\otimes_\Theta \mathbb{Z}[\Theta^3]$$

is a 2-cycle on Θ with values in $\mathbb{Z}$. Tracing through the construction of the connecting homomorphism, one computes that $\partial[\xi]$ is represented by the cycle

$$\sum_i (\gamma_i\tau - \delta_i\tau) \otimes (\delta_i, \varepsilon_i).$$

Therefore,

$$\langle j_*\partial\xi, \varphi_g^{\pm}\rangle^{\times} = \prod_i \times\!\!\!\!\!\!\int_X \left(\frac{z-\gamma_i\tau}{z-\delta_i\tau}\right) \varphi_g^{\pm}(\delta_i, \epsilon_i).$$

By the definition of the map of the universal coefficients theorem, the image of $\kappa(\varphi_g^{\pm})$ in $\mathrm{Hom}(H_2(\Theta,\mathbb{Z}), E^{\times})$ sends ξ to

$$\prod_i \kappa(\varphi_g^{\pm})(\gamma_i, \delta_i, \epsilon_i) = \prod_i \times\!\!\!\!\!\!\int_X \left(\frac{z-\delta_i\tau}{z-\gamma_i\tau}\right) \varphi_g^{\pm}(\delta_i, \epsilon_i).$$

The result follows. $\square$

Therefore, we may set

$$L_g^{\pm} = \langle j_*\partial H_2(\Theta,\mathbb{Z}), \varphi^{\pm}\rangle^{\times} = \langle H_2(\Theta,\mathbb{Z}), \kappa(\varphi_g^{\pm})\rangle \subset E^{\times}.$$

Proposition 22 ([8, Proposition 30]). *$L_g^{\pm}$ is a lattice in $E^{\times}$.*

Therefore, there is a unique $\mathscr{L}(L_g^{\pm}) \in E$ such that $\log_{\mathscr{L}}(L_g^{\pm}) = 0$. We call this quantity the *$\mathscr{L}$-invariant of* $L_g^{\pm}$. We wish to prove the following:

Proposition 23. *$\mathscr{L}(L_g^{\pm}) = \mathscr{L}^{D}(\varphi_g^{\pm})$.*

Proof. By the universal coefficients theorem,

$$\log_{\mathscr{L}}(L_g^{\pm}) = \log_{\mathscr{L}}\langle H_2(\Theta,\mathbb{Z}), \kappa(\varphi_g^{\pm})\rangle = \langle H_2(\Theta,\mathbb{Z}), \kappa_{\mathscr{L}}(\varphi_g^{\pm})\rangle = 0$$

if and only if $\kappa_{\mathscr{L}}(\varphi_g^{\pm}) = 0$. But this occurs if and only if $\mathscr{L} = \mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm})$. $\square$

Corollary 24 ([8, Conjecture 2]). *Let q be the Tate period of the elliptic curve $\mathcal{E}/\mathbb{Q}$ associated to f. Then*

$$\mathscr{L}(L_g^{\pm}) = \log_p(q)/\operatorname{ord}_p(q).$$

Proof. By Proposition 22 and Theorem 2, $\mathscr{L}(L_g^{\pm}) = \mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm}) = \mathscr{L}^{\mathrm{GS}}(f)$. By the Galois-theoretic portion of the proof of the Greenberg-Stevens theorem [9, Theorem 3.18], we have $\mathscr{L}^{\mathrm{GS}}(f) = \log_p(q)/\operatorname{ord}_p(q)$. $\square$

In [8], the second author gave a consruction of local *Stark-Heegner points* on $E^{\times}/L_g^{\pm}$. We conjectured that the elliptic curve $E^{\times}/L_g^{\pm}$ is isogenous to $\mathcal{E}/E$, yielding a construction of local points on $\mathcal{E}$. Corollary 24 makes proves this conjecture and makes the construction undconditional. In the following section, we will apply the above techniques further to obtain a formula for the formal group logarithms of these Stark-Heegner points in terms of Hida families.

9 Abel-Jacobi maps and Stark-Heegner points

Let g be as in §2 and set $\mathscr{L} = \mathscr{L}^{\mathrm{D}}(\varphi_g^{\pm})$. The natural $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant pairing

$$\langle \cdot\,, \cdot \rangle : M^0(X) \times C(X)/E \longrightarrow \mathbb{C}_p$$

induces a pairing

$$H^1(\Gamma, M^0(X)) \times H_1(\Gamma, C(X)/E) \longrightarrow \mathbb{C}_p. \tag{45}$$

Define $j_{\mathscr{L}} : \Delta^0 \to C(X)/E$ by

$$j_{\mathscr{L}}(\{\tau_2\} - \{\tau_1\})(z) = \log_{\mathscr{L}} \left(\frac{z - \tau_2}{z - \tau_1} \right).$$

Since it is Γ-equivariant, $j_{\mathscr{L}}$ induces a homomorphism

$$j_{\mathscr{L}*} : H_1(\Gamma, \Delta^0) \longrightarrow H_1(\Gamma, C(X)/E).$$

Let $\mathbb{T}^{(p)}$ be the Hecke algebra away p. There is a natural action of $\mathbb{T}^{(p)}$ on $H_1(\Gamma, \Delta^0)$ described by double cosets such that, endowing $\mathrm{Hom}(H_1(\Gamma, \Delta^0), E)$ with the corresponding dual action, the map

$$H^1(\Gamma, \Delta^0) \longrightarrow \mathrm{Hom}(H_1(\Gamma, \Delta^0), E), \quad \varphi \mapsto \big(\xi \mapsto \langle \varphi, j_{\mathscr{L}*}\xi \rangle_{\mathscr{L}}\big)$$

induced by the pairing (45) is $\mathbb{T}^{(p)}$-equivariant. Let $A_g^{\pm} = \langle \mathrm{Res}_{\Gamma}^{\Theta} \varphi_g^{\pm}, \cdot \rangle$ be the image of $\mathrm{Res}_{\Gamma}^{\Theta} \varphi_g^{\pm}$ in $\mathrm{Hom}(H_1(\Gamma, \Delta^0), E)$. Writing $\mathrm{Hom}(H_1(\Gamma, \Delta^0), E)^{g,\pm})$ as the eigenspace on which $\mathbb{T}^{(p)}$ acts via the Hecke eigenvalues of g and W_∞ acts as ± 1, we see that $A_g^{\pm}$ belongs to $\mathrm{Hom}(H_1(\Gamma, \Delta^0), E)^{g,\pm}$.

Proposition 25. *There is a unique homomorphism* $\mathrm{AJ}_g^{\pm} \in \mathrm{Hom}(H_1(\Gamma, \Delta), E)^{g,\pm}$ *such that the diagram*

$$\begin{tikzcd}
H_1(\Gamma, \Delta^0) \arrow[rr] \arrow[dr, "A_g^{\pm}"'] & & H_1(\Gamma, \Delta) \arrow[dl, "\mathrm{AJ}_g^{\pm}"] \\
& E &
\end{tikzcd}$$

Remark 26. We have chosen the notation $\mathrm{AJ}_g^{\pm}$ for this map because it formally resembles an Abel-Jacobi map.

Proof. The existence of $\mathrm{AJ}_g^{\pm}$ is shown in [8, §10] as follows: The long exact sequence in Θ-cohomology associated to the short exact sequence

$$0 \longrightarrow \Delta^0 \xrightarrow{i} \Delta \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0 \tag{46}$$

takes the form

$$\cdots \longrightarrow H_2(\Theta, \mathbb{Z}) \xrightarrow{\partial} H_1(\Theta, \Delta^0) \longrightarrow H_1(\Theta, \Delta) \longrightarrow H_1(\Theta, \mathbb{Z}) \longrightarrow \cdots .$$

Applying $\mathrm{Hom}(-, E)$, we obtain the short exact sequence

$$0 \to \mathrm{Hom}(H_1(\Theta, \mathbb{Z}), E) \to \mathrm{Hom}(H_1(\Theta, \Delta), E) \to \mathrm{Hom}(H_1(\Theta, \Delta^0)/\partial H_2(\Theta, \mathbb{Z}), E) \to 0$$

By [8, Proposition 17], $H_1(\Theta, \mathbb{Z})$ is torsion and therefore, $\mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), E) = 0$. Since the homomorphism $j_{\mathscr{L}*}\partial\xi \mapsto \langle \xi, \varphi_g^{\pm} \rangle$ annihilates $H_2(\Theta, \mathbb{Z})$, it extends uniquely to an element of $\mathrm{Hom}(H_1(\Theta, \Delta), E)^{g,\pm}$. We may thus take $\mathrm{AJ}_g^{\pm}$ to be the composite of this homomorphism with the corestrition map $H_1(\Gamma, \Delta) \to H_1(\Theta, \Delta)$.

We now show that we do not lose uniqueness by corestricting from Θ to Γ. From the long exact sequence in Γ-cohomology associated to (46), we extract the short exact sequence

$$0 \longrightarrow P \xrightarrow{i_*} H_1(\Gamma, \Delta) \xrightarrow{\deg_*} Q \longrightarrow 0,$$

where $P = H_1(\Gamma, \Delta^0)/\partial H_2(\Gamma, \mathbb{Q})$ and $Q = \mathrm{im} \deg_*$. Since $H_2(\Gamma, \mathbb{Q})$ is Eisenstein, the natural image of $A_g^{\pm}$ in $\mathrm{Hom}(\partial H_2(\Gamma, \mathbb{Q}), E)$ is zero. Therefore, $A_g^{\pm}$ may be viewed as an element of $\mathrm{Hom}(P, E)$. Dualizing and passing to eigenspaces, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{im} \deg_*, E)^{g,\pm} \longrightarrow \mathrm{Hom}(H_1(\Gamma, \Delta), E)^{g,\pm} \longrightarrow \mathrm{Hom}(P, E)^{g,\pm}.$$

As $\mathrm{Hom}(\mathrm{im} \deg_*, E)$ is a quotient of $\mathrm{Hom}(H_1(\Gamma, \mathbb{Q}), E) = H^1(\Gamma, E)$ and g is p-new, the system of Hecke eigenvalues of g cannot occur in $\mathrm{Hom}(\mathrm{im} \deg_*, E)$, i.e., the eigenspace $\mathrm{Hom}(\mathrm{im} \deg_*, E)^{g,\pm}$ is zero. Therefore, both maps in the sequence

$$\mathrm{Hom}(H_1(\Gamma, \Delta), E)^{g,\pm} \longrightarrow \mathrm{Hom}(P, E)^{g,\pm} \longrightarrow \mathrm{Hom}(H_1(\Gamma, \Delta^0), E)$$

are injective, establishing the uniqueness claim. ☐

Define $J_{\mathscr{L}} : \Delta \to C(\mathbb{X})/E$ by

$$J_{\mathscr{L}}(\{\tau\})(x, y) = \log_{\mathscr{L}}(x - y\tau).$$

Since it is Γ-equivariant, $J_{\mathscr{L}}$ induces a homomorphism

$$J_{\mathscr{L}*} : H_1(\Gamma, \Delta) \longrightarrow H_1(\Gamma, C(\mathbb{X})/E).$$

The natural Γ-equivariant pairing

$$C(\mathbb{X})/E \times M^0(\mathbb{X}) \longrightarrow E$$

induces a corresponding pairing

$$H^1(\Gamma, M^0(\mathbb{X})) \times C(\mathbb{X})/E \longrightarrow E.$$

Corollary 27. *The map* $\mathrm{AJ}_g^{\pm} : H_1(\Gamma, \Delta) \to E$ *is given by*

$$\mathrm{AJ}_g^{\pm}(\xi) = \langle \Phi_g^{\pm}, J_{\mathscr{L}*}\xi \rangle.$$

Proof. It is easy to see that the element $\widetilde{\mathrm{AJ}}_g^{\pm}$ of $\mathrm{Hom}(H_1(\Gamma, \Delta), E)$ defined by $\xi \mapsto \langle \Phi_g^{\pm}, J_{\mathscr{L}*}\xi \rangle$ belongs to the $(g, \pm)$-eigenspace. Since $\pi_* \Phi_g^{\pm} = \mathrm{Res}_{\Gamma}^{\Theta} \varphi_g^{\pm}$, the diagram

$$\begin{tikzcd}
H_1(\Gamma, \Delta^0) \arrow[r, "j_{\mathscr{L}*}"] \arrow[dd] & H_1(\Gamma, C(X)/E) \arrow[dd, "\pi^*"'] \arrow[dr, "\langle \mathrm{Res}_{\Gamma}^{\Theta} \varphi_g^{\pm}, \cdot \rangle"] & \\
 & & E \\
H_1(\Gamma, \Delta) \arrow[r, "J_{\mathscr{L}*}"'] & H_1(\Gamma, C(\mathbb{X})/E) \arrow[ur, "\langle \Phi_g^{\pm}, \cdot \rangle"'] &
\end{tikzcd}$$

commutes, implying that

$$\begin{tikzcd}
H_1(\Gamma, \Delta^0) \arrow[rr] \arrow[dr, "A_g^{\pm}"'] & & H_1(\Gamma, \Delta) \arrow[dl, "\widetilde{\mathrm{AJ}}_g^{\pm}"] \\
 & E &
\end{tikzcd}$$

commutes as well. Therefore, by Proposition 25, $\mathrm{AJ}_g^{\pm} = \widetilde{\mathrm{AJ}}_g^{\pm}$. $\square$

Let K be a real quadratic field and let $\mathcal{O} \subset K$ be an order such that $(\mathrm{disc}\,\mathcal{O}, Np) = 1$. Then there is an embedding

$$\psi : K \longrightarrow B$$

such that $\psi(\mathcal{O}) = \psi(K) \cap R$. Suppose further that p is inert in K. Then $\psi(K)$ acts on $\mathbb{P}^1(E)$ via ι_p with two fixed points τ_ψ and $\overline{\tau}_\psi$ in $\mathcal{H}_p$, conjugate under the action of $\mathrm{Gal}(E/\mathbb{Q}_p)$. Let ε be a generator of the unit group of $\mathcal{O}$. Then since $\psi(\varepsilon)\tau_\psi = \tau_\psi$, we have

$$\{\tau_\psi\} \otimes (1, \psi(\varepsilon)) \in Z_1(\Gamma, \Delta).$$

Let $C_{[\psi]}$ be the corresponding class in $H_1(\Gamma, \Delta)$. The brackets around ψ indicate that $C_{[\psi]}$ depends only on the Γ-conjugacy class of the embedding ψ. Since we assume that the Hecke eigenvalues of g lie in $\mathbb{Z}$, we may associate an elliptic curve $\mathcal{E}/\mathbb{Q}$ to g by the Eichler-Shimura construction. Let

$$\log_{\mathcal{E}} : \mathcal{E}(E) \xrightarrow{\sim} E^{\times}/q^{\mathbb{Z}} \xrightarrow{\log_{\mathscr{L}}} E$$

denote the p-adic formal group logarithm on $\mathcal{E}$, where the first arrow is the Tate uniformization of $\mathcal{E}/E$. The points $\mathrm{AJ}_g^{\pm}(C_{[\psi]}) \in E = \log_{\mathcal{E}} \mathcal{E}(E)$ are called *Stark–Heegner points* on $\mathcal{E}$. We conjecture in [8, §10] that the locally defined points $\mathrm{AJ}_g^{\pm}(C_{[\psi]})$ in fact belong to $\log_{\mathcal{E}}(\mathcal{E}(H_{\mathcal{O}}))$, where $H_{\mathcal{O}}$ is the ring class field of K associated to the order $\mathcal{O}$. By the results of this section, we have the following formula for $\mathrm{AJ}_g^{\pm}(C_{[\psi]})$ in terms of the Hida family $\Phi_g^{\pm}$:

Corollary 28.

$$\mathrm{AJ}_g^{\pm}(C_{[\psi]}) = \langle J_* C_{[\psi]}, \Phi_g^{\pm}, \rangle.$$

We hope to apply this formula with the methods of [4] to prove partial results towards the rationality of the Stark–Heegner points $\mathrm{AJ}_g^{\pm}(C_{[\psi]})$ over $H_{\mathcal{O}}$ in future work.

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