Ribet's converse to Herbrand and the weak Gross–Stark conjecture

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Setup for Stark's conjecture

$F =$ totally real field
$H =$ CM abelian extension of F

$\mathfrak{p} =$ prime ideal of F, splits completely in H, lies above p
$G = \mathrm{Gal}(H/F)$. Let $\chi \in \widehat{G}$.

For $\mathrm{Re}(s) > 1$, define

$$L(\chi, s) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}.$$

Analytic continuation to $\mathbf{C}$.

A basic proposition

$U_\mathfrak{p} := \{u \in H^\times : |u|_w = 1 \text{ for all } w \nmid \mathfrak{p}\}$
Fix a prime $\mathfrak{P}$ of H above $\mathfrak{p}$.

Proposition 1. *There exists a unique $u \in U_\mathfrak{p} \otimes \mathbf{Q}$ such that*

$$L(\chi, 0) = \sum_\sigma \chi(\sigma)\, \mathrm{ord}_\mathfrak{P}(u^\sigma)$$

for all $\chi \in \widehat{G}$.

Proof. (Sketch) Find an integer m such that

$$\prod_\sigma (\mathfrak{P}^{\sigma^{-1}})^{\zeta(\sigma,0)\cdot m}$$

is a principal ideal, generated by some α. Define $u = \alpha \otimes \frac{1}{m}$. $\square$

A conjecture on integrality

Conjecture 1 (Brumer–Stark–Tate)**.** *Let* e *be the number of roots of unity in* H*. There exists a* $u \in U_{\mathfrak{p}}$ *such that*

$$L(\chi, 0) = \frac{1}{e} \sum_{\sigma} \chi(\sigma) \operatorname{ord}_{\mathfrak{P}}(u^{\sigma})$$

for all $\chi \in \widehat{G}$*. Furthermore* $H(u^{1/e})/F$ *is an abelian extension.*

The p-adic L-function

$E = \mathbf{Q}_p(\chi)$. Teichmuller character $\omega : \mathrm{Gal}(F(\mu_{2p})/F) \to \mathbf{Z}_p^\times$.

View χ as having modulus divisible by all $\mathfrak{q} \mid p$, $\mathfrak{q} \neq \mathfrak{p}$.

χ_p denotes χ viewed with modulus divisible by all $\mathfrak{q} \mid p$.

Theorem (Deligne-Ribet, Pi. Cassou-Nogues, Barsky)**.** *There exists an E-valued function $L_p(\chi\omega, s)$, meromorphic on $\mathbf{Z}_p$ and regular outside $s = 1$, such that*

$$L_p(\chi\omega, n) = L(\chi_p\omega^n, n)$$

for all $n \in \mathbf{Z}^{\leq 0}$.

Note that

$$\begin{aligned} L_p(\chi\omega, 0) &= L(\chi_p, 0) \\ &= (1 - \chi(\mathfrak{p}))L(\chi, 0) \\ &= 0. \end{aligned}$$

The weak Gross–Stark conjecture

Conjecture 2 (Gross). *Let* $u \in U_{\mathfrak{p}} \otimes \mathbf{Q}$ *be as in Proposition 1. Then*

$$L_p'(\chi\omega, 0) = -\sum_{\sigma} \chi(\sigma) \log_p \mathrm{Norm}_{H_{\mathfrak{P}}/\mathbf{Q}_p}(u^{\sigma}).$$

$\log_p : \mathbf{Q}_p^{\times} \to \mathbf{Z}_p$ is the Iwasawa branch, $\log_p(p) = 0$.

$\log_p \circ \mathrm{Norm}_{H_{\mathfrak{P}}/\mathbf{Q}_p} : U_{\mathfrak{p}} \to \mathbf{Z}_p$ has been extended to $U_{\mathfrak{p}} \otimes \mathbf{Q} \to \mathbf{Q}_p$.

Further Refinements

* There is a Conjecture 3 (Gross), the "strong Gross–Stark conjecture," that simultaneously generalizes Conjectures 1 and 2.

* There is a Conjecture 4 (D—) strengthening Conjecture 3 by giving an exact formula for $u \in H_{\mathfrak{P}}^{\times}$. This uses methods of Shintani.

Ribet's converse to Herbrand

Theorem (Ribet). *Let p be prime and k an even integer such that $2 \leq k \leq p-3$. Suppose that $p \mid B_k$. There exists a Galois extension $E/\mathbf{Q}$ containing $\mathbf{Q}(\mu_p)$ such that:*
(1) $E/\mathbf{Q}(\mu_p)$ is unramified of degree p, and
(2) $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ *acts on* $\mathrm{Gal}(E/\mathbf{Q}(\mu_p))$ *via the character ω^{1-k}.*

Reformulation. If $p \mid B_k$, there exists a non-trivial cohomology class $\kappa \in H^1(G_{\mathbf{Q}}, \mathbf{F}_p(\omega^{1-k}))$ that is everywhere unramified.

κ arises via a representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathbf{GL}_2(\mathbf{F}_p)$ of the form

$$\overline{\rho} \cong \begin{pmatrix} 1 & * \\ 0 & \omega^{k-1} \end{pmatrix}$$

with $\kappa(\sigma) := *(\sigma)\omega^{1-k}(\sigma)$.

Reformulating Gross's Conjecture, algebraic side

$S = \{$arch primes, those ramifying in H, those dividing $p\}$.
$F_S =$ maximal extension of F unramified outside S.
$G_S = \mathrm{Gal}(F_S/F)$.

For each $\mathfrak{q} \in S$, there is a restriction map

$$H^1(G_S, E(\chi^{-1})) \to H^1(F_\mathfrak{q}, E(\chi^{-1})).$$

We will construct a $\kappa \in H^1(G_S, E(\chi^{-1}))$ such that

$$\kappa_\mathfrak{q} = 0 \text{ for all } \mathfrak{q} \in S, \mathfrak{q} \neq \mathfrak{p},$$

and

$$\kappa_\mathfrak{p}(\mathrm{art}(v)) = \log_p \mathrm{Norm}_{F_\mathfrak{p}/\mathbf{Q}_p}(v)$$

for all $v \in \mathcal{O}_\mathfrak{p}^\times$, where art : $F_\mathfrak{p}^\times \to \mathrm{Gal}(\overline{F}_\mathfrak{p}/F_\mathfrak{p})^{\mathrm{ab}}$ is the local Artin map.

Define

$$\mathscr{L}_{alg}(\chi) = \kappa_\mathfrak{p}(\mathrm{art}(p)).$$

Reformulating Gross's Conjecture

Define

$$\mathscr{L}_{an}(\chi) = \frac{L_p'(\chi\omega, 0)}{L(\chi, 0)}.$$

Proposition 2. *Conjecture 2 for* χ *is equivalent to*

$$\mathscr{L}_{alg}(\chi) = \mathscr{L}_{an}(\chi).$$

Proof. Kummer theory:

$$(U_{\mathfrak{p}} \otimes E)^{\chi^{-1}} \subset H^1(G_S, E(\chi)(1)).$$

Poitou-Tate duality:

relates $H^1(G_S, E(\chi)(1))$ and $H^1(G_S, E(\chi^{-1}))$.

□

Eisenstein Series

$k \geq 1$ integer.
$M_k(\mathfrak{n}, \psi)$ = Hilbert modular forms for F of level $\mathfrak{n}$, character ψ.

η, ψ narrow ray class characters modulo $\mathfrak{a}, \mathfrak{b}$, possibly imprimitive.

$$\begin{aligned} \eta((v)) &= \mathrm{sgn}_1(v)^{q_1} \cdots \mathrm{sgn}_n(v)^{q_n} \qquad \text{for } v \equiv 1 \pmod{\mathfrak{a}} \\ \psi((v)) &= \mathrm{sgn}_1(v)^{r_1} \cdots \mathrm{sgn}_n(v)^{r_n} \qquad \text{for } v \equiv 1 \pmod{\mathfrak{b}}, \end{aligned}$$

with $q_i + r_i \equiv k \pmod 2$ for $i = 1, \ldots, n$.

There exists $E_k(\eta, \psi) \in M_k(\mathfrak{ab}, \eta\psi)$ with Fourier coefficients

$$c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{n} | \mathfrak{m}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right) \psi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{k-1}$$

and constant coefficients ($k > 1$)

$$c_\lambda(0, E_k(\eta, \psi)) = \begin{cases} 2^{-n} \overline{\eta}(\lambda) L(\psi\overline{\eta}, 1-k) & \text{if } \mathfrak{a} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

λ ranges over narrow class group of F.

A p-adic family

Let $\eta\psi$ be totally odd.

$$E_k^*(\eta, \psi) := E_k(\eta, \psi_p \omega^{1-k}).$$

Fourier coefficients

$$c(\mathfrak{m}, E_k^*(\eta, \psi)) = \sum_{\substack{\mathfrak{n} | \mathfrak{m} \\ (\mathfrak{n}, p)=1}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right) \psi(\mathfrak{n}) \langle \mathrm{N}\mathfrak{n} \rangle^{k-1}.$$

Constant terms

$$c_\lambda(0, E_k^*(\eta, \psi)) = 0$$

if $\mathfrak{a} \neq 1$ and

$$c_\lambda(0, E_k^*(\eta, \psi)) = 2^{-n} \overline{\eta}(\lambda) L_p(\psi\overline{\eta}\omega, 1-k)$$

if $\mathfrak{a} = 1$.

All Fourier coefficients of $E_k^*(\eta, \psi)$ are p-adic analytic functions of k.

For integer weights $k \geq 2$, have $E_k^*(\eta, \psi) \in M_k(\mathfrak{a}\mathfrak{b}p, \eta\psi\omega^{1-k})$.

Λ-adic forms

$\mathcal{O} = \mathbf{Z}_p[\chi]$, $\Lambda = \mathcal{O}[[T]]$.
For $k \geq 2$ integer, $\zeta \in \mu_{p^\infty}$, define

$$\nu_{k,\zeta} : \Lambda \to \mathcal{O}[\zeta]$$

by $\nu_{k,\zeta}(T) = \zeta\gamma^{k-2} - 1$, where γ is a top gen of $1 + p\mathbf{Z}_p$.

A Λ-adic form $\mathcal{F}$ of level $\mathfrak{n}$ is a set of elements of Λ

$$\begin{cases} c(\mathfrak{m}, \mathcal{F}) \text{ for integral ideals } \mathfrak{m} \\ c_\lambda(0, \mathcal{F}) \text{ for } \lambda \in \mathrm{Cl}^+(F), \end{cases}$$

such that for all but finitely many (k, ζ), there is a classical modular form of weight k and level $\mathfrak{n}p^r$ with Fourier coefficients $\nu_{k,\zeta} \circ c(\mathfrak{m}, \mathcal{F})$ and constant terms $\nu_{k,\zeta} \circ c_\lambda(0, \mathcal{F})$.

Let $\mathbf{T}$ denote the Hecke algebra of Λ-adic forms.

Our previous example

For characters η, ψ as before, have a Λ-adic Eisenstein series $\mathcal{E}(\eta, \psi)$ defined by

$$c(\mathfrak{m}, \mathcal{E}(\eta, \psi)) = \sum_{\substack{\mathfrak{n}|\mathfrak{m} \\ (\mathfrak{n},p)=1}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right) \psi(\mathfrak{n}) \langle \mathrm{N}\mathfrak{n} \rangle (1+T)^{\alpha},$$

where α is defined by $\langle \mathrm{N}\mathfrak{n} \rangle = \gamma^{\alpha}$, and

$$c_{\lambda}(0, \mathcal{E}(\eta, \psi)) = \begin{cases} 2^{-n}\overline{\eta}(\lambda)\mathcal{L}(\psi\overline{\eta}\omega) & \text{if } \mathfrak{a} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Have

$$\mathcal{E}_k(\eta, \psi) := \nu_{k,1} \circ \mathcal{E}(\eta, \psi) = E_k^*(\eta, \psi).$$

Special Cases

Let χ be as in the Gross–Stark conjecture.

Consider $\mathcal{E}(1,\chi)$, with constant terms $2^{-n}\mathcal{L}(\chi\omega)$, where

$$\nu_{k,1} \circ \mathcal{L}(\chi\omega) = L_p(\chi\omega, 1-k).$$

Let

$$\mathcal{G} = \frac{\mathcal{E}(1,\omega^{-1})}{2^{-n}\mathcal{L}(1)},$$

with constant terms 1.

Leopoldt's conjecture for F implies that the p-adic zeta-function $\mathcal{L}(1)$ has a pole at $k=1$, so $\mathcal{G}_0 = 1$.

Ribet's construction

Suppose $p \mid B_k$. Then p divides the constant term c of $E_2(1, \omega^{k-2})$. Consider

$$f = E_2(1, \omega^{k-2}) - \frac{c}{d} E_1(1, \omega^n) E_1(1, \omega^m)$$

where $n + m \equiv k - 2 \pmod{p-1}$ and d is the constant term of $E_1(1, \omega^n) E_1(1, \omega^m)$.

The modular form f has no constant term, it is "semi-cusp form." Furthermore, it is congruent to $E_2(1, \omega^{k-2})$ modulo p.

A Ribet-style construction

Define

$$\mathcal{H} = \mathcal{E}(1,\chi) - \frac{\mathcal{L}(\chi\omega)}{L(\chi,0)} E_1(1,\chi)\mathcal{G}(u^{-1}(1+T)-1).$$

In other words,

$$\mathcal{H}_k = E_k^*(1,\chi) - \frac{L_p(\chi\omega, 1-k)}{L(\chi,0)} E_1(1,\chi)\mathcal{G}_{k-1}.$$

$\mathcal{H}$ is a "semi-cusp form": it is cuspidal at the infinity cusps.

Note $\mathcal{H}_1 = E_1^*(1,\chi)$ is an eigenform.

WWRD

Rob Pollack: "What would Ribet do?"

What Ribet did

By the lemma of Deligne–Serre there is a semi-cuspidal eigenform f' that is congruent to f mod p.

The eigenforms of weight 2 and character ω^{k-2} are the cuspidal eigenforms and the Eisenstein series $E_2(1, \omega^{k-2})$ and $E_2(\omega^{k-2}, 1)$.

$E_2(1, \omega^{k-2})$ is not a semi-cusp form.

$E_2(\omega^{k-2}, 1)$ is not congruent to f mod p.

So f' is a cusp form.

A cusp form

Assume that $\mathscr{L}_{an}(\chi^{-1}) \neq -\mathscr{L}_{an}(\chi)$.

Proposition 3. *Consider the Λ-adic form*

$$\begin{aligned} \mathcal{H} \;&=\; \mathcal{E}(1,\chi) - \frac{\mathcal{L}(\chi\omega)}{L(\chi,0)} E_1(1,\chi)\mathcal{G}(u^{-1}(1+T)-1) \\ &\quad + \frac{L(\chi^{-1},0)}{L(\chi,0)} \cdot \frac{\mathcal{L}(\chi\omega)}{\mathcal{L}(\chi^{-1}\omega)} \mathcal{E}(\chi,1). \end{aligned}$$

There exists a $t \in \mathbf{T}$ such that $\mathcal{F} = t\mathcal{H}$ is an ordinary Λ-adic cusp form with $c(1,\mathcal{F}) = 1$.

Now

$$\mathcal{H}_1 = \frac{L(\chi^{-1},0)}{L(\chi,0)} \left(1 + \frac{\mathscr{L}_{an}(\chi^{-1})}{\mathscr{L}_{an}(\chi)}\right) E_1^*(1,\chi).$$

The assumption makes this non-zero.
$\mathcal{F}_1 = E_1^*(1,\chi)$.

A "mod $(k-1)^2$-eigenform"

Let U generate the kernel of $\nu_{1,1}$.

Proposition 4. *The Λ-adic form $\mathcal{H}$ is a "mod U^2-eigenform," that is, there exist $a_\ell, a_p \in \Lambda/U^2$ such that*

$$\begin{aligned} T_\ell \mathcal{H} &\equiv a_\ell \mathcal{H} \pmod{U^2} \\ U_p \mathcal{H} &\equiv a_p \mathcal{H} \pmod{U^2} \end{aligned}$$

The same is true of $\mathcal{F}$.

The mod U^2-eigenvalues of $\mathcal{F}$

The mod U^2-eigenvalues of $\mathcal{F}$ are

$$\begin{aligned} a_\ell &= 1 + \chi(\ell) + (\log_p \mathsf{N}\ell)\frac{\chi(\ell)\mathscr{L}_{an}(\chi^{-1}) + \mathscr{L}_{an}(\chi)}{\mathscr{L}_{an}(\chi^{-1}) + \mathscr{L}_{an}(\chi)}U, \\ a_p &= 1 - \frac{\mathscr{L}_{an}(\chi^{-1})\mathscr{L}_{an}(\chi)}{\mathscr{L}_{an}(\chi^{-1}) + \mathscr{L}_{an}(\chi)}U. \end{aligned}$$

Galois Representations

Suppose that $\mathcal{F}$ can be chosen to be a cuspidal eigenform, not just a mod U^2-eigenform. By work of Hida and Wiles, there exists a Galois representation

$$\rho : G_S \to GL_2(L),$$

where L is a finite extension of the fraction field of Λ, and such that the characteristic polynomial of $\rho(\mathrm{Frob}_\ell)$ for $\ell \nmid \mathfrak{n}p$ is

$$x^2 - c(\ell, \mathcal{F})x + \chi(\ell)\langle \mathrm{N}\ell \rangle^{k-1}.$$

Recall $c(\ell, \mathcal{F}) \equiv a_\ell \pmod{U^2}$.

Specialization to $k = 1$

Since $\mathcal{F}_1 = E_1^*(1, \chi)$, it follows that we can arrange for

$$\rho \bmod U = \rho_1 \cong \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix}$$

Then

$$\kappa(\sigma) = *(\sigma)\chi^{-1}(\sigma)$$

is a 1-cocycle representing a class in $H^1(G_S, E(\chi^{-1}))$.

Ordinary Representations

We have

$$\rho|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \eta_1 & * \\ 0 & \eta_2 \end{pmatrix},$$

where η_2 is unramified and

$$\eta_2(\mathrm{art}(p)) = c(p, \mathcal{F}).$$

Recall $c(p, \mathcal{F}) \equiv a_p \pmod{U^2}$.

The Wrench — comparing local and global

By comparing the two forms of $\rho|_{D_\mathfrak{p}}$ using a change of basis matrix, one finds

$$\kappa_\mathfrak{p}(\mathrm{art}(v)) = c_\kappa \frac{\mathscr{L}_{an}(\chi^{-1})}{\mathscr{L}_{an}(\chi^{-1}) + \mathscr{L}_{an}(\chi)} (\log_p \mathrm{N}v)$$

for $v \in \mathcal{O}_\mathfrak{p}^\times$ and

$$\kappa_\mathfrak{p}(\mathrm{art}(p)) = c_\kappa \frac{\mathscr{L}_{an}(\chi^{-1})\mathscr{L}_{an}(\chi)}{\mathscr{L}_{an}(\chi^{-1}) + \mathscr{L}_{an}(\chi)}.$$

for some $c_\kappa \neq 0$.

Thus

$$\mathscr{L}_{alg}(\chi) = \mathscr{L}_{an}(\chi).$$

Results

Theorem. *Assume Leopoldt's conjecture for F. If $\mathfrak{p}$ is the only prime of F above p, assume that $\mathscr{L}_{an}(\chi^{-1}) \neq -\mathscr{L}_{an}(\chi)$. Then the weak Gross–Stark conjecture is true for χ.*

Corollary. *Let F be a real quadratic field. Let H be a narrow ring class field extension of F. Then the weak Gross–Stark conjecture is true for H/F.*