1. **Section 11.5**

1. Prove that if $M$ is a cyclic $R$-module then $\mathcal{T}(M) = \mathcal{S}(M)$, i.e., the tensor algebra $\mathcal{T}(M)$ is commutative.

Let $m$ be a generator for $M$, and let $m_1, \ldots, m_k, m'_1, \ldots, m'_l \in M$. It follows that $m_1 \otimes \cdots \otimes m_k = a_1 m \otimes \cdots \otimes a_k m = a_1 a_2 \cdots a_k (m \otimes \cdots \otimes m)$ for some $a_1, \ldots, a_k \in R$, so

$$
(m_1 \otimes \cdots \otimes m_k) \cdot (m'_1 \otimes \cdots \otimes m'_l) = (a_1 \cdots a_k) (m_1 \otimes \cdots \otimes m) (a'_1 \cdots a'_l) (m'_1 \otimes \cdots \otimes m) \\
= (a_1 \cdots a_k) (a'_1 \cdots a'_l) (m \otimes \cdots \otimes m) \\
= (a'_1 \cdots a'_l) (m_1 \otimes \cdots \otimes m) (a_1 \cdots a_k) (m'_1 \otimes \cdots \otimes m) \\
= (m'_1 \otimes \cdots \otimes m'_l) \cdot (m_1 \otimes \cdots \otimes m_k).
$$

Therefore, single tensors commute, and thus all elements of $\mathcal{T}(M)$ commute and $\mathcal{T}(M) = \mathcal{S}(M)$.

6. If $A$ is any $R$-algebra in which $a^2 = 0$ for all $a \in A$ and $\varphi : M \to A$ is an $R$-module homomorphism, prove there is a unique $R$-algebra homomorphism $\Phi : \wedge(M) \to A$ such that $\Phi|_M = \varphi$.

Define $\tilde{\Phi}_k : M \times \cdots \times M \to A$ such that $\tilde{\Phi}_k(m_1, \ldots, m_k) = \varphi(m_1) \cdots \varphi(m_k)$. This is clearly bilinear, and it is alternating since if $m_i = m_{i+1}$, then $\varphi(m_i) = \varphi(m_{i+1})$ and $\varphi(m_i)^2 = 0$, so $\varphi(m_1) \cdots \varphi(m_k) = 0$. By the universal property, it extends to a map $\Phi_k : \wedge^k(M) \to A$ which is unique with the property that $\Phi_k|_M = \varphi$. It follows that any $R$-module morphism $\Phi : \wedge(M) \to A$ exists as the direct sum of $\Phi_k$’s with the property that $\Phi|_M = \varphi$. Furthermore $\Phi$ is easily seen to be an $R$-algebra homomorphism. Note that $\Phi$ restricted to $\wedge^k M$ is determined by $\Phi|_M = \varphi$. Since $\Phi$ is an $R$-algebra homomorphism $\Phi$ is unique.

8. Let $R$ be an integral domain and let $F$ be its field of fractions.

(a) Considering $F$ as an $R$-module, prove that $\wedge^2 F = 0$.

Let $\frac{a_1}{b_1}, \frac{a_2}{b_2} \in F$. It follows that

$$
\frac{a_1}{b_1} \wedge \frac{a_2}{b_2} = a_1 a_2 \left( \frac{1}{b_1} \wedge \frac{1}{b_2} \right) = a_1 a_2 b_1 b_2 \left( \frac{1}{b_1 b_2} \wedge \frac{1}{b_1 b_2} \right) = 0.
$$

So, $\wedge^2(F) = 0$.

(b) Let $I$ be an $R$-submodule of $F$ (for example, any ideal in $R$). Prove that $\wedge^i I$ is a torsion $R$-module for $i \geq 2$ (i.e., for every $x \in \wedge^i I$ there is some nonzero $r \in R$ such that $r x = 0$).

It suffices to prove the result for simple tensors in $\wedge^i I$ since these generate the $R$-module $\wedge^i I$ and the set of torsion elements is an $R$-module.
since \( R \) is an integral domain (Midterm \#1). Let \( \frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k} \) be a \( k \)-tensor in \( \bigwedge^k I \) and let \( r = a_1 a_2 b_1 b_2 \). However,

\[
r \cdot \left( \frac{a_1}{b_1} \wedge \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k} \right) = a_1 a_2 b_1 b_2 \cdot \left( a_1 \frac{a_2}{b_2} \wedge \cdots \wedge \frac{a_k}{b_k} \right)
= a_1 a_2 \wedge a_1 a_2 \wedge \cdots \wedge \frac{a_k}{b_k} = 0,
\]

so every element of \( \bigwedge^k I \) is torsion.

(c) Give an example of an integral domain \( R \) and an \( R \)-module \( I \) in \( F \) with \( \bigwedge^k I \neq 0 \) for every \( k \geq 0 \).

Consider \( \mathbb{C}[x_1, x_2, \ldots] \), the polynomial ring over \( \mathbb{C} \) in a countably infinite number of variables. The ideal \( (x_1, x_2, \ldots) \) of rank 1 has the desired property that \( \bigwedge^k I \neq 0 \) for any \( k \in \mathbb{N} \).

12. (a) Prove that if \( f(x, y) \) is an alternating bilinear map on \( V \) (i.e., \( f(x, x) = 0 \) for all \( x \in V \)) then \( f(x, y) = -f(y, x) \) for all \( x, y \in V \).

Let \( x, y \in V \). \( f(x + y, x + y) = f(x, x) + f(x, y) + f(y, x) + f(y, y) = 0 \) since \( f \) is alternating, but \( f(x, x) = f(y, y) = 0 \), so \( f(x, y) = -f(y, x) \).

(b) Suppose that \(-1 \neq 1 \) in \( F \). Prove that \( f(x, y) \) is an alternating bilinear map on \( V \) (i.e., \( f(x, x) = 0 \) for all \( x \in V \)) if and only if \( f(x, y) = -f(y, x) \) for all \( x, y \in V \).

The forward direction is done by part (a). To see the reverse direction, assume \( f(x, y) = -f(y, x) \), then \( f(x, x) = -f(x, x) \) and \( 2f(x, y) = 0 \). Because \( F \) has characteristic not equal to 2, and \( F \) is a field, \( f(x, x) = 0 \) and \( f \) is therefore alternating.

(c) Suppose that \(-1 = 1 \) in \( F \). Prove that every alternating bilinear map on \( V \) is symmetric (i.e., \( f(x, y) = f(y, x) \) for all \( x, y \in V \)). Prove that there is a symmetric bilinear map on \( V \) that is not alternating.

By part (a), \( f(x, y) = -f(y, x) = f(y, x) \), so every alternating form is symmetric. To find a symmetric form that is not alternating, let \( (x_1, \ldots, x_n) \) and \((y_1, \ldots, y_n)\) be elements of \( V \) after choosing a basis, and define \( f : V \times V \to F \) such that \( f((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 y_1 + \cdots + x_n y_n \). It is easy to verify that this is a symmetric bilinear form that is not alternating.

14. Prove that if \( M \) is an \( R \)-module direct factor of the \( R \)-module \( N \) then \( T(M) \) (respectively, \( S(M) \) and \( \bigwedge(M) \)) is an \( R \)-subalgebra of \( T(N) \) (respectively, \( S(M) \) and \( \bigwedge(M) \)).

Let \( N = M \oplus P \). Tensor products distribute across direct sums so we have \( N \otimes N = (M \oplus P) \otimes (M \oplus P) = (M \otimes M) \oplus (M \otimes P) \oplus (P \otimes M) \oplus (P \otimes P) \). It is then clear that \( M \otimes M \subseteq N \otimes N \). This generalizes easily for any \( k \), i.e., \( \otimes^k M \subseteq \otimes^k N \), and it is straightforward to show that \( T(M) \subseteq T(N) \) as a subalgebra. The argument is similar for \( S(M) \) and \( \bigwedge(M) \).

2. Written Problems

1. Show that if \( R \) is an integral domain and \( M \) is an \( R \)-module, then my definition of \( \text{rank}_R(M) \) and the book’s definition agree, i.e., show that \( \text{dim}_F(F \otimes M) = \text{max} \) the maximum number of \( R \)-linearly independent elements of \( M \).
Let \( n \) be the maximum number of \( R \)-linearly independent elements of \( M \), and let such elements be \( m_1, m_2, \ldots, m_n \). Accordingly, if \( r_1 m_1 + r_2 m_2 + \cdots + r_n m_n = 0 \), then \( r_i = 0 \) for all \( i \). Now consider the collection \( 1 \otimes m_1, 1 \otimes m_2, \ldots, 1 \otimes m_n \), and assume

\[
0 = r_1(1 \otimes m_1) + r_2(1 \otimes m_2) + \cdots + r_n(1 \otimes m_n)
= 1 \otimes r_1 m_1 + 1 \otimes r_2 m_2 + \cdots + 1 \otimes r_n m_n
= 1 \otimes (r_1 m_1 + r_2 m_2 + \cdots + r_n m_n).
\]

By a previous exercise, this occurs only when \( r(r_1 + \cdots + r_n m_n) = 0 \) for some nonzero \( r \in R \). However the \( m_i \) are linearly independent over \( R \) and \( R \) is an integral domain, so this implies that \( r_i = 0 \) for all \( i \). Ergo, \( 1 \otimes m_1, 1 \otimes m_2, \ldots, 1 \otimes m_n \) are linearly independent, and \( \dim_F(F \otimes M) \geq n \).

Let \( \dim_F(F \otimes M) = l \) and assume \( \frac{1}{b_1} \otimes m_1, \frac{1}{b_2} \otimes m_2, \ldots, \frac{1}{b_l} \otimes m_l \) is a basis for \( F \otimes M \) as an \( F \)-vector space. Since \( F \) is a field, we can scale each element of the basis by any nonzero element of \( F \) and still have a basis. Scale the element \( \frac{1}{b_i} \otimes m_i \) by \( b_i \) for \( 1 \leq i \leq l \). The new basis is then \( 1 \otimes m_1, 1 \otimes m_2, \ldots, 1 \otimes m_n \). Since this is a basis, it is linearly independent. Assume that \( m_1, \ldots, m_l \) are \( R \)-linearly independent in \( M \). Then \( r_1 m_1 + \cdots + r_l m_l = 0 \) for some \( r_1, \ldots, r_l \in R \) not all zero, and

\[
0 = 1 \otimes 0
= 1 \otimes (r_1 m_1 + \cdots + r_l m_l)
= (1 \otimes r_1 m_1) + \cdots + (1 \otimes r_l m_l)
= r_1(1 \otimes m_1) + \cdots + r_l(1 \otimes m_l)
\]

which is a contradiction, as \( 1 \otimes m_1, 1 \otimes m_2, \ldots, 1 \otimes m_n \) is a basis for \( F \otimes M \). Therefore, \( m_1, \ldots, m_l \) are linearly independent, and \( n \geq l = \dim_F(F \otimes M) \geq n \). So \( n = l \) which proves the definitions are equivalent.

3. Section 12.1

1. Let \( M \) be a module over the integral domain \( R \).

(a) Suppose \( x \) is a nonzero torsion element in \( M \). Show that \( x \) and 0 are “linearly dependent.” Conclude that the rank of \( M_{\text{tor}} \) is 0, so that in particular any torsion \( R \)-module has rank 0.

Since \( x \) is torsion, there exists some nonzero \( r \in R \) such that \( rx = 0 \). Let \( r' \in R \) be nonzero. \( rx + r'0 = 0 \), so \( x \) and 0 are “linearly dependent.”

(b) Show that the rank of \( M \) is the same as the rank of the (torsion free) quotient \( M/M_{\text{tor}} \).

Let \( F \) be the field of fractions for \( R \). We show that \( F \otimes M \) and \( F \otimes M/M_{\text{tor}} \) are isomorphic as \( F \)-vector spaces, and conclude that the rank of \( M \) is equal to that of \( M/M_{\text{tor}} \). Define \( f : F \otimes_R M \rightarrow F \otimes_R M/M_{\text{tor}} \) such that \( f\left(\frac{a}{b} \otimes m\right) = f\left(\frac{a}{b} \otimes \bar{m}\right) \) for \( \frac{a}{b} \in F \) and \( m \in M \). It is easy to verify that \( f \) is linear and surjective. To check injectivity, recall that every element of \( F \otimes_R M \) can be written as \( \frac{1}{r} \otimes m \) for some nonzero \( r \in R \). Then \( f\left(\frac{1}{r} \otimes m\right) = \frac{1}{r} \otimes \bar{m} \). Recall that from a previous homework problem that this is zero only if \( \bar{m} \) is torsion in \( M/M_{\text{tor}} \). If there exists a nonzero \( s \in R \) such that \( sm = 0 \) in \( M/M_{\text{tor}} \), then \( sm \in M_{\text{tor}} \) which means \( s'(sm) = 0 \) for some nonzero \( s' \in R \). This implies that \( m \in M_{\text{tor}} \) since \( R \) is an integral
domain. Accordingly $\frac{1}{r} \otimes m = 0$ in $F \otimes_R M$, so $f$ is injective and therefore an isomorphism of vector spaces. As such, the dimensions are equal, and so are the ranks.

3. Let $R$ be an integral domain and $A$ and $B$ be $R$-modules of ranks $m$ and $n$ respectively. Prove that the rank of $A \oplus B$ is $m + n$.

Let $F$ be the field of fractions of $R$, and consider $F \otimes (A \oplus B)$ as an $F$-vector space. By vector space theory,

$$F \otimes (A \oplus B) = (F \otimes A) \oplus (F \otimes B),$$

from which it follows that

$$\dim_F(F \otimes (A \oplus B)) = \dim_F(F \otimes A) + \dim_F(F \otimes B).$$

The rank of $A \oplus B$ is therefore $m + n$.

4. Let $R$ be an integral domain, let $M$ be an $R$-module, and let $N$ be a submodule of $M$. Suppose $M$ has a rank of $n$, $N$ has a rank $r$, and the quotient $M/N$ has rank $s$. Prove that $n = r + s$.

It is simple to verify that the following is a short exact sequence of $R$-modules with the canonical injection and projection mappings:

$$0 \to N \to M \to M/N \to 0.$$ 

Let $F$ be the field of fractions of $R$. Since $F$ is a flat $R$-module,

$$0 \to F \otimes N \to F \otimes M \to F \otimes M/N \to 0$$

is short exact, and since it is a short exact sequence of $F$-vector spaces, it splits. Therefore, $F \otimes M = (F \otimes N) \oplus (F \otimes M/N)$. It follows that $n = r + s$ after taking the dimension of both sides, and distributing across the direct sum.

13. If $M$ is a finitely generated module over the P.I.D. $R$, describe the structure of $M/M_{\text{tor}}$.

By problem 1, part (b), $M/M_{\text{tor}}$ is a module of the same rank as $M$, but with no torsion elements. By the fundamental theorem, $M/M_{\text{tor}}$ is a free module of rank $n$ where $n$ is the rank of $M$, i.e. $M/M_{\text{tor}} \cong R^n$.

15. Prove that if $R$ is a Noetherian ring then $R^n$ is a Noetherian $R$-module.

This proof proceeds by induction on $n$.

The case where $n = 1$ is trivial, as the definitions of a Noetherian ring and a Noetherian $R$-module coincide when viewing $R$ as an $R$-module.

For $n > 1$, assume that $e_1, e_2, \ldots, e_n$ is a basis for $R^n$ and let $N \subseteq R^n$ be a submodule. Let $L = Re_1$, the cyclic $R$-submodule generated by $e_1$, and consider $R^n/L \cong R^{n-1}$. By induction, the image $\bar{N}$ of $N$ in $R^n/L$ is finitely generated. Let $n_1, \ldots, n_m \in N$ be elements such that their image in $R^n/L$ generates $\bar{N}$. Also by induction, $N \cap L \subseteq L$ is finitely generated as a submodule, so let $l_1, \ldots, l_p$ be generators for $N \cap L$. If $n \in N$, then $\bar{n} = r_1 \bar{n}_1 + \cdots + r_m \bar{n}_m$ for $r_i \in R$. Subtracting $r_1 n_1 + \cdots + r_m n_m$ from $n$ yields an element of $N \cap L$ which can be written as a linear combination of $l_1, \ldots, l_p$. Therefore, every element of $N$ can be written as a linear combination of $n_1, \ldots, n_m, l_1, \ldots, l_p$, and $N$ is finitely generated. It follows that $R^n$ is Noetherian as an $R$-module.