Math 202 — Homework #5

Due May 12, 2011

The page and problem numbers below refer to the third edition of Dummit and Foote's "Abstract Algebra."

- Sec 13.1, #1, 4, 5, 8.
- Sec 13.2, #1, 3, 7, 15, 19, 22.
- Sec 13.3, #1.
- Sec. 13.4, #1, 5, 6. Notes on #5: A field extension K/F such that every irreducible polynomial in $F[x]$ with a root in K splits completely into linear factors in K is called a *normal extension*. This problem proves that a finite extension is normal if and only if it is a splitting field. The book does the nonstandard thing of defining a normal extension to be a splitting field of a collection of polynomials, and leaving this exercise to show that it is equivalent to the usual definition.

  Also, I found the book's hint a little confusing, so let me be more explicit: showing that if K is normal, then K is a splitting field is easy. For the converse, suppose that K/F is a splitting field, and first prove the following lemma: let K' be an extension of K and let $\sigma : K \to K'$ be a field homomorphism that is the identity on F. Then $\sigma(K) = K$. Now we can prove that K/F is normal. Suppose that $\alpha \in K$ is a root of the irreducible polynomial $f(x) \in F[x]$. Suppose that β is another root of $f(x)$ in some extension K' of K. We want to show that in fact $\beta \in K$. Now Theorem 8 says that there is an isomorphism $\sigma : F(\alpha) \cong F(\beta)$ that is the identity on F. Furthermore, K is the splitting field of some polynomial $g(x)$ over F, and hence over $F(\alpha)$ too. It is clear that $K(\beta)$ is the splitting field of $\sigma(g)(x) = g(x)$ over $F(\beta)$. By Theorem 27, σ can be extended to an isomorphism $K \to K(\beta)$, so by the lemma, we have $K = K(\beta)$, i.e. $\beta \in K$. (This follows just by looking at degrees too, but the lemma is nice to have.)