Math 202 — Homework #1

Due April 9, 2015

The page and problem numbers below refer to the third edition of Dummit and Foote’s “Abstract Algebra.” Write up solutions to the unstarred problems. The starred problems are suggested but will not be graded.

• Sec 13.1, #1, 4, 5, 8*.
• Sec 13.2, #1*, 3, 7*, 15, 19, 22.
• Sec 13.3, #1.
• Sec. 13.4, #1*, 5, 6. Notes on #5: A field extension $K/F$ such that every irreducible polynomial in $F[x]$ with a root in $K$ splits completely into linear factors in $K$ is called a normal extension. This problem proves that a finite extension is normal if and only if it is a splitting field. The book does the nonstandard thing of defining a normal extension to be a splitting field of a collection of polynomials, and leaving this exercise to show that it is equivalent to the usual definition.

Also, I found the book’s hint a little confusing, so let me be more explicit: showing that if $K$ is normal, then $K$ is a splitting field is easy. For the converse, suppose that $K/F$ is a splitting field, and first prove the following lemma: let $K'$ be an extension of $K$ and let $\sigma : K \to K'$ be a field homomorphism that is the identity on $F$. Then $\sigma(K) = K$. Now we can prove that $K/F$ is normal. Suppose that $\alpha \in K$ is a root of the irreducible polynomial $f(x) \in F[x]$. Suppose that $\beta$ is another root of $f(x)$ in some extension $K'$ of $K$. We want to show that in fact $\beta \in K$. Now Theorem 8 says that there is an isomorphism $\sigma : F(\alpha) \cong F(\beta)$ that is the identity on $F$. Furthermore, $K$ is the splitting field of some polynomial $g(x)$ over $F$, and hence over $F(\alpha)$ too. It is clear that $K(\beta)$ is the splitting field of $\sigma(g)(x) = g(x)$ over $F(\beta)$. By Theorem 27, $\sigma$ can be extended to an isomorphism $K \to K(\beta)$, so by the lemma, we have $K = K(\beta)$, i.e. $\beta \in K$. (This follows just by looking at degrees too, but the lemma is nice to have.)