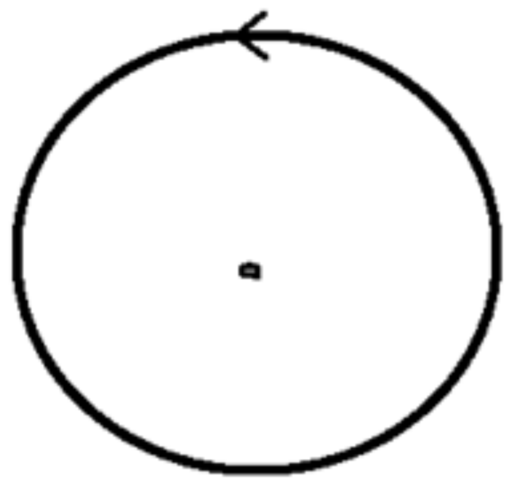
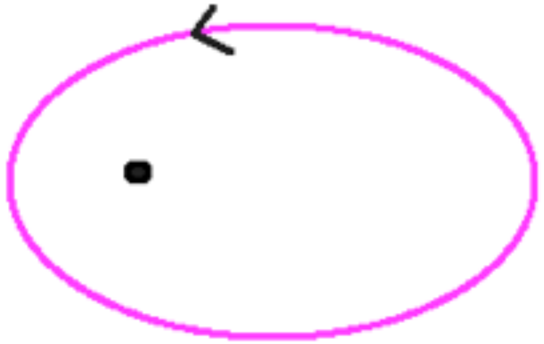
N-body Tour

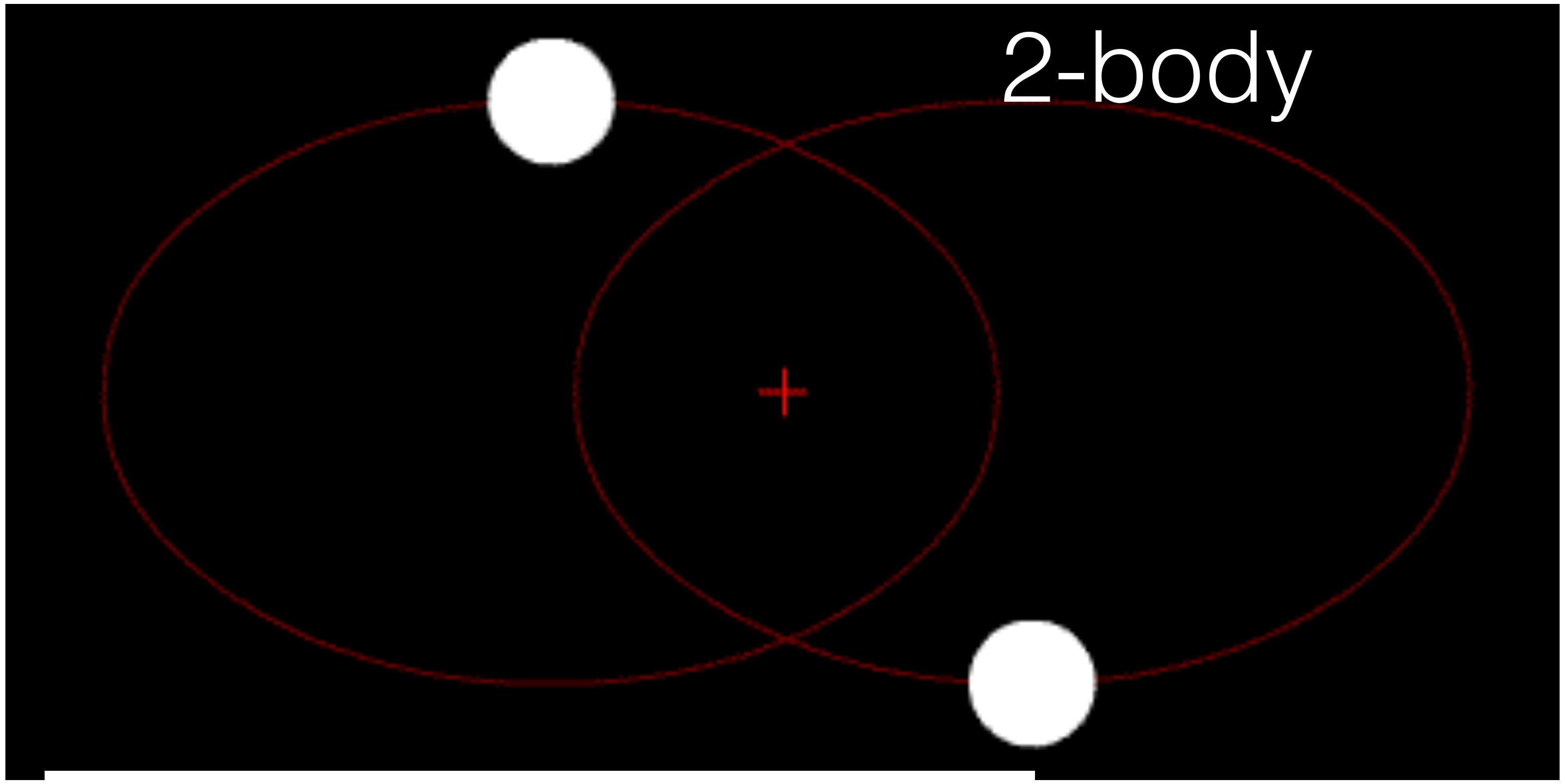
(N=2, 3)

gracias a:
Rick Moeckel, Greg Laughlin, Danya Rose.
y Alain Chenciner y Albouy





Kepler: conics.   …planetary motions..



NEWTON'S EQNS

$$F_{ji} = -F_{ji}, \ |F_{ij}| \sim \frac{1}{r_{ij}^2}, \ r_{ij} = |q_i - q_j|$$

$$m_1 \ddot{q}_1 = F_{21},$$

$$\implies F_{ji} = \frac{m_j m_i (q_j - q_i)}{r_{ji}^3},$$

$$m_2 \ddot{q}_2 = F_{12}$$

Lagrange 1772

ESSAI

SUR

LE PROBLÈME DES TROIS CORPS.

Juvat integros accedere fontes.
Lucr.

(Prix de l'Académie Royale des Sciences de Paris, tome IX, 1772.)

AVERTISSEMENT.

Ces Recherches renferment une Méthode pour résoudre le Pr
des trois Corps, différente de toutes celles qui ont été données
présent. Elle consiste à n'employer dans la détermination de l'or
chaque Corps d'autres éléments que les distances entre les trois
c'est-à-dire, le triangle formé par ces Corps à chaque instant. Pou
il faut d'abord trouver les équations qui déterminent ces mêu
tances par le temps; ensuite, en supposant les distances connues, il faut
en déduire le mouvement relatif des Corps par rapport à un plan fixe

Euler 1767

DE MOTV RECTILINEO TRIVM CORPORVM SE MVTVO ATTRAHENTIVM.

Auctore

L. EVLERO.

1.

Sint A, B, C massae trium corporum eorumque distantiae a puncto fixo O ad datum tempus t ponantur

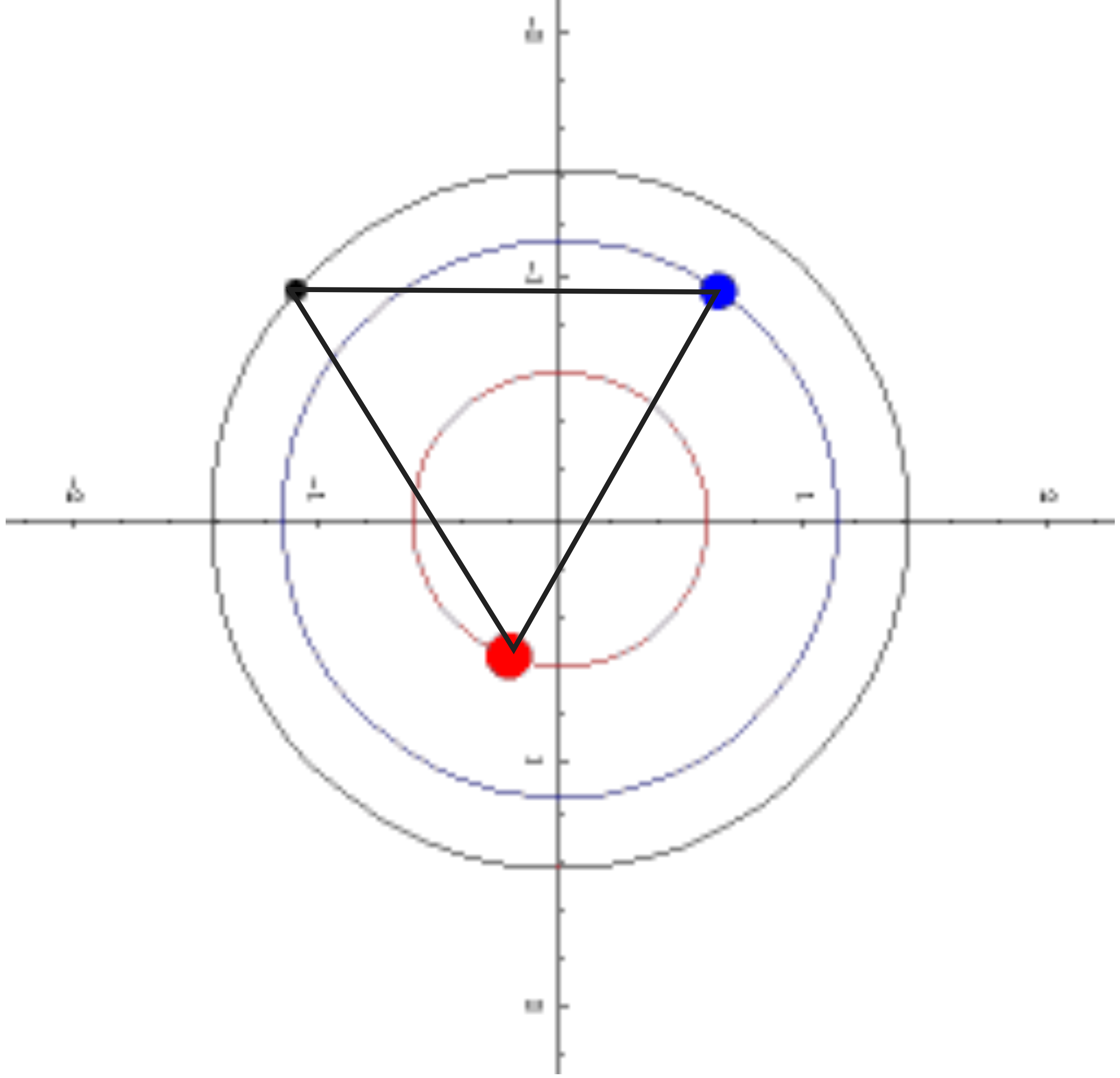
$$OA = x, \quad OB = y \quad \text{et} \quad OC = z$$

vbiquidem sumitur $y > x$ et $z > y$. Hinc motus principia praebent has tres aequationes:

$$\text{I. } \frac{ddx}{dt^2} = \frac{B}{(y-x)^2} + \frac{C}{(z-x)^2};$$

$$\text{II. } \frac{ddy}{dt^2} = \frac{-A}{(y-x)^2} + \frac{C}{(z-y)^2}$$

$$\text{III. } \frac{ddz}{dt^2} = \frac{-A}{(z-x)^2} - \frac{B}{(z-y)^2}$$



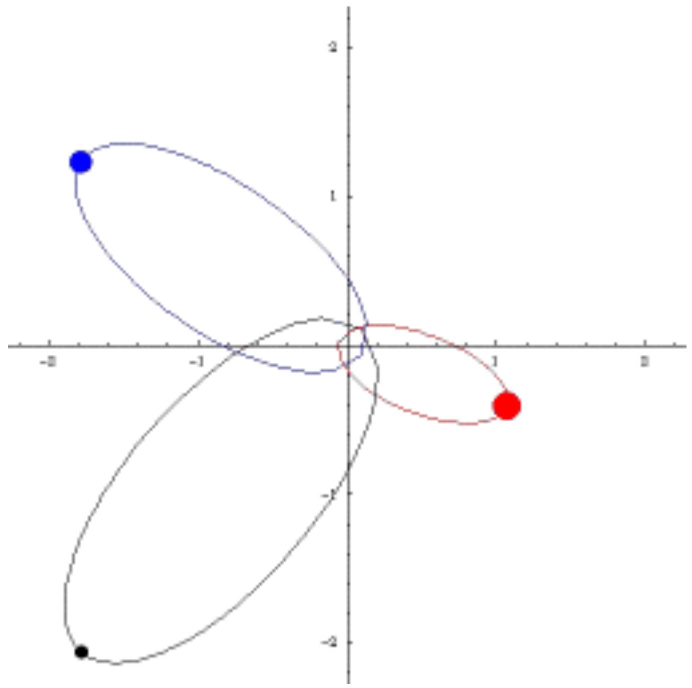
3-body.
Lagrange's solution(s)
Equilateral triangles.

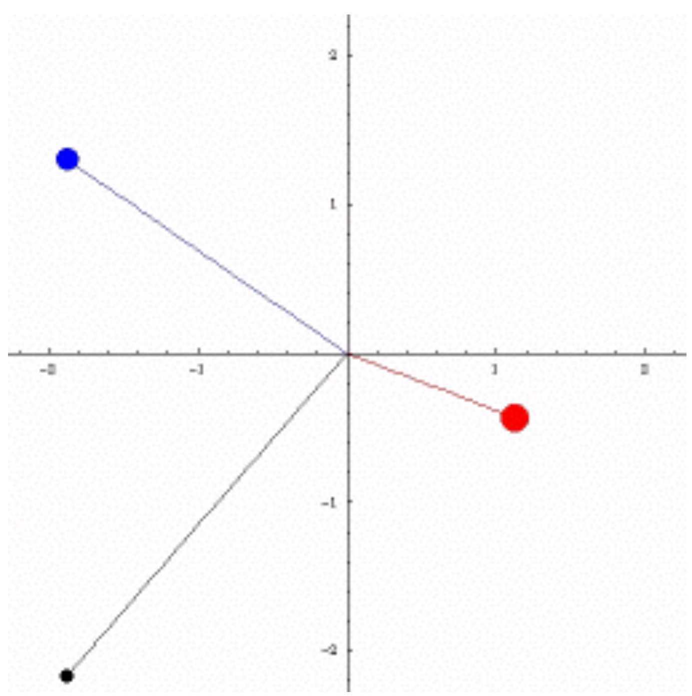
1772,

$$m_1\ddot{q}_1 = F_{21} + F_{31}$$

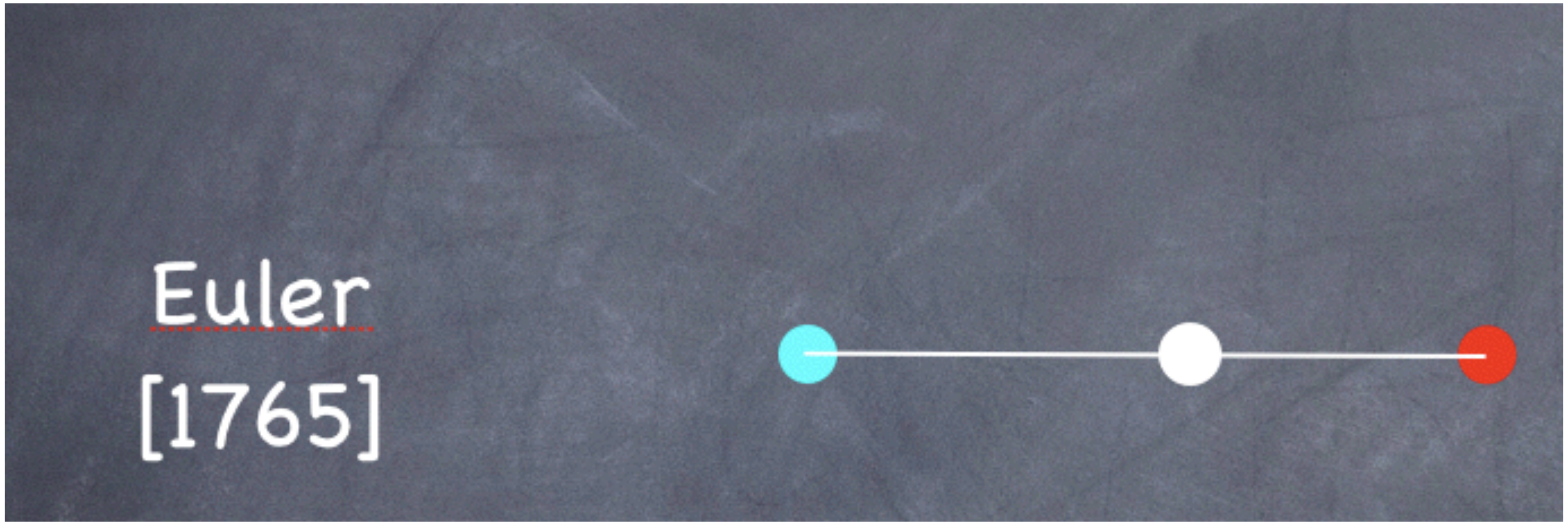
$$m_2\ddot{q}_2 = F_{12} + F_{32}$$

$$m_3\ddot{q}_3 = F_{23} + F_{13}$$



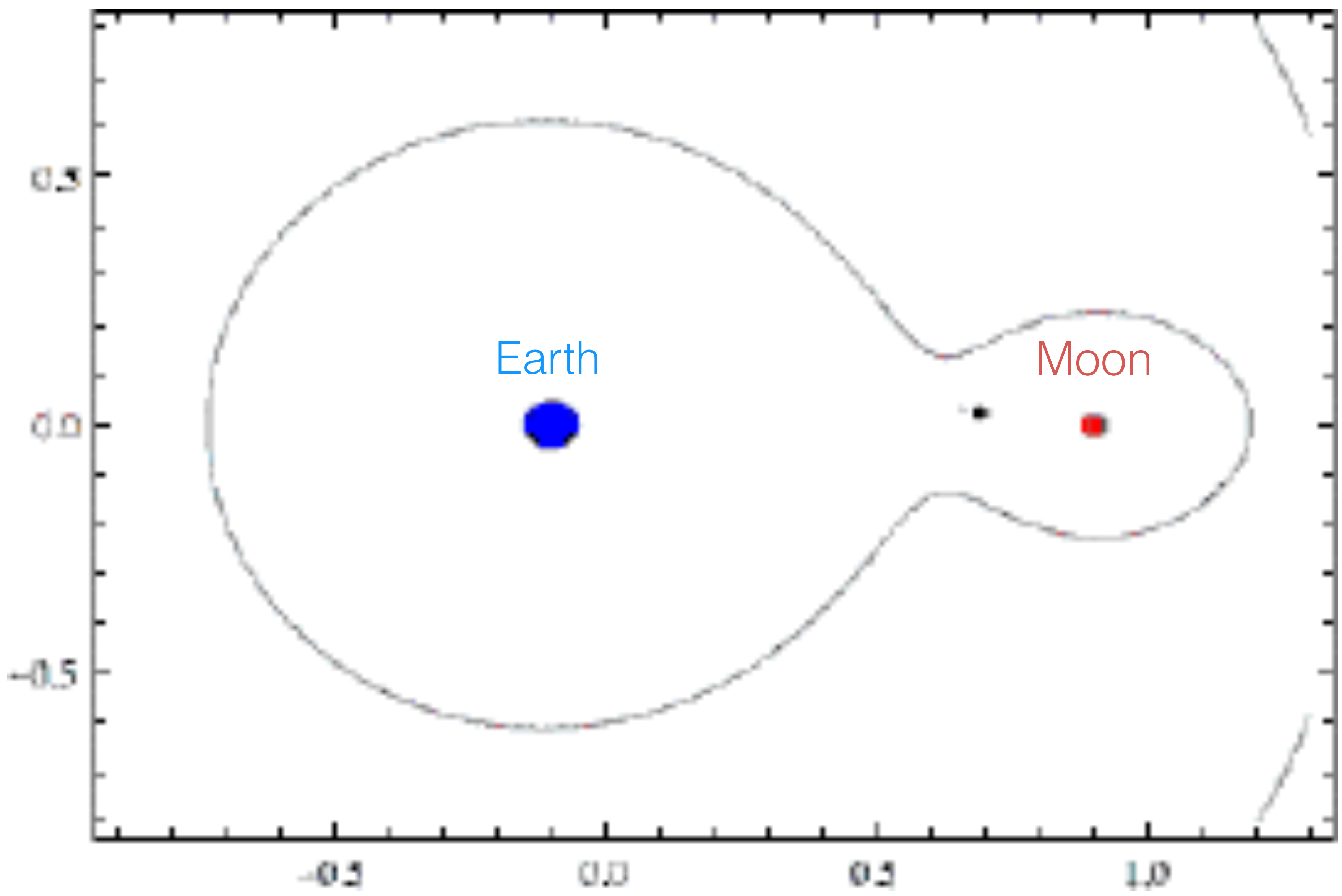


for each mass distribution
for each Kepler conic…

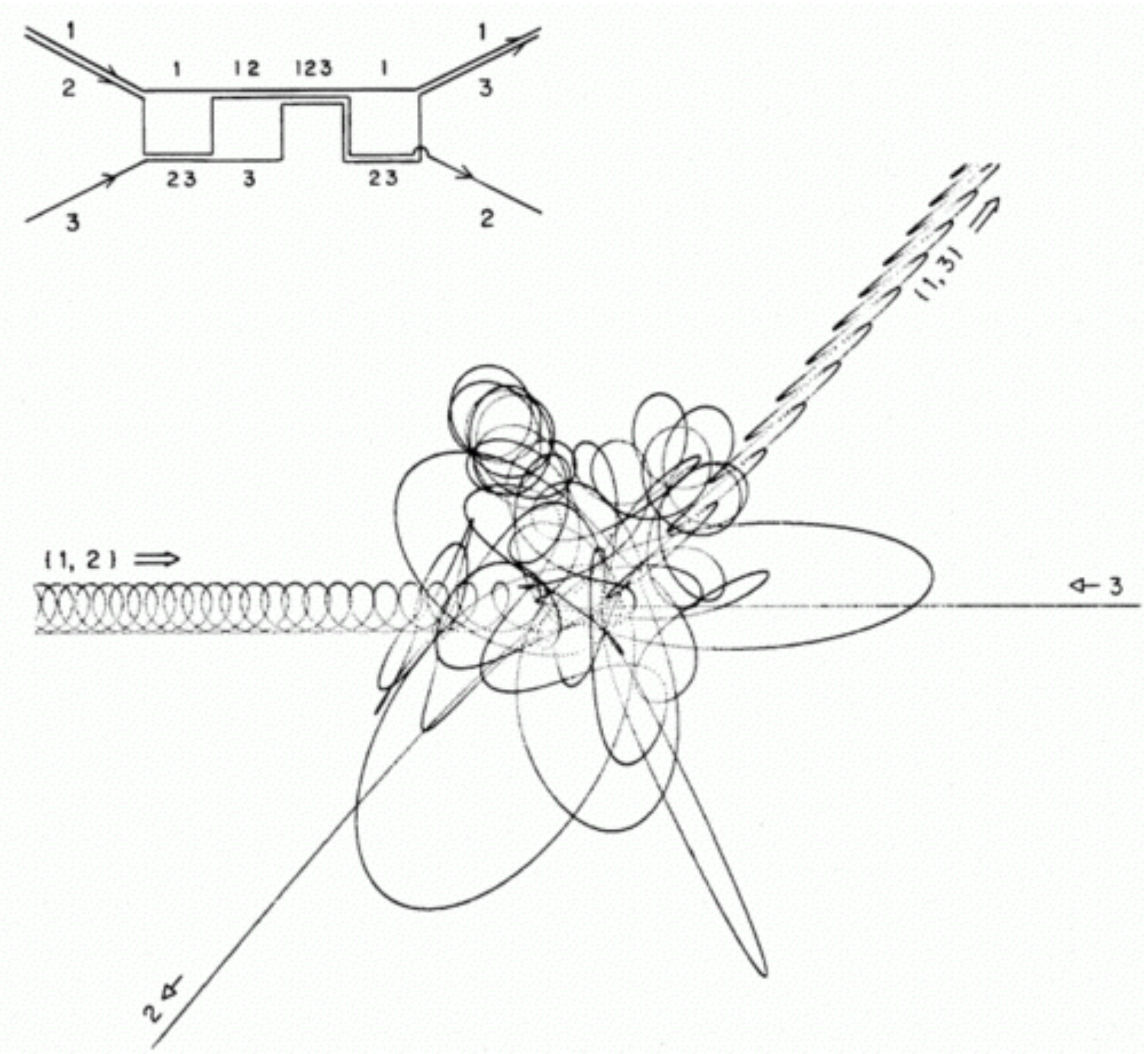


for each mass distribution
for each Kepler conic…

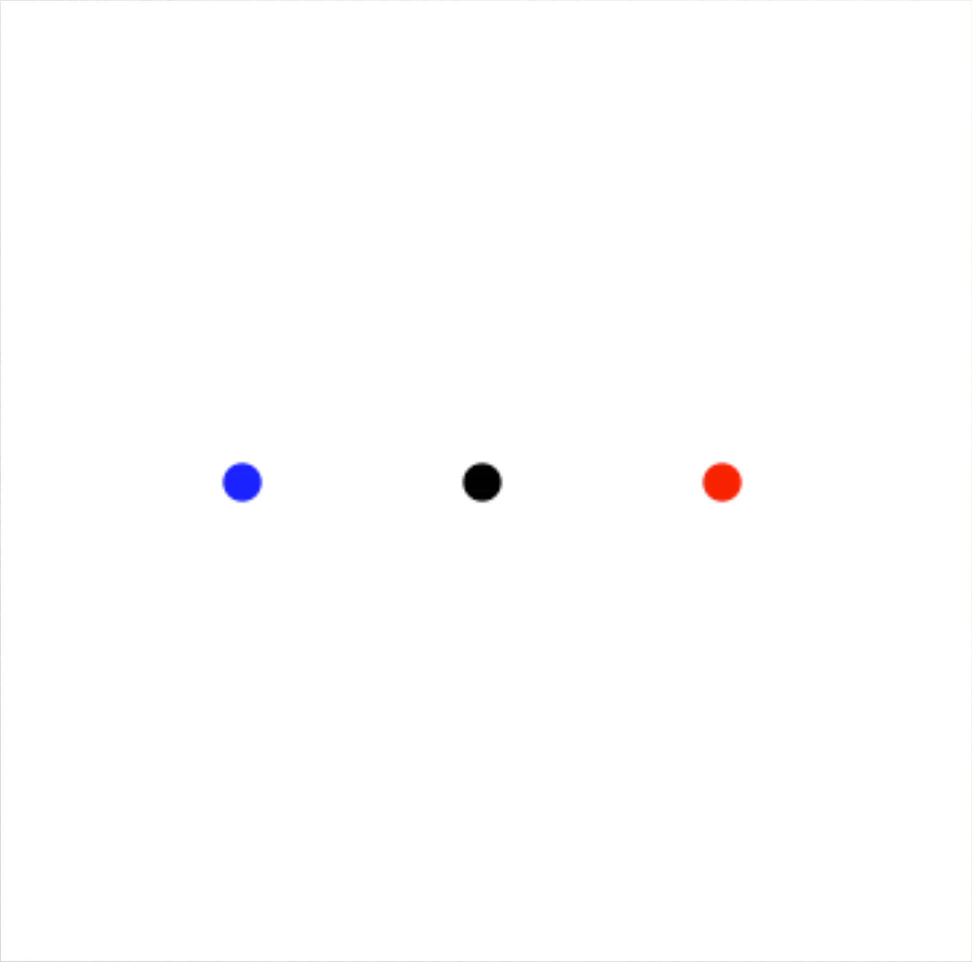
Poincare. 1892: Chaos in Restricted three-body problem. Shown here: a transit orbit, in a rotating frame

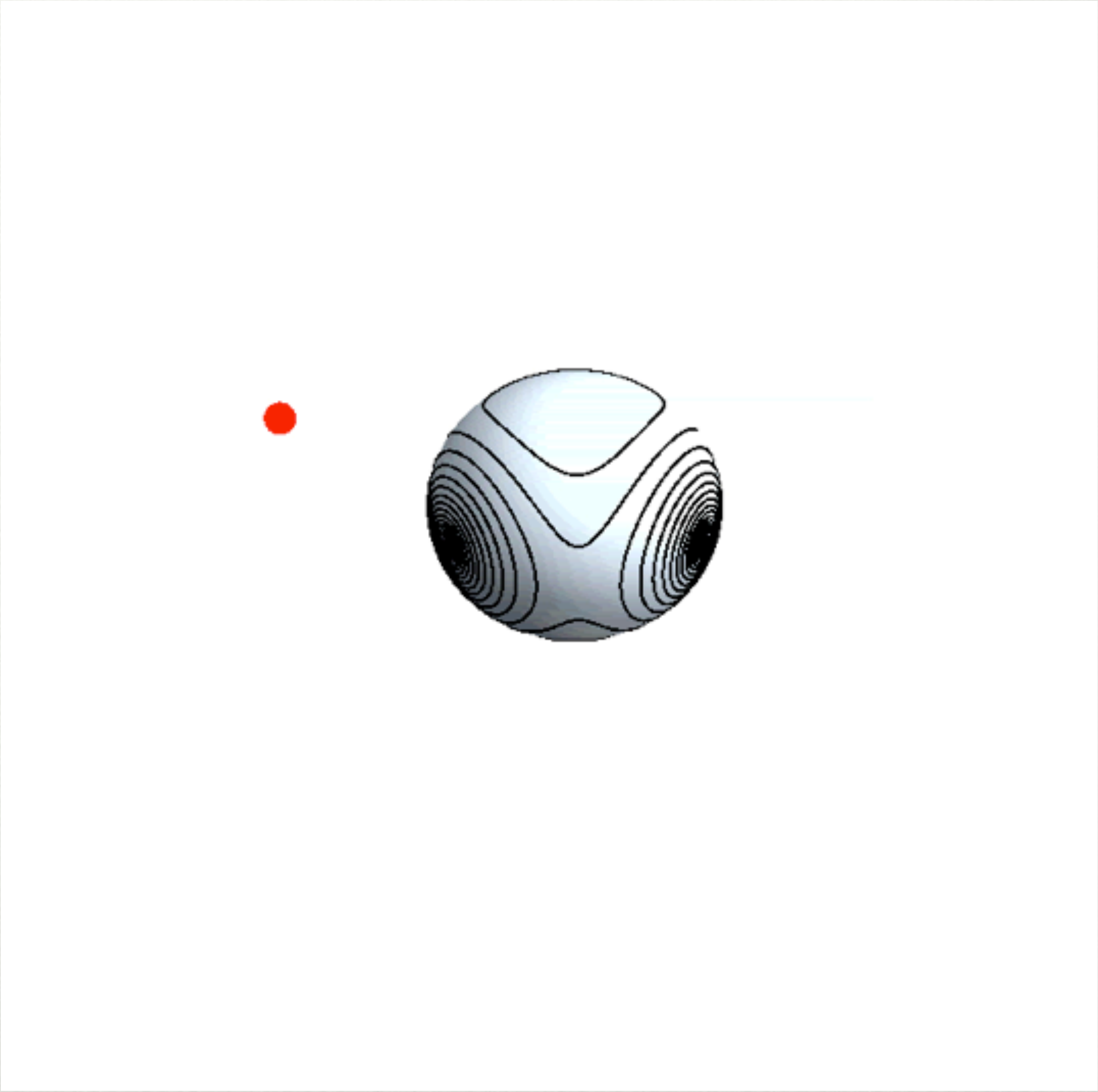


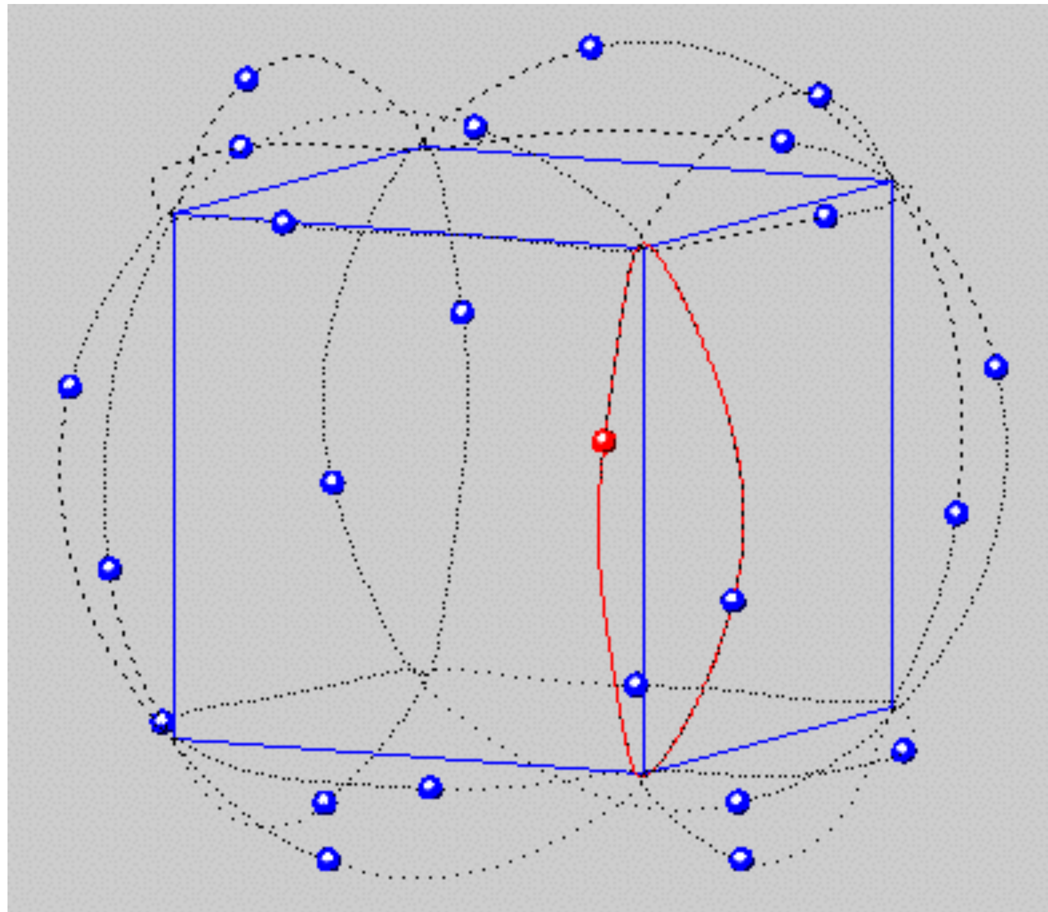
Hut. 1970s



Cris Moore 1994; Chenciner-Montgomery 2000





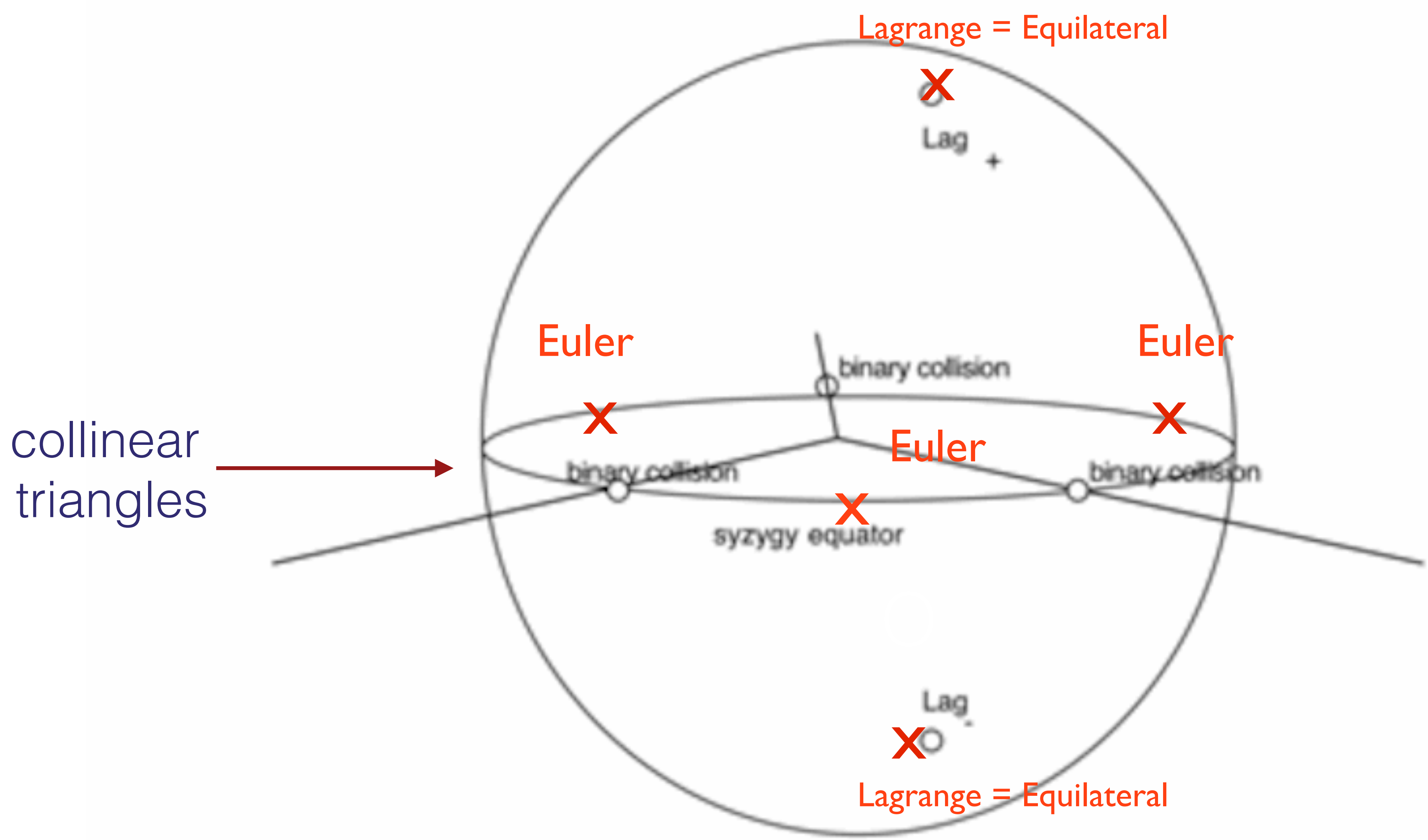


and about 100 more
6 or more per
platonic solid

Fusco,Gronchi, Negrini.
Platonic polyhedra, topological constraints
and periodic orbits of the classical N-body problem' ,
Invent. Math., Vol.285/2, 283-332. 2011

End Tour

Begin Shape Sphere



SHAPE SPHERE	⊂	**SHAPE SPACE**
Oriented similarity classes of triangles		Oriented congruence classes of triangles

Galileo:

1600

The laws of physics are invariant under the group G of isometries of space

G: translations, rotations, reflections

(N-body) Eqns: dN coupled 2nd order ODEs on the …

Configuration space Q = N-fold product of space

$$G \circlearrowright Q$$

GENERAL PRINCIPLE:

Q a manifold. G a Lie group acting on Q.

Then any G-invariant ODE (eg the N-body problem) on Q reduces to an equivalent ODE on the quotient "shape space" Q/G

OUR CASE:

$Q = (\mathbb{R}^2)^3 =$ Configuration space of planar 3-body problem.

$G = ISO_+(\mathbb{R}^2) =$ translations, rotations, ~~reflections~~.

$Q/G =$ Oriented congruence classes of triangles $= \mathbb{R}^3$

[3D: SSS] =Shape space;

CONTAINS: shape sphere =oriented SIMILARITY classes of triangles (set I = size = 1)

so expect: 2nd order ODEs on shape space (parameteized by J ...->)

Structures on $\mathbb{E} = (\mathbb{R}^2)^3 = \mathbb{C}^3$.

Points: $q = (q_1, q_2, q_3)$ with $q_a \in \mathbb{R}^2 = \mathbb{C}$; $(x, y) = x + iy$.

Mass metric: $\langle q, w \rangle = \Sigma m_a q_a \cdot w_a$, so Kinetic Energy $= K = \frac{1}{2}\langle \dot{q}, \dot{q} \rangle$

Newton's eqn: $\ddot{q} = \nabla U(q)$ (2nd order ODE on $\mathbb{E}$)

$U = \frac{m_1 m_2}{r_{12}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} =$ neg. potential energy,

$\nabla =$ gradient associated with mass metric: $dU(q)(v) = \langle \nabla U(q), v \rangle$

$CONSTANT:$ Energy $= K - U$

Size $=$ moment of inertia $= I(q) = \langle q, q \rangle$

ALL DESCEND TO SHAPE SPACE: $\mathbb{E}/G$

$CONSTANT:$ $J = \Sigma m_a q_a \wedge v_a$

$$q_a \wedge v_a = Im(q_a \bar{v}_a) = \text{Signed area}$$

= Angular momentum

Quotient map: $\mathbb{C}^3 \to \mathbb{R}^3$ from Configurations to shapes

$\mathbb{C}^3 \to^{modtranslations} \mathbb{C}^2 \to^{modrotations} \mathbb{R}^3$ is:

$\mathbb{C}^3 \to^{Jacobi} \mathbb{C}^2 \to^{Normalization} \mathbb{C}^2 \to^{\text{'}Hopf'} \mathbb{R}^3$

Jacobi:

$(q_1, q_2, q_3) \mapsto (q_2 - q_1, q_3 - (\frac{m_1}{m_1+m_2} q_1 + \frac{m_2}{m_1+m_2} q_2)) = (Y_0, Y_1)$

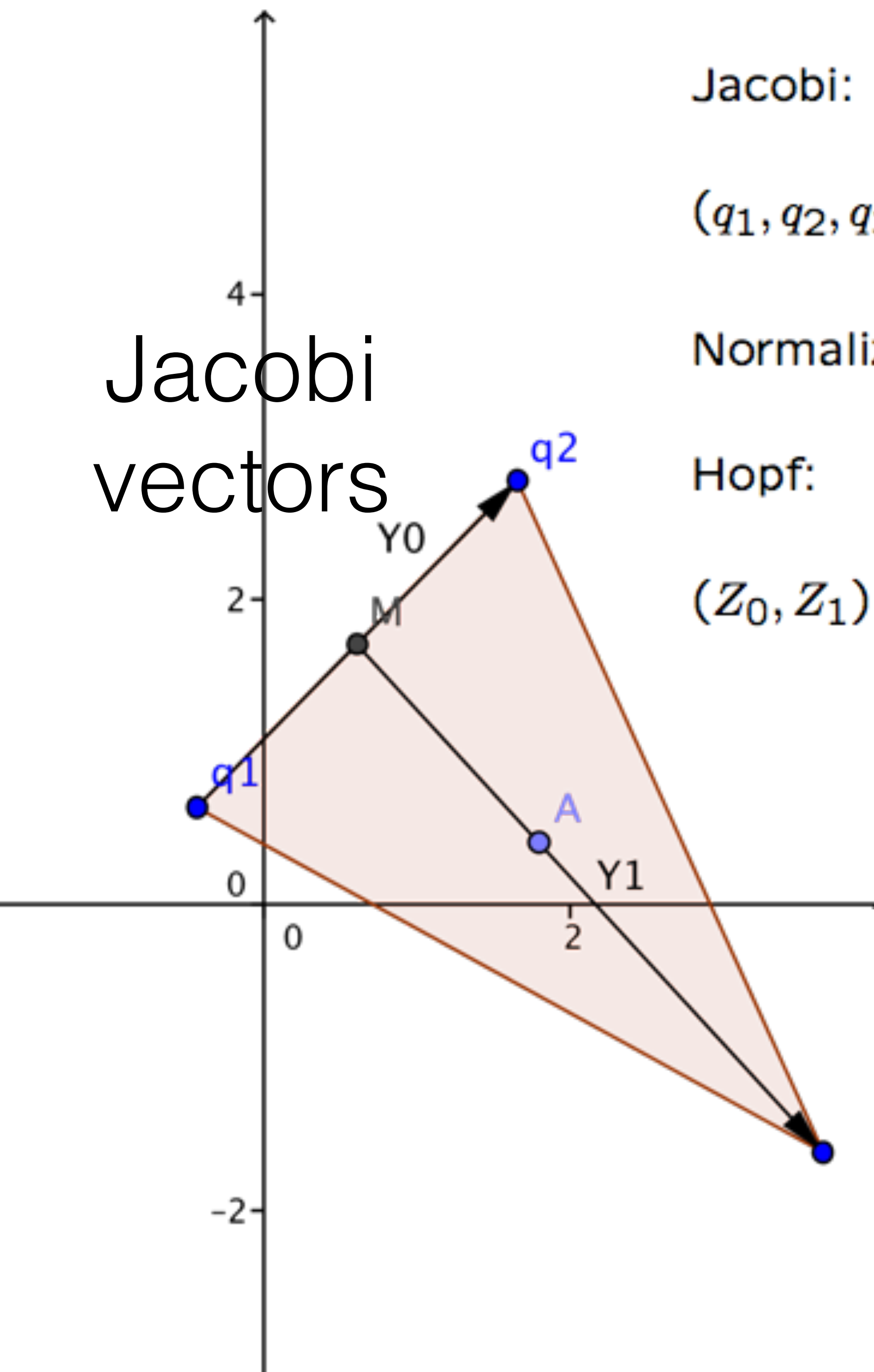
Normalization: $(Y_0, Y_1) \mapsto (\frac{1}{\mu_1} Y_0, \mapsto (\frac{1}{\mu_2} Y_1) = (Z_0, Z_1)$

Hopf:

$(Z_0, Z_1) \mapsto (|Z_0|^2 - |Z_1|^2, 2 Z_0 \bar{Z}_1) = (|Z_0|^2 - |Z_1|^2, 2 Z_0 \cdot Z_1, 2 Z_0 \wedge Z_1)$

real imag.

Jacobi vectors

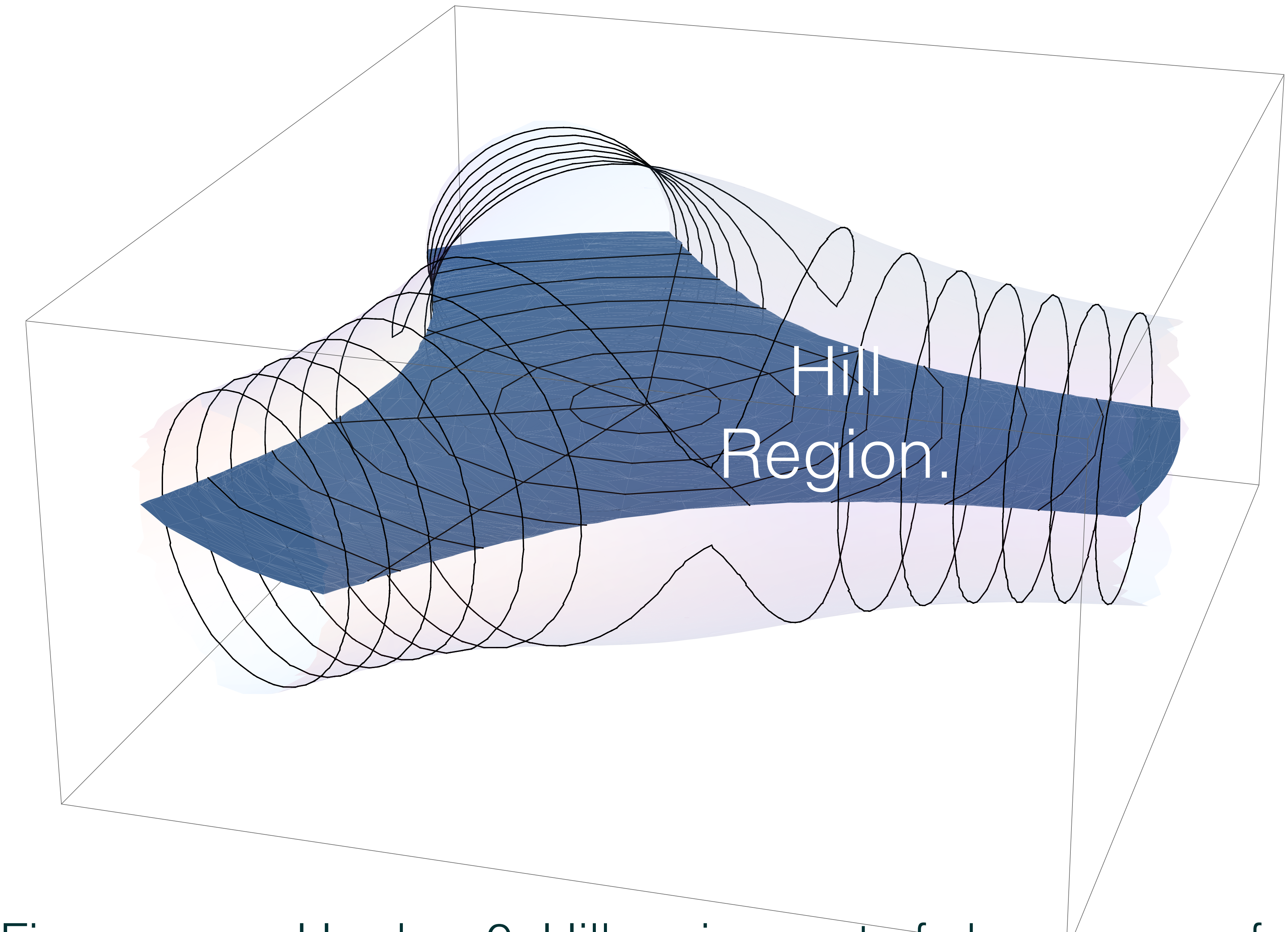


Honest Hopf: $\mathbb{C}^2 \to S^2 =$ shape sphere $\subset \mathbb{R}^3$

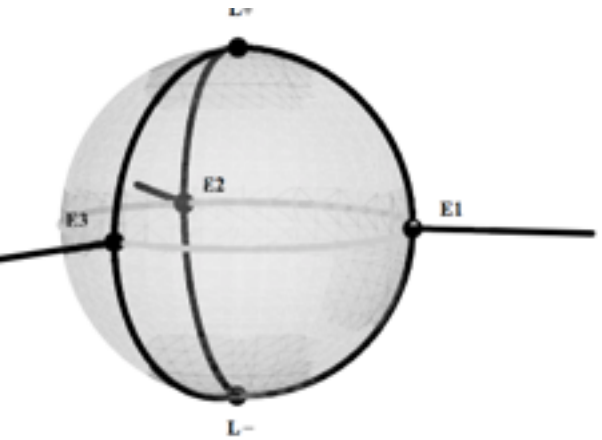
$(Z_0, Z_1) \mapsto \frac{1}{I}(|Z_0|^2 - |Z_1|^2, 2 Z_0 \bar{Z}_1)$

with $I =$ mom. of inertia $= |Z_0|^2 + |Z_1|^2 = \langle q, q \rangle$

go to UNAM2016/shapesphereB.ggb
and or shapespaceB.ggb



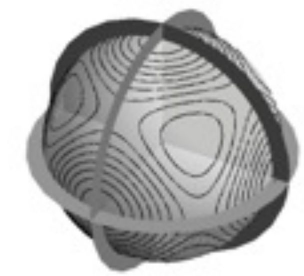
Fix energy = H =-h < 0. Hill region:part of shape space for which there is a v and H(q,v) = -h. Domain where motion occurs. Identical to region with U(q) > +h



all syz. seq
realized

blow-up

Reg. &
Grav. Billiards

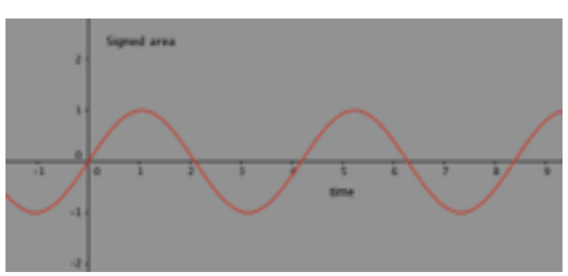


Bestiary of
Danya Rose

Mechanical intuition ,
conformal transf.
`dispositiones’
…

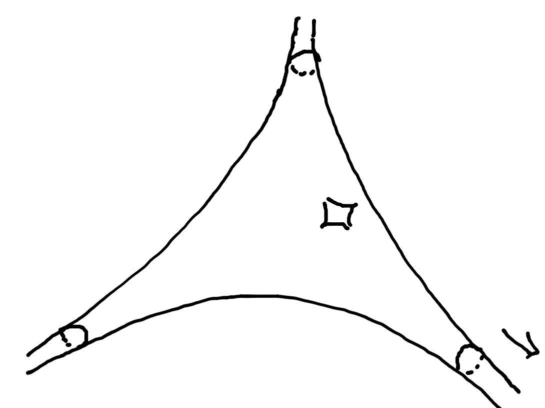
variational methods
plus discrete symmetries

Jacobi-Mauptertuis metric,
Riem geom: curvature



`Infinitely many…’
Ang. mom = 0 &
Energy < 0 implies;
infinitely many syzygies..
(w a single exception)

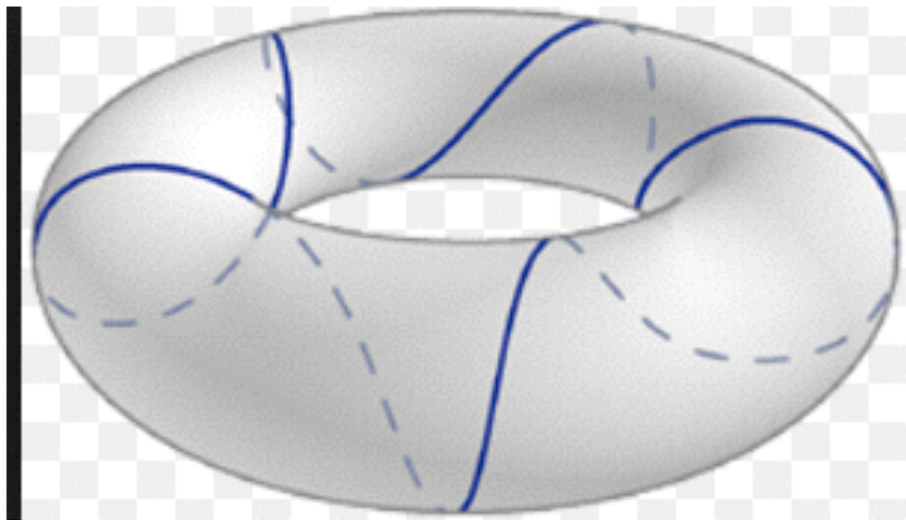
`A remarkable…’
surprising new solutions;
infinite families of
``designer’’ solutions



`Hyperbolic Pants`…’
UNIQUENESS of figure eight;
hyperbolic flow: if masses equal
and we `cheat”
1/r —> 1/(r*r)

Figure Eight:

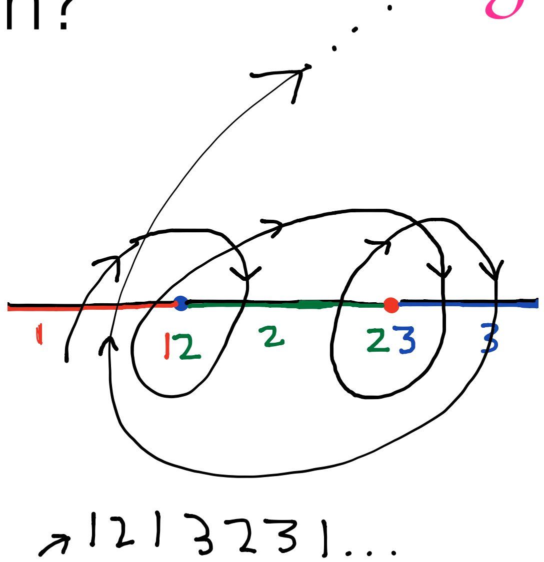
Motivating Question:



inspired by (compact) Riemannian geometry

Is every free homotopy class of loops realized by a periodic solution?

Direct Method! minimize length over given class



INSTEAD of *length*: integral of Lagrangian:

$$\text{Action} = A(q(\cdot)) = \int_0^T K(\dot{q}(t)) + U(q(t))dt$$

..Every free homotopy class…

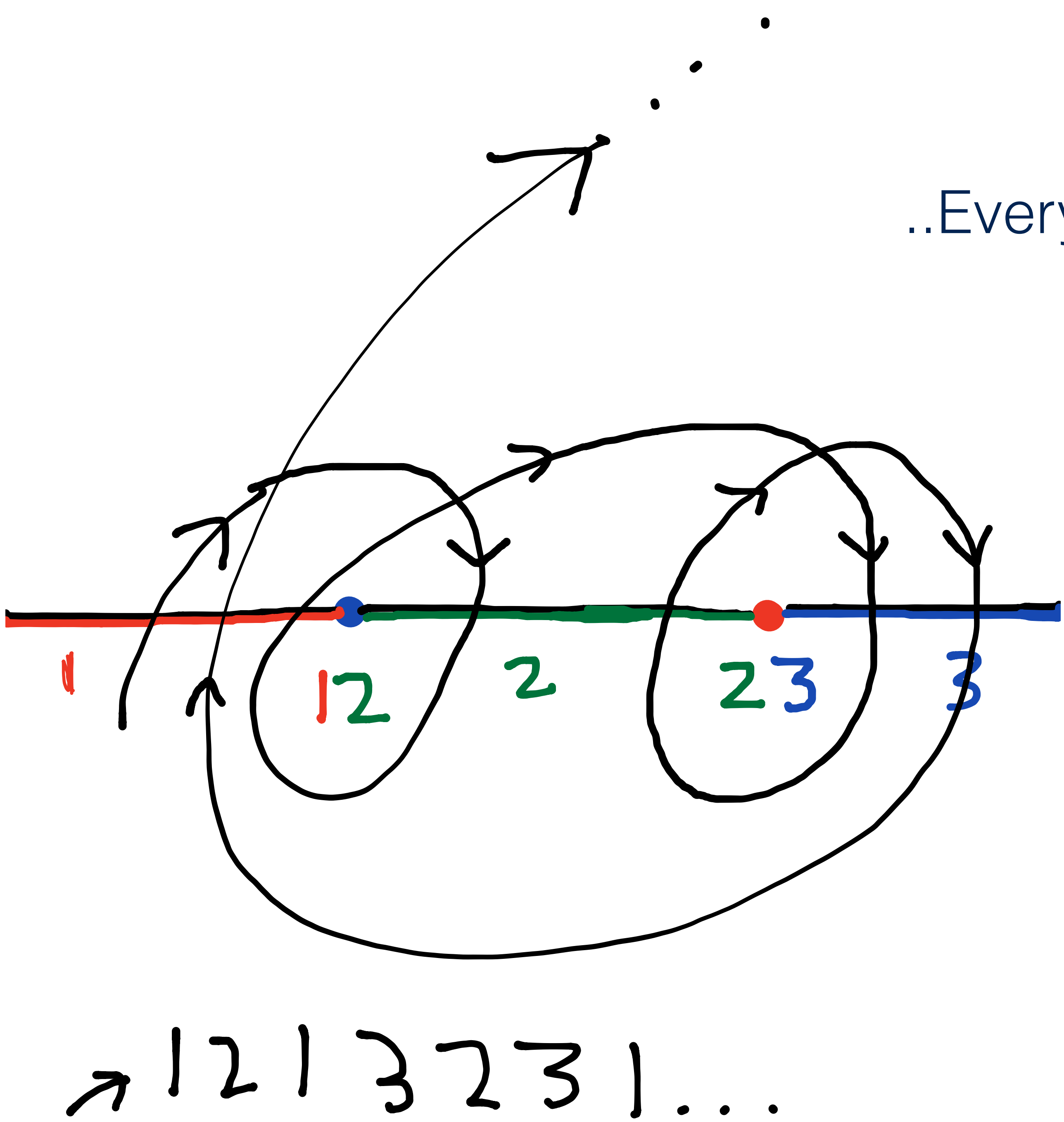


Figure 8: 123123

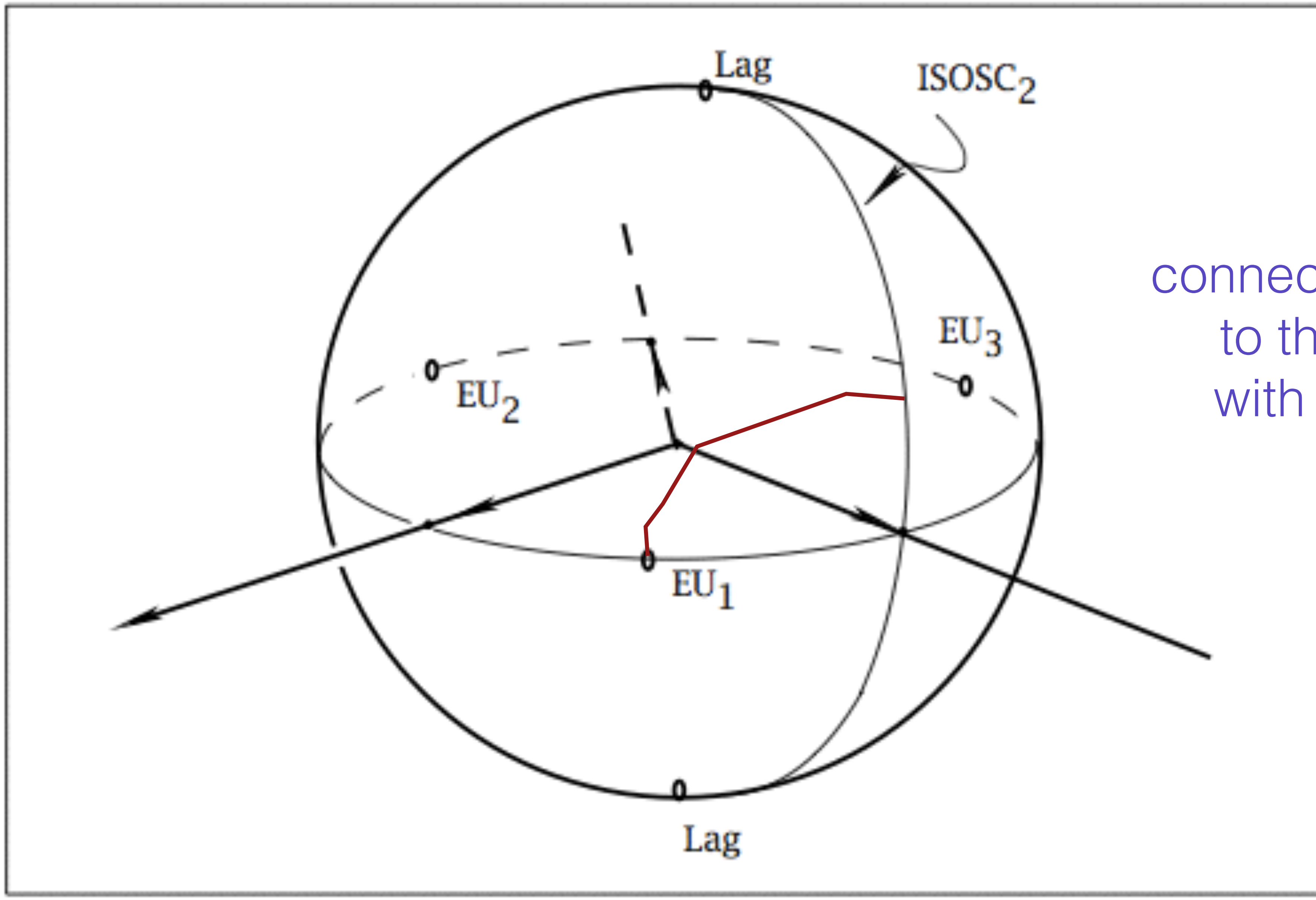


Figure 4. The shape sphere.

Min. Action A(q)
among all paths
q in shape space
connecting the Euler Ray (Eu_1)
to the Isosceles subspace
with 2 as vertex (ISOSC_2)

go to UNAM2016/reflec3.ggb

Danya Rose's …

F Bestiary of periodic orbits

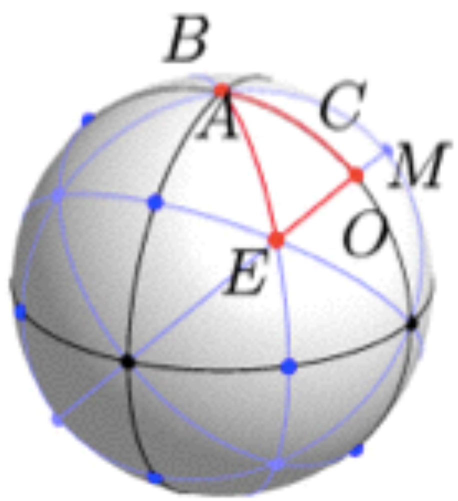
by Danya Rose. U. of Sydney. 2016 thesis

This appendix contains a complete listing (or "bestiary") of the orbits discovered during the numerical search discussed in section 5.3, with interesting examples and observations discussed in detail in Chapter 6. Each orbit is named according to its collision class or the fixed set in which it lives (isosceles or rectilinear), the order of its isotropy subgroup, and an index.

Sorting is by collision class or invariant subspace, order of isotropy subgroup, stability, sequence class, and, finally, by physical period.

Levi-Civita ; Lemaitre … Heggie…
Waldvogen; .. Moeckel-Montgomery

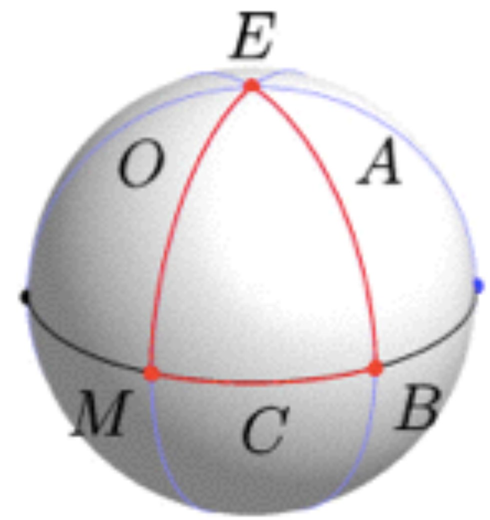
regularize
binary collisions
and reduce



(a) Regularised shape sphere.

4:1

branched
over binaries
2:1



(b) Classical shape sphere.

48-fold symmetry

12-fold symmetry

F.1.2.5 B-mode, Unstable: t0 (8, 5)

Isotropy subgroup: $\{(I,0), (\tau\rho\sigma_2, \frac{1}{2}), (\tau\rho s_1, \frac{1}{4}), (\sigma_2 s_1, \frac{1}{4}), (s_2, \frac{1}{2}), (\tau\rho\sigma_2 s_2, 0), (\tau\rho s_3, \frac{3}{4}), (\sigma_2 s_3, \frac{3}{4})\}$

Sequence type: $(A\Omega C\Omega')^4$

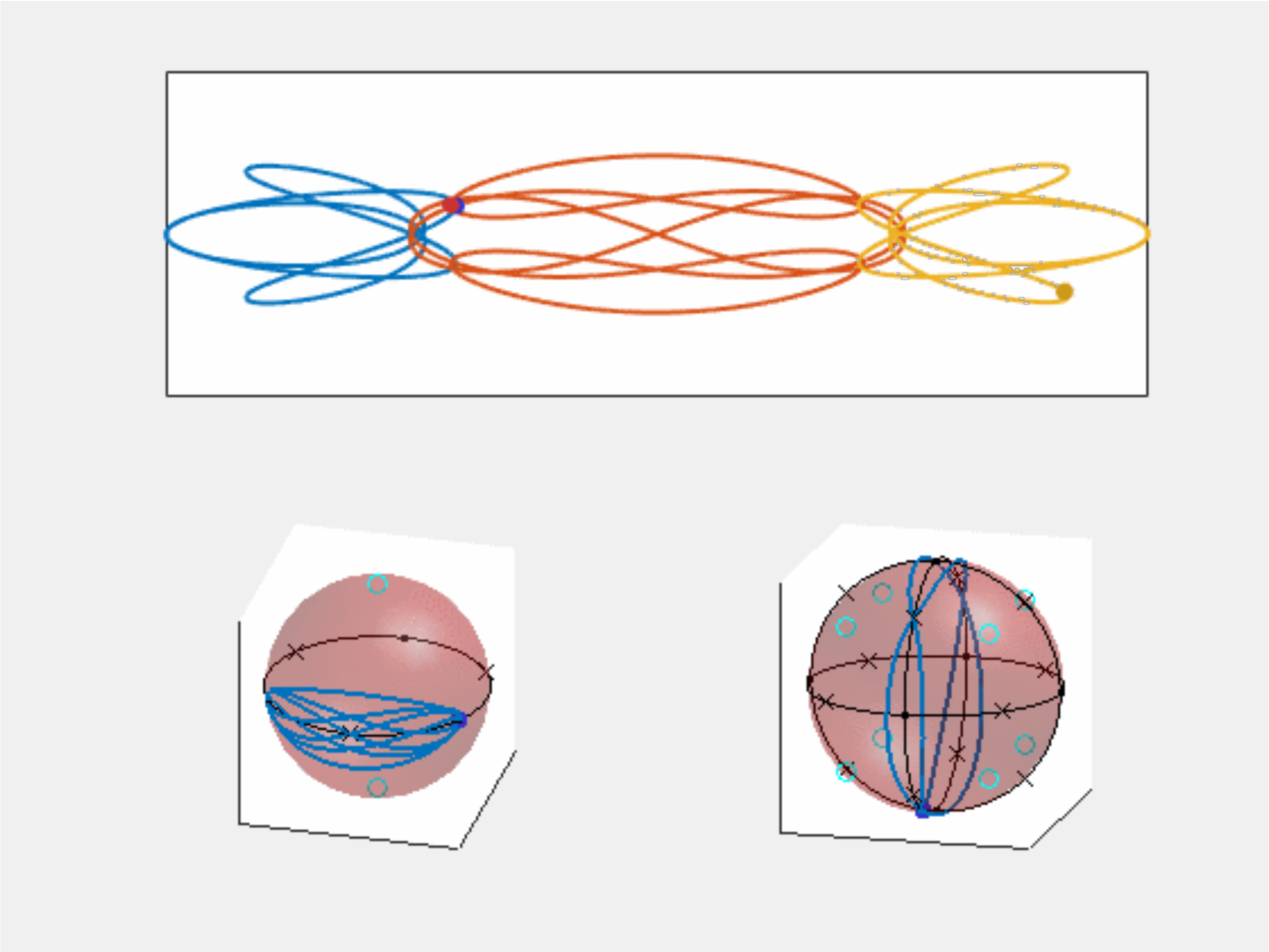
Ω: $OCACOACAOACACOA$

$$T_p = 67.01921804$$
$$T_r = 14.96879087$$
$$\Delta G = 12.36682876$$
$$\Delta\phi = -8.57712227$$
$$W = 0.00000000$$

$$z_1 = \begin{pmatrix} -0.94542673 \\ +1.08224817 \\ +0.94542673 \\ +1.17170924 \\ -0.00000000 \\ +1.17170924 \end{pmatrix} \qquad z_2 = \begin{pmatrix} -1.41582468 \\ +0.96447581 \\ -0.00000000 \\ +0.00000000 \\ -0.00000000 \\ +1.72244433 \end{pmatrix}$$

$$R_1 = \tau\rho\sigma_2 s_2 \qquad R_2 = \tau\rho s_3$$

$$\lambda = \begin{pmatrix} +25.42672460 + 0.00000000i \\ +1.00000000 + 0.00000000i \\ +0.25229380 + 0.96765068i \\ +0.25229380 - 0.96765068i \\ +1.00000000 + 0.00000000i \\ +0.03932870 + 0.00000000i \end{pmatrix} \qquad |\lambda| = \begin{pmatrix} +25.42672460 \\ +1.00000000 \\ +1.00000000 \\ +1.00000000 \\ +1.00000000 \\ +0.03932870 \end{pmatrix}$$



FINI

Overflow!

`Burrau' or
Pythagorean 3-4-5
three body problem (*)

(*): Greg Laughlin, UCSC made film w
Burlisch-Stoer integrator