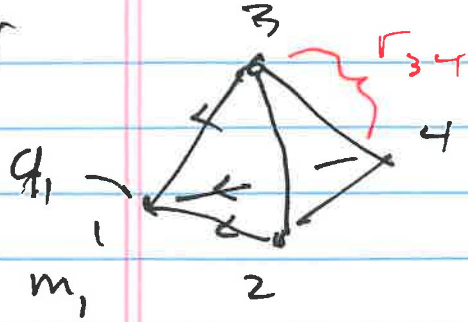
1



4 body problem in space (point masses) $(\mathbb{R}^3)$.

$$r_{ab} = |q_a - q_b|.$$

$$\left.\begin{aligned} m_1 \ddot{q}_1 &= F_{12} + F_{13} + F_{14} \\ m_2 \ddot{q}_2 &= F_{21} + F_{23} + F_{24} \end{aligned}\right\} \quad (N)$$

etc

$$F_{ab} = -G \frac{m_a m_b}{r_{ab}^2} \cancel{\exp} \frac{(q_a - q_b)}{r_{ab}}.$$

Conserv. laws: $\boxed{\sum m_a q_a \times \dot{q}_a = J(q, \dot{q})}$
$= 0$
← assume.

May assume: $\sum m_a \dot{q}_a = 0$, $\sum m_a q_a$

Config: $[q_1, q_2, q_3, q_4] \in M(3,4) = M$

Def. Config q is 'degenerate' or 'coplanar' if all the q_a lie on the same plane.

$\Sigma \subset M$ = hypersurface of deg. configs

Def Soln $q(t)$ is bounded if $\exists\, c > 0$ s.t. $\boxed{r_{ab}(t) \le c}$ $\forall\, t$ ~~in its domain~~ (B)

"Thm: Σ is a global slice for the $J=0$ bdd solns"

Quantitative: $M = \Sigma m_a$.

$\omega^2 = GM/c^3$; ω has units $\frac{1}{\text{Time}}$.

Thm [M-, 2018]. If $q: I \to M$ solves, has $J = 0$ & bound (B) then within every subinterval $[t_0, t_1] \subset I$ of size $\Delta t = t_1 - t_0 \leq \pi/\omega$ q goes coplanar: $\exists t_*, t_0 \leq t_* \leq t_1, q(t_*) \in \Sigma$.

Cor. If $I = [0, \infty)$ then q goes coplanar ∞ly after.

Inspirations

1. 3 bodies in the plane, Σ = collinear configs.

Same ~~Thm~~ Cor.

Thm [M-, 2002] ~~⊗⊗~~ $J = 0$, soln bdd defined on $(0, \infty)$. Then soln becomes collinear ∞ly after.

"syzygies!"

2. Conf: Shape Space for $N = 3$ in the plane:

$Sh(2,3) = M(2,3)/SE(2) = \mathbb{R}^3$

$[q_1, q_2, q_3]$ Isometries of plane pres. orient.

Littlejohn: $Sh(3,4) := M(3,4)/SE(3) = \mathbb{R}^6$!

aside $Sh(d,d+1) = \text{---} = \mathbb{R}^{??}$

Set-up. $\langle\ ,\ \rangle_m$ on $M = M(3,4)$.

$\frac{1}{2}\langle \dot{q},\dot{q}\rangle_m = K.E. = \frac{1}{2}\sum m_a \dot{q}_a \cdot \dot{q}_a$

so $\langle v, w\rangle = \sum m_a v_a \cdot w_a$.

$V: M \to \mathbb{R}$; $V = -G\sum \frac{m_a m_b}{r_{ab}}$.

$\nabla V(q)$ by $dV(q)(w) = \langle \nabla V(q), w\rangle_m$.

(N): $\ddot{q} = -\nabla V(q)$.

3B invariant under $SE(3)$ = transl + rotns

(also $E(3)$)

$q_a \mapsto q_a + c$; $c \in \mathbb{R}^3$

$q_a \mapsto g q_a$ $g \in SO(3)$

or $q \mapsto gq$.

$$
\begin{array}{lll}
\Sigma \subset M(3,4) & \langle\ ,\ \rangle_m & \\
\quad \pi \downarrow \mathbb{R}^3 & & \\
\Sigma \subset M(3,3) & \cdot\text{(c. of mass zero)} & \text{metric submersion.} \\
\det q = 0. \quad \pi \downarrow SO(3) & & d_M(\pi^{-1}(\sigma_1), \pi^{-1}(\sigma_2)) \\
\Sigma \subset Sh(3,4) & & = d_{Sh}(\sigma_1, \sigma_2)
\end{array}
$$

$G = SE(3)$ w $SO(3)$

Upstairs: M $\qquad \ddot q = -\nabla V(q)$

$\pi \downarrow$

Downstairs: Sh. $\qquad \nabla_{\dot\gamma}\dot\gamma = -\nabla V(\gamma)$ $\quad$ if $J=0$

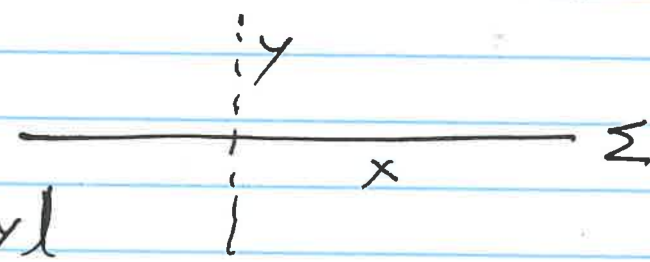
here V is G-inv; $\pi^* V = V$
by abuse..

Σ: tot. geod. downstairs i.e.
(not " " up ")

geod tgt to Σ stay in Σ.
solns to (N) starting planar stay planar.

$S(q) =$ signed dist (q, Σ)
S is also G-inv. $\pi^* S = S$.

Why signed dist?



$\text{dist}((x,y),\Sigma) = |y|$
$\text{signed dist}((x,y),\Sigma) = y$. smooth across Σ!

$S(t) = S(q(t)) = s(\gamma(t))$.

Thm A If $q(t)$ avoids $\text{Sing}\, S := \{q : S \text{ not diffble at } q\}$.
Thm.. $\ddot S = -Sg$ w $g>0$ & if (B) holds for q
$g \geq \omega^2$

Thm B codim $(\text{Sing}(S)) = 2$.

Pf of A. $\dot{S} = \langle \nabla S, \dot\gamma \rangle$

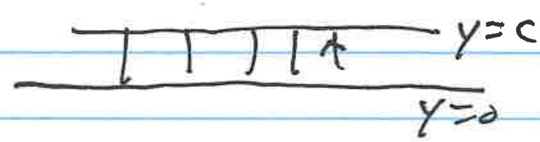
$$\ddot{S} = \langle \nabla S, \nabla_{\dot\gamma}\dot\gamma \rangle + \langle \nabla_{\dot\gamma} \nabla S, \dot\gamma \rangle$$
$$= \langle \nabla S, -\nabla V(\gamma) \rangle + Q(\dot\gamma, \dot\gamma)$$
$$= \underbrace{-S g_1}_{I} \qquad \underbrace{-S g_2}_{II}$$

(I): $g_1 > 0$ & (B) $\Rightarrow g_1 \geq \omega^2$.
(II): $g_2 \geq 0$

Pf II). Use $\|\nabla S\| = 1$ ($\Leftrightarrow$ HJ in the case $V=0$: $H(q, dS(q)) = c$ or $\frac{1}{2}\|\nabla S(q)\|^2 = c$.

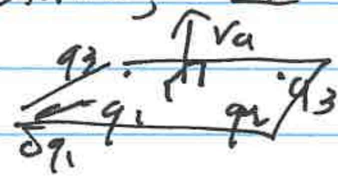
so: $\int$ curves of the fn $\frac{dq}{d\tau} = \nabla S(q(\tau))$ are unit speed geod $\perp \Sigma$.

Why? True for any hypersurface Σ in any Riem mfd M.

eg. 

x-axis in $\mathbb{R}^2$

Us: $\int$ curves: $q(\tau) = q + \tau v$. lines upstairs; $\perp$ to Σ: $q \in \Sigma$.



$\mathbb{R}^2 \subset \mathbb{R}^3$ $\delta q \in T_q \Sigma$: $\delta q_a \in \mathbb{R}^2$ arb.
$\langle \delta q, v \rangle = 0 \ \forall \delta q$
$\Longrightarrow v_a \in \mathbb{R} e_3$

Then $r_{ab}(\tau)^2 = |(q_a + \tau v_a) - (q_b + \tau v_b)|^2 \Rightarrow 0!$
$$= |q_{ab}|^2 + 2\tau \cancel{q_{ab} \cdot v_{ab}} + \tau^2 |v_{ab}|^2$$

so. $\langle \nabla S(q) - \nabla V(q) \rangle = \frac{d}{d\tau}(-V(q(\tau)))$

$$\& \ \frac{d}{d\tau} r_{ab}(\tau)^2 = 2\tau\, |v_{ab}|^2.$$
$$= 2S\, |v_{ab}|^2$$
$$\uparrow \quad !!$$

so

$$\frac{d}{d\tau}(-V) = \frac{d}{d\tau}\, G \sum \frac{m_a m_b}{r_{ab}}$$
$$= G \sum m_a m_b \left(\frac{-2S|v_{ab}|^2}{r_{ab}^3} \right),$$
$$= -S\, G \sum \frac{m_a m_b |v_{ab}|^2}{r_{ab}^3}.$$
$$= -S g_1$$

$g_1 > 0$ ✓.

(B) ~~But $\|\nabla V\| \geq \|\nabla S\|$~~; why (B) $\Rightarrow g_1 \geq \omega^2$?

$\|v\| = \|\nabla S\| = 1$ ✓ $\quad \sum m_a v_a = 0$.

Lagrange identity: $\Rightarrow \|v\|^2 = \frac{\sum m_a m_b |v_{ab}|^2}{\sum m_a}$

so

$$r_{ab} \leq c \Rightarrow \frac{1}{r_{ab}^3} \geq \frac{1}{c^3}.$$

$\sum m_a m_b |v_{ab}|^2 = M,$

$$g_1 \geq -S\frac{G}{c^3}\left(\sum m_a m_b |v_{ab}|^2\right) = -S\left(\frac{GM}{c^3}\right).$$

II. $Q_\gamma(v,v) = \langle \nabla_v \nabla S, v\rangle =$

Claim: $= -S H_\gamma(v,v)$, $H \geq 0$.

use ~~Geom~~ of Sh.

1) $K_{Sh} \geq 0$. non neg curved. (1)

Pf: O'neill fmla

$\left\{ \begin{array}{ll} M, K_M & \text{Riem} \\ \downarrow & \text{sub} \\ B \quad K_B. & \\ K_B \geq K_M & \\ \text{indeed} & \\ K_B = K_M + \frac{3}{4}\|\text{curv}\|^2 & \end{array} \right.$

2. Q is ess'lly the 2nd f.f. II of $\Sigma_c = \{S = c\}$, $c = S(\gamma)$.

3. Rel'n betwee (1) & (2) i.e. ~~ext &~~ intr & extr curv.

2. Σ_c, $\Sigma = \Sigma_0$. ∇S = unit 'outward' normal to Σ_c $= N$ outward to $\{0 \leq S \leq c\}$.

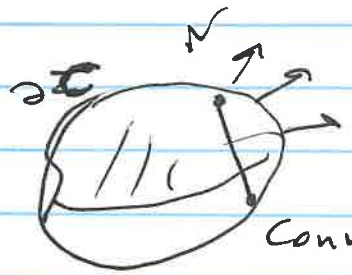
so $II(v,v) = \langle \nabla_v N, v\rangle = \langle \nabla_v \nabla S, v\rangle$, $v \perp \nabla S$, ie $v \parallel \Sigma_c$.

while if $v = N$ differentiate $\langle N, N\rangle = 1$ to get $\langle \nabla_v N, N\rangle = \langle \nabla_v N, v\rangle = 0$

$\Rightarrow Q_\gamma(v,v) = \begin{cases} II_{S(\gamma)}(v,v), & v \perp \nabla S \\ 0 & v \parallel \nabla S. \end{cases}$

~~Al~~ Also: if $\gamma \in \Sigma$, $II_\gamma = 0$ since Σ is t.t. geod, $\Rightarrow Q = -S H$ ← some quad form.

3. "Sign & the Meaning of Curvature"
— Gromov



convex, in $\mathbb{R}^n$

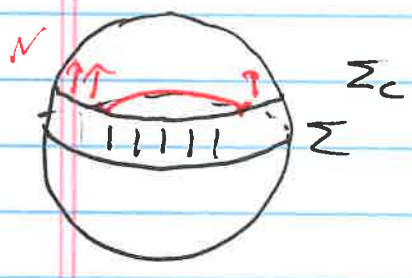
use outward normal

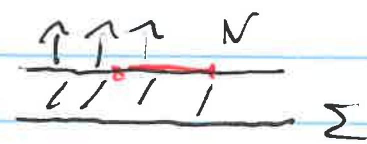
$II_N \geq 0$

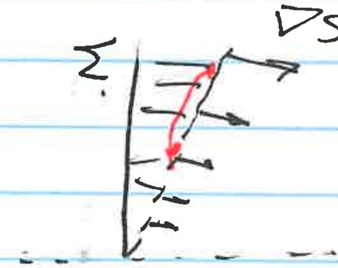
if strictly convex

$II_N > 0$.

But note: $N \to -N \Rightarrow II \to -II$







$K > 0$	$K = 0$	$K < 0$
$II_c < 0$	$II_c = 0$	$II_c > 0$.
~~convex:~~ $\{S_0 \leq S \leq c\}$ is concave	barely convex	convex

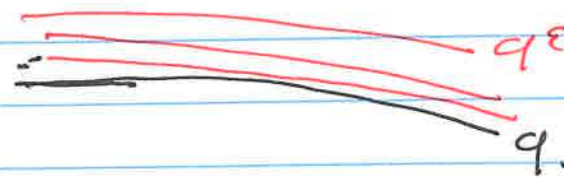
Thm Σ = tot geod hypersurf. in an ambient space B ($= Sh.$). S = signed dist$(\cdot, \Sigma)$. Then

$$II_{\nabla S,\ \{S = \text{const}\}}(\ ,\ ) = -S\,H(\ ,\ )$$

with $H \geq 0$ if $K_B \geq 0$
& $H \leq 0$ if $K_B \leq 0$.

QED for ~~Thm~~ II & Thm A

How Thm A + Thm B (codim Sing(S) = 2)
$\Rightarrow$ Main Thm.



$J(q^\varepsilon, \dot{q}^\varepsilon) = 0.$

$q^\varepsilon \xrightarrow[C^\infty]{} q$

q^ε miss $\mathrm{Sing}(S)$
possible since codim 2 q !
would not be poss. if codim 1!

Now Thm A applies to q^ε.

$\therefore$ If q sat bd $r_{ab} \le c$
q^ε " " $r^\varepsilon_{ab} < c + O(\varepsilon)$

But $S^\varepsilon(t) := S(q^\varepsilon(t)) \longrightarrow S(t) := S(q(t))$

& S^ε have ≥ 1 zeros in every interval of size $\Delta t = \frac{\pi}{c + O(\varepsilon)} + O(\varepsilon)$

$\therefore$ S " zero in every interval of size Δt.

Thm B. $\mathrm{codim}(\mathrm{Sing}(S)) = 2$.

$\Sigma \subset M(3,3)$ $\quad \|q\|^2_m = \mathrm{tr}(q q^T)$.

Σ & $\|\cdot\|^2$ invariant under
$q \mapsto g_1 q g_1^T, \; g_1, g_2 \in O(3)$

N.B: $\Sigma = \{q : \det q = 0\}$.

$\therefore$ $S(g_1 q g_2) = S(q)$, $g_1, g_2 \in SO(3)$

(Rmk on democ: graph)

use sing value decomp.

$D =$ diags a 'global slice' for tes action:

$g_1 q g_2^{+} = d = \mathrm{diag}(x_1, x_2, x_3) \in D.$

moreover can arrange.

$$x_1 \geq x_2 \geq |x_3| \geq 0$$

if $g_i \in O(3)$ can ensure $x_3 > 0$

But

$$\det(g_1 q g_2^{+}) = (\det g_1)(\det g_2)\det q.$$

$\mathcal{M}$ ~~det under~~

$|x_3| = |S|$, $x_3 = S$.

ps - sing value [the ; statistic of Shape sp].

Shorten?

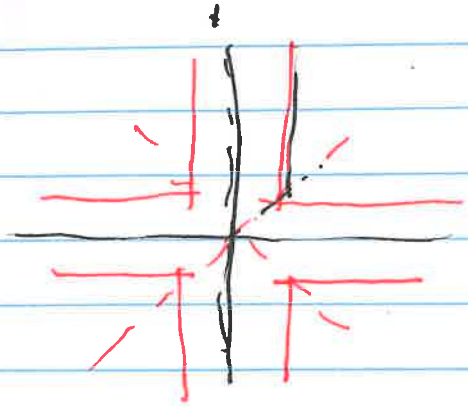
To understand Sing do not insist on ordering x_i:

$$\Sigma AD = \{(x_1, x_2, x_3) : x_1 x_2 x_3 = 0\}$$

$$|S| = \min_i |x_i|.$$

Easiest to see in case $N=2$.

$D=(x_1, x_2)$



Σ

$\mathrm{Sing}(S) \cap D = \{x : x_1 = \pm x_2\}$.

Shorter ?

~~Point~~ pts $x_1 = \pm x_2$ have extra symm rel $SO(2)\times SO(2)$ action.

$D = x I$

$N=3$: $D \cap \mathrm{Sing}(S) = \begin{pmatrix} x_3 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}$

more bodry rot sym,

~~compact~~ orbits thru these pts have dim $\underline{1}$ less than orbits thru gen pt.

$-(1+1) \longrightarrow 2$.

(QED for Main Thm

generalizations: $d+1$ bodies in $\mathbb{R}^d$.

- any attractive pair potl force:

$$V = +G \sum m_a m_a f_{ab}(r_{ab}).$$

$$f'_{ab}(r) > 0.$$

$$f''_{ab}(r) < 0.$$

eg $\frac{1}{r^\alpha}$ $\quad 0 < \alpha < \infty$

---

w/ explicit nods,

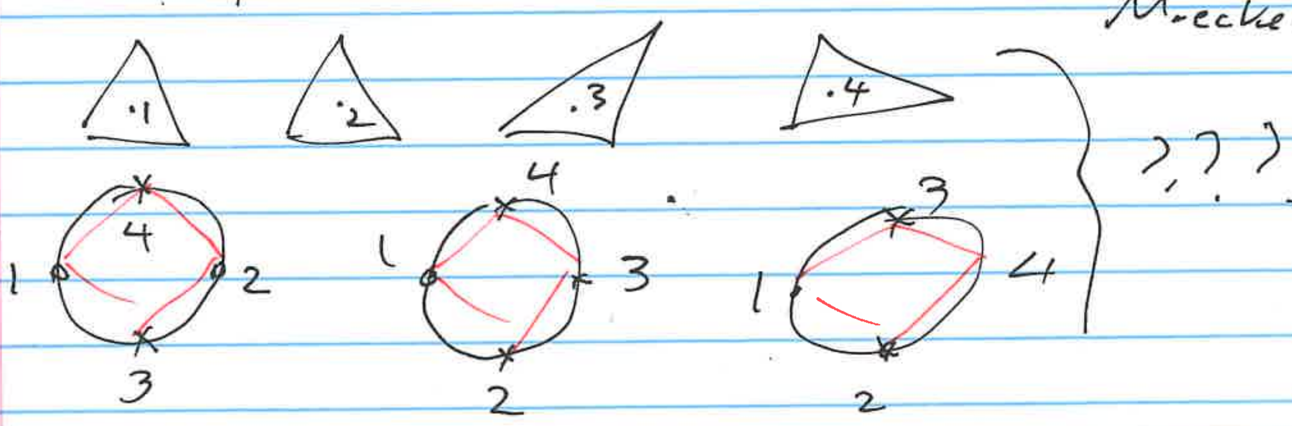
Dyn. outlook.

$N = 3$ types of syzygies? collinearity,

which symbol seq possible?

- Fig 8
- hardly/ Moeckel.

$N = 4$



Finally: For $d=3$, $N=d+1=4$

is the hyp. of $J=0$ needed?

For $d=2$, or even

yes $J=0$ needed.