Points and curves in the monster tower

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Abstract. Cartan introduced the method of prolongation which can be applied either to manifolds with distributions (Pfaffian systems) or integral curves to these distributions. Repeated application of prolongation to the plane endowed with its tangent bundle yields the monster tower, a sequence of manifolds, each a circle bundle over the previous one, each endowed with a rank 2 distribution. In [**MZ**] we proved that the problem of classifying points in the monster tower up to symmetry is the same as the problem of classifying Goursat distribution flags up to local diffeomorphism. The first level of the monster tower is a three-dimensional contact manifold and its integral curves are Legendrian curves. The philosophy driving the current work is that all questions regarding the monster tower (and hence regarding Goursat distribution germs) can be reduced to problems regarding Legendrian curve singularities. Here we establish a canonical correspondence between points of the monster tower and finite jets of Legendrian curves. We show that each point of the monster can be realized by evaluating the k-fold prolongation of an analytic Legendrian curve. Singular points arise from singular curves. The first prolongation of a point, i.e. a constant curve, is the circle fiber over that point. These curves are called vertical curves. Further prolongations of vertical curves we call tangency curves. The union of the vertical and tangency curves form precisely the abnormal curves (in the sense of sub-Riemannian geometry) for the monster distribution. Using these curves we define three types of points - regular (R), vertical (V) , and tangency (T) and from them associated singularity classes, the RVT classes. The RVT classes corresponds to singularity classes in the space of germs of Legendrian curves. All previous classification results for Goursat flags (many obtained by long calculation) now follow from this correspondence as corollaries of well-known results in the classification of Legendrian curve germs. Using the same correspondence we go beyond the known results and obtain the determination and classification of all simple points of the monster, and hence all simple Goursat germs. Finally, as spin-off to these ideas we prove that any plane curve singularity admits a resolution via a finite number of prolongations.

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Preface

This paper is a natural continuation of the previous paper [**MZ**] where we studied a class of objects called Goursat distributions – certain 2-plane fields in n-space – using Cartan's method of prolongation. The class of Goursat germs have interesting singularities which get exponentially deeper and more complicated with increasing n. In that paper we constructed a sequence of circle bundles called the "monster tower" such that any Goursat singularity in dimension n can be found in the tower at the same dimension. After writing that paper it became clear that their must be a dictionary between singularities of Legendrian curves (dimension 3), Goursat singularities, and points of the Monster tower (any dimension). The current paper develops this dictionary and uses it to prove a host of new classification results concerning Goursat singularities.

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CHAPTER 1

Introduction

1.1. The Monster Construction

The Monster is a sequence of circle bundle projections

$$\ldots \to \mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2 \to \ldots \to \mathbb{P}^2\mathbb{R}^2 \to \mathbb{P}^1\mathbb{R}^2 \to \mathbb{P}^0\mathbb{R}^2 = \mathbb{R}^2$$

between manifolds $\mathbb{P}^i\mathbb{R}^2$ of dimension $i+2$, $i = 0, 1, 2, \ldots$, each endowed with a rank 2 distribution $\Delta^i \subset T\mathbb{P}^i\mathbb{R}^2$. The construction of $(\mathbb{P}^i\mathbb{R}^2, \Delta^i)$ is inductive.

(1) $\mathbb{P}^0\mathbb{R}^2 = \mathbb{R}^2$, $\Delta^0 = T\mathbb{R}^2$.

(2) $\mathbb{P}^{i+1}\mathbb{R}^2$ is the following circle bundle over $\mathbb{P}^i\mathbb{R}^2$: a point of $\mathbb{P}^{i+1}\mathbb{R}^2$ is a pair (m, ℓ), where $m \in \mathbb{P}^i\mathbb{R}^2$ and ℓ is a 1-dimensional subspace of the plane $\Delta^i(m)$.

    The 2-distribution Δ^{i+1} is defined in terms of smooth curves tangent to Δ^{i+1}. A smooth curve $\gamma : (a, b) \to \mathbb{P}^i\mathbb{R}^2$ is said to be *tangent* to Δ^i, or *integral*, if $\gamma'(t) \in \Delta^i\big(\gamma(t)\big)$, $t \in (a, b)$.

(3) a curve $t \to \big(m(t), \ell(t)\big)$ in $\mathbb{P}^{i+1}\mathbb{R}^2$ is tangent to Δ^{i+1} if the curve $t \to m(t)$ in $\mathbb{P}^i\mathbb{R}^2$ is tangent to Δ^i, and $m'(t) \in \ell(t)$ for all t.

The construction of $\mathbb{P}^{i+1}\mathbb{R}^2$ from $\mathbb{P}^i\mathbb{R}^2$ is an instance of a general procedure called *prolongation* due to E. Cartan [**C1, C2, C3**] and beautifully explained in section 3 of [**B**]. (Cartan invented several procedures now known as 'prolongation'. At a coordinate level, these prolongation procedures involve extending previously defined objects by adding derivatives.) The description we have given is repeated from our earlier work [**MZ**]. The symbol $\mathbb{P}$ is used to denote projectivization : $\mathbb{P}^{i+1}\mathbb{R}^2$ is the projectivization $\mathbb{P}\Delta^i$ of the rank 2 vector bundle Δ^i over $P^i\mathbb{R}^2$. When we refer to the Monster at level i, or the ith Monster , we mean $\mathbb{P}^i\mathbb{R}^2$ endowed with the distribution Δ^i.

1.2. Symmetries. Equivalence of points of the Monster

A local symmetry of the Monster at level i is a local diffeomorphism $\Phi : U \to \widetilde{U}$, where U and $\widetilde{U}$ are open sets of $\mathbb{P}^i\mathbb{R}^2$ and $d\Phi$ brings Δ^i restricted to U to Δ^i restricted to $\widetilde{U}$.

Convention. All objects, diffeomorphisms, curves, etc., are assumed to be real-analytic.

Two points $p, \tilde{p}$ of the Monster manifold $\mathbb{P}^i\mathbb{R}^2$ are equivalent if there exists a local symmetry of $\mathbb{P}^i\mathbb{R}^2$ sending p to $\tilde{p}$.

1.3. Prolonging symmetries

The prolongation of a local symmetry Φ at level i is the local symmetry Φ^1 at level $i+1$ defined by

$$\Phi^1(m,l) = \big(\Phi(m), d\Phi_m(\ell)\big).$$

Prolongation preserves the fibers of the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. The process of prolongation can be iterated. The k-step-prolongation of Φ is the

one-step-prolongation of the Φ^{k-1}.

1.4. The basic theorem

Our whole approach hinges on the basic theorem:

THEOREM 1.1. *For $i > 1$ every local symmetry at level i is the prolongation of a symmetry at level $i-1$.*

Upon applying the theorem repeatedly, we eventually arrive at level 1, which is the contact manifold $(\mathbb{P}^1\mathbb{R}^2, \Delta^1)$. (See for example [**A1**] regarding this contact manifold.) The symmetries of a contact manifold are called contact transformations, or contactomorphisms. Thus Theorem 1.1 asserts that the $(i-1)$-fold prolongation is an isomorphism between the pseudogroup of contact transformations (level 1) and the pseudogroup of local symmetries at level i. The theorem expressly excludes the isomorphism between the $i=0$ and $i=1$ pseudogroups. Indeed the pseudogroup of contact transformations is strictly larger than the 1st prolongation of the pseudogroup of local diffeomorphisms of the plane, see [**A1**].

Theorem 1.1 can be deduced from our earlier work [**MZ**], namely from the “sandwich lemma“ for Goursat distribution and the theorem (Theorem 1.2 in the next section) on realizing all Goursat distribution germs as points within the Monster tower. In order to be self-contained we present a simple proof of Theorem 1.1 in section 1.7 in purely Monster terms.

REMARK. Our Monster construction started with 1-dimensional contact elements for the plane. The construction generalized by starting instead with k-dimensional contact elements on an n-manifold. The generalization of Theorem 1.1 remains valid, and holds even for $i = 1$ when $k > 1$. This generalization is sometimes called the Backlund theorem, and is tied up with the local symmetries associated to the canonical distribution for jet spaces. See for example [**B**] and [**Y**].

1.5. The Monster and Goursat distributions

Our earlier work [**MZ**], where the Monster was introduced, was motivated by the problem of classifying Goursat distributions. Given a distribution $D \subset TM$ on a manifold M we can form its ‘square’ $D^2 = [D, D]$, where $[\cdot,\cdot]$ denotes Lie bracket. Iterate, forming $D^{j+1} = [D^j, D^j]$. The distribution is called ‘Goursat’ if the D^j have constant rank, and this rank increase by one at each step $rank(D^{j+1}) = 1 + rank(D^j)$, up until the final step j at which point $D^j = TM$.

THEOREM 1.2 ([**MZ**]).

1. *The distribution Δ^i on $\mathbb{P}^i\mathbb{R}^2$ is Goursat for $i \geq 1$.*
2. *Any germ of any rank 2 Goursat distribution on a $(2+i)$-dimensional manifold appears somewhere in the Monster manifold $(\mathbb{P}^i\mathbb{R}^2, \Delta^i)$: this germ is diffeomorphic to the germ of Δ^i at some point of $\mathbb{P}^i\mathbb{R}^2$.*

This theorem asserts that the problems of classifying points of the Monster and of classifying germs of Goursat 2-distributions are the same problem.

1.5.1. Darboux, Engel, and Cartan theorems in Monster terms. A rank 2 Goursat distribution on a 3-manifold is a contact structure. A rank 2 Goursat distribution on a 4-manifold is called an Engel structure. Classical theorems of Darboux and Engel assert that all contact structures are locally diffeomorphic and that all Engel structures are locally diffeomorphic. (See, for example [**A1**], [**VG**], [**Z3**]). In Monster terms:

THEOREM 1.3 (Darboux and Engel theorems in Monster terms). *All points of $\mathbb{P}^1\mathbb{R}^2$ are equivalent. All points of $\mathbb{P}^2\mathbb{R}^2$ are equivalent.*

For $i \geq 2$ not all points of $\mathbb{P}^i\mathbb{R}^2$ are equivalent, but there is a single open dense equivalence class. Cartan found the normal form for the points of this class, i.e. he wrote down the generic Goursat germ. In Monster terms:

THEOREM 1.4 (Cartan theorem in Monster terms). *There is a single equivalence class of points in $\mathbb{P}^i\mathbb{R}^2$ which is open and dense.*

1.5.2. Some history. Cartan [**C1**], [**C2**], [**C3**] asserts a version of Theorem 1.2. However, Cartan defines prolongation by taking the usual derivatives of coordinates, which is to say, his prolongation is affine and does not consider the possibility of tangent lines "going vertical". When interpreted in his affine sense, Cartan's assertions are, apparently, the (false) theorem that the open dense germ of Theorem 1.4 is everything, i.e. that all Goursat distributions of a fixed dimension are diffeomorphic to a fixed model. Indeed, in his famous five-variables paper, [**C4**] Cartan seems to say explicitly that the only Goursat germ is the open and dense one. In [**GKR**] the authors found a counterexample to exactly this assertion, and with it the first appearing Goursat singularity. Goursat in his book [**Go**] presented the assertion of Theorem 1.4. A number of decades later Bryant and Hsu [**B**] redefined Cartan's prolongation in projective terms, which is the prolongation we have just used in defining the Monster. For more on the history, see the introduction to [**GKR**] and section 3.1 of [**B**].

1.5.3. Normal forms for Goursat distributions. Giaro, Kumpera, and Ruiz discovered in [**GKR**] the first Goursat germ not covered by Cartan. In so doing they initiated the study of singular Goursat germs. Kumpera and Ruiz introduced special coordinates in dimension $2 + i$ with associated Goursat normal forms depending on $(i-2)$ real parameters which covered all rank 2 Goursat germs. They and their followers calculated which parameters could be "killed", which could be reduced to 1 or -1 and which must be left continuous (moduli). The outcome of these computations is that the set of equivalence classes of Goursat germs on $\mathbb{R}^{2+i}$ is finite for $i \leq 7$. Consequently the set of equivalence classes of points of $\mathbb{P}^i\mathbb{R}^2$, $i \leq 7$ is finite. This number is $2, 5, 13, 34, 93$ for $i = 3, 4, 5, 6, 7$. See the works [**GKR**], [**KR**], [**Ga**], [**Mor1**], [**Mor2**] of Giaro, Kumpera, and Ruiz ($i = 3, 4$), Gaspar ($i = 5$) and Mormul ($i = 6, 7$). Mormul discovered [**Mor2**], [**Mor3**], [**Mor7**] the first moduli, which appears at $i = 8$. The length of these computations increases exponentially with i. Beyond Cartan's theorem, the only results which are valid for all dimensions are Mormul's classifications of codimension one singularities in [**Mor4**] and his classification of the simplest codimension two singularities in [**Mor5**].

1.6. Our approach

We reduce the problem of classifying points in the Monster to a well-studied classification problem: that of germs of Legendrian curves.

1.6.1. Integral curves. By a curve in $\mathbb{P}^i\mathbb{R}^2$ we mean a map $\gamma : (a,b) \to \mathbb{P}^i\mathbb{R}^2$. A curve in $\mathbb{P}^i\mathbb{R}^2$ tangent to the 2-distribution Δ^i is called *integral curve*. Integral curves in a contact 3-manifold such as $\mathbb{P}^1\mathbb{R}^2$ are called Legendrian curves, and their germs are called Legendrian germs.

1.6.2. Equivalent curves. Two germs of curves $\gamma : (\mathbb{R},0) \to (\mathbb{P}^i\mathbb{R}^2, p)$ and $\widetilde{\gamma} : (\mathbb{R},0) \to (\mathbb{P}^i\mathbb{R}^2, \tilde{p})$ are called equivalent if there exists a local symmetry $\Phi : (\mathbb{P}^i\mathbb{R}^2, p) \to (\mathbb{P}^i\mathbb{R}^2, \tilde{p})$ and a local diffeomorphism $\phi : (\mathbb{R},0) \to (\mathbb{R},0)$ (a reparameterization of a curve) such that $\widetilde{\gamma} = \Phi \circ \gamma \circ \phi$.

Remark. When $i = 1$ we will call this equivalence RL-contact-equivalence ('RL' is for Right-Left, see for example [**AVG**]).

1.6.3. Classification problems. Consider the following problems.

(1) Classify germs of integral curves in the Monster tower
(2) Classify germs of Legendrian curves
(3) Classify points in the Monster tower
(4) Classify finite jets of Legendrian curves

We will show 1. and 2. are equivalent problems, and we will show that 3. and 4. are equivalent problems. The equivalence between 3. and 4. allows us to translate well-known results regarding Legendrian germs into classification results on points of the Monster which leads to a number of new classification results and gives simple unified proofs of previously disparate results. For example, Mormul's classification [**Mor4**] of the codimension one singularities of Goursat 2-distributions now becomes a corollary of the contact classification of the simplest (A-type) singularities of Legendrian germs, see Example 3.27 and Corollary 4.42.

1.6.4. Simple points of the Monster. A point of the Monster is called *simple* if it is contained in a neighborhood which is the union of a *finite* number of equivalence classes. We go on to solve the basic classification problem:

(*) Determine and classify the simple points of the Monster tower.

The earlier classification results for levels $i = 2, \ldots, 7$ mentioned above (see section 1.5.3) comprise a small part of our solution to (*).

1.7. Proof of the basic theorem

The proof of Theorem 1.1 is based on the fact that for $i \geq 1$ the fibers of the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$ can be intrinsically defined and so any symmetry maps fibers to fibers. This intrinsic definition is in terms of characteristic vector fields for the distribution $(\Delta^i)^2 = [\Delta^i, \Delta^i]$.

1.7.1. The distribution $(\Delta^i)^2$. The following lemma gives a simple relation between the 2-distribution Δ^{i-1} and the 3-distribution $(\Delta^i)^2$.

Lemma 1.5. *For $i \geq 1$ the distribution $(\Delta^i)^2$ has constant rank 3. A curve in $\mathbb{P}^i\mathbb{R}^2$ is tangent to $(\Delta^i)^2$ if and only if its projection to $\mathbb{P}^{i-1}\mathbb{R}^2$ is tangent to the 2-distribution Δ^{i-1}.*

PROOF. Fix a point $(m, \ell) \in \mathbb{P}^i\mathbb{R}^2$. Choose a local frame (X_1, X_2) of Δ^{i-1} which is defined near m and is such that $X_1(m) \in \ell$. Given a point $(\widetilde{m}, \widetilde{\ell})$ close to (m, ℓ) one has $\widetilde{\ell} = \operatorname{span}\big(X_1(\widetilde{m}) + tX_2(\widetilde{m})\big)$, where $t \in \mathbb{R}$ is a small number parameterizing the line $\widetilde{\ell}$. Choose local coordinates x on $\mathbb{P}^{i-1}\mathbb{R}^2$ defined near m. Then (x, t) form a local coordinate system on $\mathbb{P}^i\mathbb{R}^2$ near (m, ℓ). In this local coordinate system

$$\Delta^i = \operatorname{span}\Big(X_1(x) + t \cdot X_2(x),\ \partial/\partial t\Big)$$

and consequently

$$(\Delta^i)^2 = \operatorname{span}\Big(X_1(x),\ X_2(x),\ \partial/\partial t\Big) \tag{1.1}$$

which proves Lemma 1.5. □

1.7.2. Characteristic vector field for the distribution $(\Delta^i)^2$.

DEFINITION 1.6. A vector field C is a characteristic vector field for a distribution D if C is tangent to D and $[C, D] \subseteq D$.

Take local coordinates (x, t) as in the proof of Lemma 1.5. Then $\partial/\partial t$ is a characteristic

vector field for the 3-distribution $(\Delta^i)^2$ since $[\partial/\partial t,\ X_1(x)] = [\partial/\partial t, X_2(x)] \equiv 0$. These relations imply that any vector field of the form $f(x, t)\partial/\partial t$ is a characteristic vector field. Now, a vector field has the form $f(x, t)\partial/\partial t$ if and only if it is tangent to the fibers of the fibration $\mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i-1}\mathbb{R}^2$, so we have shown that any vector field tangent to the fibration is a characteristic vector field. The following lemma states that there are no other characteristic vector fields provided that $i \geq 2$.

LEMMA 1.7. *If $i \geq 2$ then a vector field on $\mathbb{P}^i\mathbb{R}^2$ is a characteristic vector field for the distribution $(\Delta^i)^2$ if and only if this vector field is tangent to the fibers of the fibration $\mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i-1}\mathbb{R}^2$..*

Note that $(\Delta^1)^2 = T\mathbb{P}^1\mathbb{R}^2$ (this follows, for example, from Lemma 1.5) and consequently Lemma 1.7 does not hold for $i = 1$.

PROOF. Take local coordinates (x, t) as in the proof of Lemma 1.5. Let C be a characteristic vector field for the 3-distribution (1.1). This means that C has the form

$$C = g_1(x, t)X_1(x) + g_2(x, t)X_2(x) + f(x, t)\partial/\partial t$$

and that

$$[C,\ X_1(x)],\ [C,\ X_2(x)],\ [C,\ \partial/\partial t] \in \operatorname{span}\Big(X_1(x),\ X_2(x),\ \partial/\partial t\Big) \tag{1.2}$$

Expanding out the Lie brackets, we find that (1.2) is equivalent to the inclusions

$$g_1(x, t) \cdot [X_1(x),\ X_2(x)],\quad g_2(x, t) \cdot [X_1(x),\ X_2(x)] \in \operatorname{span}\big(X_1(x), X_2(x)\big) \tag{1.3}$$

But $X_1(x), X_2(x), [X_1(x), X_2(x)]$ are linearly independent. (See Lemma 1.5 and use that $i \geq 1$.) Therefore (1.3) holds if and only if $g_1(x, t) = g_2(x, t) \equiv 0$, which is to say, if and only if C is tangent to the fibers of the fibration. □

1.7.3. From Lemma 1.7 to Theorem 1.1. Consider any symmetry Ψ of $\mathbb{P}^{i+1}\mathbb{R}^2$, $i \geq 1$. Ψ must preserve the 3-distribution $(\Delta^{i+1})^2$ and so, by Lemma 1.7, the fibration $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. Therefore Ψ induces a diffeomorphism Φ of $\mathbb{P}^i\mathbb{R}^2$. Since the push-down of $(\Delta^{i+1})^2$ is Δ^i, it is clear that Φ is a symmetry of $\mathbb{P}^i\mathbb{R}^2$. To prove Theorem 1.1 it remains to show that Ψ is the prolongation of Φ.

Take any point $(m, \ell) \in \mathbb{P}^{i+1}\mathbb{R}^2$. Let I be a small interval containing 0 and take any integral curve $\Gamma : t \in I \to (m(t), \ell(t))$ in $\mathbb{P}^{i+1}\mathbb{R}^2$ passing through (m, ℓ) at $t = 0$ and such that $m'(t) \neq 0$ for $t \in I$. (Such a curve exists.) Let $\widetilde{\Gamma} = \Psi \circ \Gamma : t \to (\widetilde{m}(t), \widetilde{\ell}(t))$ be the image of Γ under the symmetry Ψ. The symmetry Φ brings the projected curve $t \to m(t)$ to the curve $t \to \widetilde{m}(t)$. It follows that the prolongation Φ^1 of Φ takes Γ to a curve of the form $\widehat{\Gamma} : t \to (\widetilde{m}(t), \widehat{\ell}(t))$. Since Φ^1 and Ψ are both symmetries the curves $\widetilde{\Gamma}$ and $\widehat{\Gamma}$ are both integral. Therefore $\widetilde{m}'(t) \in \widehat{\ell}(t)$ and $\widetilde{m}'(t) \in \widetilde{\ell}(t)$. Since Φ is a diffeomorphism and $m'(t) \neq 0$ we have that $\widetilde{m}'(t) \neq 0, t \in I$ and it follows that $\widehat{\ell}(t) = \widetilde{\ell}(t) = \mathrm{span}(\widetilde{m}'(t))$ for $t \in I$. $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ are the same curve! In particular $\widehat{\Gamma}(0) = \widetilde{\Gamma}(0)$ which means that $\Phi^1(m, \ell) = \Psi(m, \ell)$. Since (m, ℓ) was arbitrary, $\Phi^1 = \Psi$.

1.8. Plan of the Paper

Our main results are formulated in chapters 2 - 4.

In chapter 2 we define two operations : Monsterization and Legendrization. Monsterization sends a Legendrian curve to a point of the Monster. Legendrization reverses this operation by sending a point in the Monster to a class of Legendrian curves. These two operations are basic tools for obtaining our results.

In section 2.1 we prove the equivalence of the classification problems 1. and 2. in section 1.6.3. The proof requires that we define the prolongation of a (non-constant) singular integral curve.

In section 2.2 we prove that any point of the Monster is touched by the prolongation of some (singular) Legendrian curve and that RL-contact-equivalent Legendrian curves touch equivalent points. These facts allows us to define Monsterization and to give one of several equivalent definitions of a non-singular point in the Monster: a point is non-singular if it is realized as the prolongation of *immersed* Legendrian curve, evaluated at a certain time. The Cartan Theorem 1.4 regarding the open dense Goursat germ becomes a direct corollary of a well-known result on the local contact equivalence of non-singular Legendrian curves.

In section 2.3 we define critical curves and Legendrization. A curve is *critical* if it is an integral curve in the Monster whose projection to the first level $\mathbb{P}^1\mathbb{R}^2$ is constant. The Legendrization of a point of the Monster is the singularity class of Legendrian curves (with respect to RL-contact equivalence) formed by the projections (to level 1) of all those integral curves passing through the given point which are not tangent to a critical curve through that point. Without this tangency restriction the Legendrization operation would yield singularity classes much too large (or "deep") to be useful.

In section 2.4 we use Legendrization to relate the equivalence problem for points in the Monster ((2) of 1.6.3) with the equivalence problem for Legendrian curve germs ((3) of 1.6.3). Theorems 2.19 and 2.20 are the starting point for establishing the equivalence of problems (3) and (4) of section 1.6.3.

In section 2.5 we use Legendrization to give a definition of the *jet-identification number* $r = r(p)$ of a point p in the Monster. This number is the integer such that the point p can be identified with a single r-jet of a Legendrian curve in the following sense: the Legendrization of p contains, along with any Legendrian curve germ, all Legendrian curve germs with the same r-jet *and* the set of r-jets of the curves in the Legendrization of p is exhausted by a single r-jet. The set of all points for which this number is defined forms a singularity class – i.e. the set is invariant under the action of the symmetry group. Within this singularity class the equivalence problem for points is shown to be equivalent, via Legendrization, to the equivalence problem for jets of Legendrian curves. To make use of this equivalence we must answer two questions: How do we effectively characterize this singularity class, i.e. the points for which the jet-identification number exist? For these points, how do we calculate this number? We answer these questions in sections 2.6 – 2.8.

In section 2.6 we partition directions ℓ in Δ^i into *regular* and *critical.* A direction is critical if there exists a critical curve tangent to it ; otherwise it is called regular. We then partition points of the Monster into *regular and critical.* A point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$, $m \in \mathbb{P}^k\mathbb{R}^2, \ell \subset \Delta^k(m)$ is critical or regular according to whether the direction ℓ is critical or regular in Δ^k. Theorem 2.29 states that the jet-identification number $r(p)$ is defined if and only if the point p is regular. In sections 2.7 and 2.8 we give a method for calculating the jet-identification number. This method requires introducing two other integer invariants, the *degree of regularity*, defined only for regular points, and the *parameterization number*, defined for all points.

Although the jet-identification number is undefined for critical points, there is a straightforward method for reducing the equivalence problem for critical points to that for regular points. This reduction is the subject of section 3.9, after we have stratified the Monster into certain natural singularity classes, the "RVT classes".

The notion of a critical curve is critical to our approach. We study critical curves in the first two sections of chapter 3. The simplest critical curves are the vertical curves – those lying in a fiber of $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. We prove that all other critical curves are prolongations of vertical curves. We also prove that a curve is critical if and only if it is a singular (or abnormal) integral curve in the sense of sub-Riemannian geometry for the distribution Δ^i.

We use the critical curves to partition points of the Monster in three types, regular (R) vertical (V) , and tangency (T). By successively projecting a point to lower levels of the Monster and applying the R V T partition at that level we can associate to each point at level i an $i-3$-tuple of letters R, V, T. (It is an $i-3$-tuple because this process stops at level 2 where there are no points of type V or T invariantly defined.) In this way we obtain a stratification of the Monster into singularity classes, the RVT classes, indexed by certain $i-3$-tuples of the letters R, V, T.

The RVT classes, expressed in Goursat terms (Theorem 1.2), coincide with classes defined earlier by Mormul [**Mor6**] using Kumpera-Ruiz coordinates. The RVT classes in the ith level of the Monster also coincide with certain classes defined by Jean [**J**] in the kinematic model of a car pulling $(i-1)$ trailers. (This model is isomorphic to $\mathbb{P}^i\mathbb{R}^2$, see [**MZ**], Appendix D.) Despite the work of our predecessors, our construction of these RVT classes is the first coordinate-free stratification of the

Monster (and consequently of Goursat distributions). The coordinate-free nature of the stratification has several advantages over coordinate definitions. One advantage is that it quickly yields two equivalent definitions of what it means for a point of the Monster to be singular (section 3.8). But the strongest advantage is realized upon combining the notion of RVT classes with the operations of Legendrization and Monsterization. This then yields a stratification of Legendrian curves which can be compared with known singularity theory results, and then carried back from the Legendrian world to the Monster world.

Sections 3.9 and 3.10 contain two reduction theorems. Call an RVT class regular or critical according to whether or not it ends in an R, or, what is the same, whether or not its points are regular. Theorem 3.20 reduces the classification of points in a critical RVT class to the classification of points in a corresponding regular RVT class. Theorem 3.23 states that the Legendrization of any RVT class coincides with that of any of its regular prolongations. This theorem thus reduces the calculation of the Legendrization of an RVT class to the case of a critical class.

Section 3.12 is devoted to the parameterization and jet-identification numbers of points of certain RVT classes. Theorems 3.42 and 3.45 state that these numbers depend only on the class, not on the specific point within the class. Combined with the results of sections 3.9, 3.10 and Chapter 2 these theorems provide an *algorithm for reducing the problems of equivalence and classification for points of the Monster to the problems of equivalence and classification for Legendrian curve jets.* This algorithm is given in section 3.13.

In chapter 4 we apply the results of chapters 2 and 3 to classify the simple points of the Monster. (Recall that a point is 'simple' if it has a neighborhood covered by a finite number of equivalence classes.) Our classification contains all known results in the classification of Goursat distributions (see section 1.5.3) and much more. We refine the notion of "simple point" into tower-simple and stage-simple. We classify the tower-simple points in terms of their RVT classes by supplying a list of those RVT classes which consist of tower-simple points. The Legendrizations of these classes are the simple singularities of Legendrian curves. We supply a list of Legendrian curve singularities with the property that a tower-simple point is equivalent to the evaluation of the prolongation of exactly one curve on the list. We then list the "fencing" RVT classes. A point is stage-simple if and only if its RVT class *does not* adjoin a fencing class. Here we use 'adjoin' in the topological sense: the set α adjoins the set β if α is contained in the closure of β. Adjacencies can be easily verified through RVT code. We then translate the tower-simple classification to the Legendrian world. We also give examples of complete classifications of points of RVT classes which are stage-simple but not tower-simple, and of RVT classes which are not stage-simple, that is to say, whose classification involves real moduli.

The results of chapter 4 require expressing Monsterization and Legendrization in local coordinates. In chapter 5 we introduce the Kumpera-Ruiz coordinates and charts and develop a Monsterization-Legendrization techniques in these coordinates.

Chapter 6 is devoted to resolving plane curve singularities by prolongation. We prove that any analytic plane curve singularity admits a resolution (or desingularization) by prolonging enough times. In this same chapter we also prove the theorems on the parameterization number and the jet-identification number whose proofs were postponed.

Chapter 7 is devoted to open questions. The principal questions concern unfolding versus prolongation and the correspondence between the RVT class and the Puiseux characteristic of a plane curve.

In Appendix A we illustrate the equivalence of problems (2) and (3) in section 1.6.3. We use Theorem 2.2, which establishes the equivalence, to classify germs of immersed integral curves in an Engel 4-manifold.

In Appendix B we summarize the known results on the RL-contact classification of Legendrian curves which we use in the present work. These results include corollaries of well-known results on the local classification of germs of plane and space curves obtained in [**BG**], [**GH**], [**A2**], as well as the results of [**Z1**], [**Z2**].

Appendix C is devoted to the proof of the part of Theorem 3.2 which asserts that an immersed curve in the Monster is critical if and only if it is singular (= abnormal) if and only if it is locally C^1-rigid.

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CHAPTER 2

Monsterization and Legendrization

2.1. From Monster curves to Legendrian curves

Throughout the paper we will use the following

NOTATION. By $\pi_{i+k,i} : \mathbb{P}^{i+k}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$ we denote the bundle projection.

By Theorem 1.1, for $k, i \geq 1$ these projections are preserved, meaning that any local symmetry maps any fiber of $\pi_{i+k,i}$ to another fiber.

DEFINITION 2.1. The k-step-projection of a curve $\Gamma : (a,b) \to \mathbb{P}^{i+k}\mathbb{R}^2$ is the curve $\Gamma_k : (a,b) \to \mathbb{P}^i\mathbb{R}^2$ such that $\Gamma_k(t) = \pi_{i+k,k}(\Gamma(t))$.

The k-step-projection of an integral curve in $\mathbb{P}^{i+k}\mathbb{R}^2$ is an integral curve in $\mathbb{P}^i\mathbb{R}^2$. In particular, the k-step-projection of an integral curve in $\mathbb{P}^{1+k}\mathbb{R}^2$ is a Legendrian curve in $\mathbb{P}^1\mathbb{R}^2$.

THEOREM 2.2. *Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$ and $\widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ be germs of integral curves. Consider the germs of Legendrian curves $\gamma = \Gamma_k$ and $\widetilde{\gamma} = \widetilde{\Gamma}_k$ and assume that they are not the germs of constant curves. Then Γ and $\widetilde{\Gamma}$ are equivalent if and only if γ and $\widetilde{\gamma}$ are* RL*-contact-equivalent.*

This theorem reduces the classification of integral curve germs in $\mathbb{P}^i\mathbb{R}^2, i \geq 2$ to the classification of Legendrian germs in $\mathbb{P}^1\mathbb{R}^2$. Transferring know results on the classification of Legendrian curves to $\mathbb{P}^i\mathbb{R}^2$ provides a series of new results on classification of integral curves in Goursat manifolds, perhaps the most important of which is the local classification of immersed curves in an Engel 4-manifold as presented in our Appendix A.

The proof of Theorem 2.2 is given in subsection 2.1.3. It is based on Theorem 1.1 and requires the following ingredients:

- the prolongation of integral curves with singularites (subsection 2.1.1);
- the prolongation and projection of symmetries of the Monster (subsection 2.1.2).

2.1.1. Prolonging curves. Prolongation is the inverse to projection. If $\Gamma : (a,b) \to \mathbb{P}^i\mathbb{R}^2$ is an immersed integral curve then its *first prolongation* Γ^1 is the integral curve in $\mathbb{P}^{i+1}\mathbb{R}^2$ defined by differentiating Γ. Thus $\Gamma^1(t) = \big(m(t), \ell(t)\big)$, where $m = \Gamma(t)$ and ℓ is the line spanned by the non-zero vector $\Gamma'(t)$. We note that Γ^1 is the unique integral curve in $\mathbb{P}^{i+1}\mathbb{R}^2$ which projects to Γ in $\mathbb{P}^i\mathbb{R}^2$. Since Γ^1 is immersed and integral, we can prolong it, and continue prolonging to obtain integral curves Γ^k at all higher levels $\mathbb{P}^{i+k}\mathbb{R}^2$.

It is essential to extend the prolongation operation to act on singular curves.

Definition 2.3. Let $\Gamma : (a,b) \to \mathbb{P}^i\mathbb{R}^2$ be a non-constant integral analytic curve. Its prolongation Γ^1 is the nonconstant analytic integral curve $(a,b) \to \mathbb{P}^{i+1}\mathbb{R}^2$ defined at $t_0 \in (a,b)$ as follows.

(1) If $\Gamma'(t_0) \neq 0$ then we define $\Gamma^1(t_0)$ as we did for immersed curve.
(2) If $\Gamma'(t_0) = 0$, then, since Γ is analytic and not constant we have $\Gamma'(t) \neq 0$ for $t \neq t_0$ close to t_0, and so $\Gamma^1(t)$ for these t is defined by step 1. Set $\Gamma^1(t_0) = lim_{t \to t_0}\Gamma^1(t)$.

Theorem 2.4. *The limit in (2) above is well-defined. The corresponding curve $\Gamma^1 : (a,b) \to \mathbb{P}^{i+1}\mathbb{R}^2$ is analytic, integral, and not constant.*

Again, we can iterate prolongation so as to form $\Gamma^k = (\Gamma^{k-1})^1$, a non-constant analytic curve germ in $\mathbb{P}^{i+k}\mathbb{R}^2$.

Proof. Let $\Gamma(t)$ be a non-constant analytic integral curve at level i which is not immersed at $t = t_0$. Fix analytic vector fields v_1, v_2 spanning the rank 2 distribution $\triangle^i$ near the point $\Gamma(t_0)$. Then for any t near t_0 one has $\Gamma'(t) = f_1(t)v_1\big(\Gamma(t)\big) + f_2(t)v_2\big(\Gamma(t)\big)$, for analytic functions germs $f_1(t), f_2(t)$. Because Γ is not immersed at t_0 we have that $f_1(t_0) = f_2(t_0) = 0$. Because Γ is analytic and not constant at least one of the function germs $f_1(t), f_2(t)$ is not the zero germ. Therefore there exist a finite number r and analytic function germs $g_1(t), g_2(t)$ such that

$$f_1(t) = t^r g_1(t),\ \ f_2(t) = t^r g_2(t),\ \ \ \big(g_1(t_0), g_2(t_0)\big) \neq (0,0).$$

At least one of the germs $g_1(t)/g_2(t),\ g_2(t)/g_1(t)$ is a well defined analytic germ. This implies the existence of the limit in the definition of $\Gamma^1(t_0)$ and the analyticity of the curve Γ^1. The integrability of Γ^1 follows from its construction. The curve Γ^1 is not constant since its projection to $\mathbb{P}^i\mathbb{R}^2$ - the curve Γ - is not constant. $\square$

From now on, all curves appearing are assumed analytic.

Proposition 2.5. *Let Γ denote an arbitrary non-constant integral curve at level i*

(i) *Projection and prolongation are inverses:*
$(\Gamma^k)_k = \Gamma$ for any $k \geq 0$.
If $k \leq i$ and Γ_k is not a constant curve then $(\Gamma_k)^k = \Gamma$.

(ii) *Projection and prolongation commute with reparameterization:*
$(\Gamma \circ \phi)^k = \Gamma^k \circ \phi$.
If $k \leq i$ then $(\Gamma \circ \phi)_k = \Gamma_k \circ \phi$.

Proof. Let (a,b) be the interval of definition of the curve Γ. We first do the case $k = 1$. The definition of projection and prolongation implies the validity of (i) and (ii) for $k = 1$ at all point $t \in (a,b)$ such that $\Gamma'(t) \neq 0$. Since Γ is analytic and not constant the points where $\Gamma'(t) = 0$ are discrete. By continuity (see Theorem 2.4) (i) and (ii) also holds for $k = 1$ at all these points, hence for all $t \in (a,b)$. Repeating this argument $k - 1$ times, we see that the validity of (i) and (iii) for $k = 1$ implies their validity for any k under the given constraints on the indices i, k. $\square$

2.1.2. Projections and prolongations of local symmetries. In section 1.3 we defined the prolongations of a local symmetry of the Monster.

The projections Φ_k, k of a local symmetry Φ of $\mathbb{P}^i\mathbb{R}^2$ are defined by Theorem 1.1, but they are only defined when $i - k \geq 1$ Here are the details. Suppose $i \geq 2$. By Theorem 1.1 Φ is the one-step-prolongation of a symmetry $\widetilde{\Phi}$ at level $(i-1)$. It is clear that $\widetilde{\Phi}$ is unique. We denote $\widetilde{\Phi}$ by Φ_1 and say that Φ_1 is the one-step projection of Φ. If $i \geq 3$ then, again by Theorem 1.1 Φ_1 is the one-step-prolongation of a symmetry $\widehat{\Phi}$ at level $(i-2)$; $\widehat{\Phi}$ is unique. We denote $\widehat{\Phi}$ by Φ_2 and say that Φ_2 is the two-step projection of Φ. Iterating we define, for any $k \leq i-1$, the projection Φ_k. It is a local symmetry at level $(i-k)$.

We do not use the projection Φ_i to level 0. This projection cannot in general be defined. (Again, see Theorem 1.1) The reason it cannot be defined is that the general contact transformation does not preserve the fibers of $\mathbb{P}^1\mathbb{R}^2 \to \mathbb{R}^2$ and so has no projection to level 0.

PROPOSITION 2.6. *Let Φ be a local symmetry at level i*

(i) *Projection is inverse to prolongation:*
$(\Phi^k)_k = \Phi$ *for* $k \geq 0$.
If $k \leq i-1$ *then* $(\Phi_k)^k = \Phi$.

(ii) *Projection and prolongation of integral curves commute with symmetries:*
$(\Phi \circ \Gamma)^k = \Phi^k \circ \Gamma^k$ *for* Γ *a non-constant analytic integral curve at level* i .
If $k \leq i-1$ *then* $(\Phi \circ \Gamma)_k = \Phi_k \circ \Gamma_k$.

PROOF. The proof is almost verbatim the same as the proof of Proposition 2.5. □

2.1.3. Proof of Theorem 2.2. Assume that the integral curve germs Γ and $\widetilde{\Gamma}$ are equivalent so that there exist a local symmetry $\Phi : (\mathbb{P}^{1+k}\mathbb{R}^2, p) \to (\mathbb{P}^{1+k}\mathbb{R}^2, \tilde{p})$ and a local diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\widetilde{\Gamma} = \Phi \circ \Gamma \circ \phi$. Let $\Psi = \Phi_k$. Then Ψ is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$. By Proposition 2.6, (ii)

$$\widetilde{\gamma} = \widetilde{\Gamma}_k = (\Phi \circ \Gamma \circ \phi)_k = \Phi_k \circ \Gamma_k \circ \phi = \Psi \circ \gamma \circ \phi,$$

i.e. their Legendrian projections γ and $\widetilde{\gamma}$ are RL-contact-equivalent.

Assume now that the integral curve germs γ and $\widetilde{\gamma}$ are RL-contact-equivalent, i.e. there exists a local contactomorphism Ψ of $\mathbb{P}^1\mathbb{R}^2$ and a local diffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\widetilde{\gamma} = \Psi \circ \gamma \circ \phi$. By Proposition 2.6, (i) $\Gamma = \gamma^k, \widetilde{\Gamma} = \widetilde{\gamma}^k$. Let $\Phi = \Psi^k$. Then Φ is a local symmetry at level $(1+k)$ and by Propositions 2.5 and 2.6

$$\widetilde{\Gamma} = \widetilde{\gamma}^k = (\Psi \circ \gamma \circ \phi)^k = \Psi^k \circ \gamma^k \circ \phi = \Phi \circ \Gamma \circ \phi,$$

i.e. the curve germs Γ and $\widetilde{\Gamma}$ are equivalent.

2.2. From curves to points. Monsterization

DEFINITION 2.7. By $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we mean the space of germs at $t = 0$ of non-constant analytic Legendrian curves $\gamma : (-\epsilon, \epsilon) \to \mathbb{P}^1\mathbb{R}^2$.

Prolongation k times defines a canonical map

$$\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2) \to \mathbb{P}^{1+k}\mathbb{R}^2 : \quad \gamma \to \gamma^k(0).$$

THEOREM 2.8.

(1) *Any point* $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ *is realized by evaluating the* k*-step-prolongation of some germ* $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$*; i.e.* $p = \gamma^k(0)$.

(2) *Let* $\gamma, \widetilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ *be equivalent germs. Then the points* $p = \gamma^k(0), \tilde{p} = \widetilde{\gamma}^k(0)$ *are equivalent in* $\mathbb{P}^{1+k}\mathbb{R}^2$.

Warning: If γ is immersed, then the knowledge of its first k derivatives at 0, i.e of $j^k\gamma(0)$, determines the value of its prolongation $\gamma^k(0)$ at 0. But if γ is singular at $t = 0$ then we may need many more of its derivatives, $j^r\gamma(0)$, for $r >> k$, before we can compute $\gamma^k(0)$. Related to this fact, we will see later on that when p is a "sufficiently singular" point of $\mathbb{P}^{1+k}\mathbb{R}^2$ then to realize it as $p = \gamma^k(0)$ as in Theorem 2.8 we may need to know $j^r\gamma(0)$ for $r >> k$. The determination of the minimum such r for a given p is related to the 'jet-identification " and 'parameterization' numbers to be introduced further on.

2.2.1. Non-singular points. We have several equivalent definitions of a singular point in the Monster. One of them is:

DEFINITION 2.9. A point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called *non-singular* if it is realized by evaluating the k-step-prolongation of an *immersed* Legendrian curve; thus $p = \gamma^k(0)$ for some $\gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ with $\gamma'(0) \neq 0$. Otherwise the point p is called *singular*.

We have the following reformulation of Cartan's theorem from section 1.5.

THEOREM 2.10. *All non-singular points in* $\mathbb{P}^{1+k}\mathbb{R}^2, k \geq 0$ *are equivalent.*

PROOF. Theorem 2.10 is a direct corollary of of the second statement of Theorem 2.8 and the well-known theorem on local contact equivalence of all non-singular equal-dimensional integral submanifolds of a contact manifold. See for example [**AG**] for this last. $\square$

2.2.2. Monsterization. Theorem 2.10 asserts that the set of non-singular points is the "Monsterization" of the class of immersed Legendrian germs. Monsterization can be applied to any singularity class.

DEFINITION 2.11. A subset $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is a *singularity class* if it is closed with respect to RL-contact equivalence of curves in $\mathbb{P}^1\mathbb{R}^2$. A subset $Q \subset \mathbb{P}^i\mathbb{R}^2$ is a singularity class if it is closed with respect to equivalence of points in the Monster tower.

DEFINITION 2.12. Let $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be a singularity class. The k-step-Monsterization of S is the following operation:

$$S \to \mathrm{Monster}^k(S) = \{\gamma^k(0),\ \gamma \in S\} \ \subset \ \mathbb{P}^{1+k}\mathbb{R}^2$$

PROPOSITION 2.13. *Let* $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ *be any singularity class containing no constant curve germs. Then* $\mathrm{Monster}^k(S)$ *is a singularity class in* $\mathbb{P}^{1+k}\mathbb{R}^2$.

PROOF. We must prove that if $p \in \mathrm{Monster}^k(S)$ and $\tilde{p}$ is equivalent to p then $\tilde{p} \in \mathrm{Monster}^k(S)$. Let $p = \gamma^k(0), \gamma \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ and let Φ be a local symmetry at level $(1 + k)$ bringing p to $\tilde{p}$. By Proposition 2.6

$$\tilde{p} = \Phi(p) = \Phi\big(\gamma^k(0)\big) = \big(\Phi \circ \gamma^k\big)(0) = \big((\Phi_k \circ \gamma)^k\big)(0).$$

The projection Φ_k is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$ and S is a singularity class in $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ so that $\Phi_k \circ \gamma \in S$ and $\tilde{p} \in \mathrm{Monster}^k(S)$. $\square$

2.3. Critical and regular curves. Legendrization

Legndrization associates to each point p of the higher Monsters $\mathbb{P}^i\mathbb{R}^2$, $i > 1$ a singularity class $Leg(p) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$.

Our first attempt at constructing the Legendrization was to project all immersed integral curves Γ through p The resulting singularity class , call it $\mathrm{LEG}(p)$, is an invariant of p, but it is too rough since it contains singularities, such as the germ of the constant curve, whose codimension is too high. To get rid of such 'deep' singularities we do not allow all immersed integral curves Γ to be projected, but rather only a certain invariant subset of these curves, the regular curves, which we now define.

Definition 2.14. Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$ be the germ of an integral curve, $k \geq 1$.

1. Γ is called *critical* if its k-step-projection is a constant curve in $\mathbb{P}^1\mathbb{R}^2$.
2. Γ is called *regular* if it is immersed and $\Gamma'(0) \neq 0$ is not tangent to any critical curve.

Remark 2.15. There are integral curve germs which are neither regular nor critical. Indeed any integral curve which is not critical, but is tangent to some critical curve is neither regular nor critical.

Notation. Given a point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ we denote by $\mathrm{Int}^{\mathrm{reg}}(p)$ the set of all regular curve germs $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{1+k}\mathbb{R}^2, p)$.

Definition 2.16. The Legendrization of a point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is the operation

$$p \to \mathrm{Leg}(p) = \{\Gamma_k, \quad \Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)\} \ \subset \ \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$$

(Recall that Γ_k is the k-step-projection of the curve Γ.) The Legendrization of a singularity class $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$ is the union of the Legendrizations of its points, i.e. it is the operation

$$Q \to \mathrm{Leg}(Q) = \{\mathrm{Leg}(p), \quad p \in Q\} \ \subset \ \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2).$$

Theorem 2.17.

1. *$Int^{reg}(p)$ and consequently* $\mathrm{Leg}(p)$ *are never empty.*
2. $\mathrm{Leg}(p)$ *is closed with respect to reparameterization of the curves.*
3. *The Legendrization of a singularity class in $\mathbb{P}^{1+k}\mathbb{R}^2$ is a singularity class in* $\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$.
4. *Monsterization is a left inverse to Legendrization:* $\mathrm{Monster}^k\big(\mathrm{Leg}(Q)\big) = Q$ *for any singularity class $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$.*

Proof. The first statement is a direct corollary of Theorem 3.8 which will be proved in section 3.3.

The second statement is a direct corollary of Proposition 2.5, (ii).

To prove the third statement, let Q be a singularity class at level $1 + k$, let $p \in Q$ and $\gamma \in \mathrm{Leg}(p)$, so that $\gamma = \Gamma_k$, where $\Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)$. Let $\widetilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ be any another Legendrian curve germ equivalent to γ, so that $\widetilde{\gamma} = \Psi \circ \gamma \circ \phi$ for some local difffeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and a local contactomorphism Ψ of $\mathbb{P}^1\mathbb{R}^2$. We must show that $\widetilde{\gamma} \in \mathrm{Leg}(Q)$. By Proposition 2.6 one has

$$\widetilde{\gamma} = \Psi \circ \Gamma_k \circ \phi = (\Psi^k)_k \circ \Gamma_k \circ \phi = (\Psi^k \circ \Gamma \circ \phi)_k = (\Phi \circ \Gamma \circ \phi)_k = \widetilde{\Gamma}_k.$$

Here $\Phi = \Psi^k$ is a local symmetry at level $(1+k)$ and we have set $\widetilde{\Gamma} = \Phi \circ \Gamma \circ \phi$. Now $\tilde{p} = \Phi(p) \in Q$ since Q is a singularity class. It remains to show that $\widetilde{\Gamma}$ is a regular integral curve through $\tilde{p}$ in order to conclude that $\widetilde{\gamma} \in \mathrm{Leg}(Q)$. $\widetilde{\Gamma}$ is integral and immersed. It is regular since Γ is regular and since symmetries of the Monster take critical curves to critical curves.

The last statement follows from Proposition 2.6, (i).

□

REMARK 2.18. If $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ then $\mathrm{Leg}\big(\mathrm{Monster}^k(S)\big) \subseteq S$, but it can happen that $\mathrm{Leg}(\mathrm{Monster}^k(S))$ does not coincide with S, i.e. Monsterization is not right inverse to Legendrization. This "pathology" happens when S contains Legendrian germs whose k-step-prolongations are not regular, i.e. either not immersed or immersed but tangent to a critical direction. However, for 'natural' singularity classes S, equality does hold "stably" and Monsterization is inverse to Legendrization. 'Stably' means that there exists a $k = k(S)$ such that for any $\gamma \in S$ the germ γ^k is immersed and its derivative $(\gamma^k)'(0)$ spans a regular direction. On such classes $\mathrm{Leg}\big(\mathrm{Monster}^k(S)\big) = S$, see chapter 5.

2.4. From points to Legendrian curves

The following theorem includes Theorem 2.8 and gives a necessary condition for the equivalence of points of the Monster tower.

THEOREM 2.19. *Let* $p, \tilde{p} \in \mathbb{P}^{1+k}\mathbb{R}^2$.

(1) *If the Legendrizations of the points* p *and* $\tilde{p}$ *contain equivalent germs* $\gamma \in \mathrm{Leg}(p)$ *and* $\widetilde{\gamma} \in \mathrm{Leg}(\tilde{p})$ *then the points* p *and* $\tilde{p}$ *are equivalent.*
(2) *If the points* p *and* $\tilde{p}$ *are equivalent then their Legendrizations* $\mathrm{Leg}(p)$ *and* $\mathrm{Leg}(\tilde{p})$ *are contactomorphic.*

To say that $\mathrm{Leg}(p)$ and $\mathrm{Leg}(\tilde{p})$ are contactomorphic means that there exists a local contactomorphism $\Psi : (\mathbb{P}^1\mathbb{R}^2, p_1) \to (\mathbb{P}^1\mathbb{R}^2, \tilde{p}_1)$, where p_1 and $\tilde{p}_1$ are the projections of p and $\tilde{p}$ to $\mathbb{P}^1\mathbb{R}^2$, such that for any $\gamma \in \mathrm{Leg}(p), \widetilde{\gamma} \in \mathrm{Leg}(\tilde{p})$ we have that $\Psi \circ \gamma \in \mathrm{Leg}(\tilde{p})$ and $\Psi^{-1} \circ \widetilde{\gamma} \in \mathrm{Leg}(\tilde{p})$.

PROOF. The first statement is a corollary of the second statement of Theorem 2.8. To prove the second statement we follow the same notation and lines as in the proof of Theorem 2.17 immediately above. If the symmetry Φ at level $(1+k)$ sends p to $\tilde{p}$ and if $\Gamma \in \mathrm{Int}^{\mathrm{reg}}(p)$ then $\widetilde{\Gamma} = \Phi \circ \Gamma$ belongs to $\mathrm{Int}^{\mathrm{reg}}(\tilde{p})$. Then $\Psi = \Phi_k$ is a local contactomorphism of $\mathbb{P}^1\mathbb{R}^2$. By Proposition 2.6, (ii) $\widetilde{\Gamma}_k = (\Phi \circ \Gamma)_k = \Psi \circ \Gamma_k$. It follows that $\Psi \circ \Gamma_k \in \mathrm{Leg}(\tilde{p})$. The same argument yields that $\Psi^{-1} \circ \widetilde{\Gamma}_k \in \mathrm{Leg}(p)$ for any curve $\widetilde{\Gamma} \in \mathrm{Int}^{\mathrm{reg}}(\tilde{p})$. □

The following statement is also a direct corollary of Theorem 2.8.

THEOREM 2.20. *Let $Q \subset \mathbb{P}^{1+k}\mathbb{R}^2$ be a singularity class whose Legendrization* $\mathrm{Leg}(Q) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ *consists of a finite number s of orbits with respect to the* RL-*contact equivalence, these orbits being represented by the Legendrian curve germs $\gamma_1, \ldots, \gamma_s$. Then any point of the class Q is equivalent to one of the points $\gamma_1^k(0), \ldots, \gamma_s^k(0)$ and consequently the class Q consists of $\tilde{s} \leq s$ orbits. In particular, if $s = 1$ then all points of the class Q are equivalent.*

2.4.1. On the Implications and Shortfalls of Theorems 2.19 and 2.20. In Chapter 3 we decompose the Monster into natural singularity classes, the RVT classes. For some of these classes Theorem 2.20 by itself immediately implies significant classification results, see Examples 3.27, 3.28, 3.29. But Theorems 2.19 and 2.20 by themselves are rather far from a complete reduction of the problem of classification of points in the Monster to the problem of RL-contact classification of Legendrian curve germs. Such a reduction is not even possible in principle. What goes wrong is that the number $\tilde{s}$ of Theorem 2.20 might be much smaller than the number s; $\tilde{s} = 1$ with s arbitrarily big is even possible. To abuse language, (that of the term 'simple'), $\tilde{s} = 1$ with $s = \infty$ is even possible, corresponding to a single orbit Q whose Legendrization $\mathrm{Leg}(Q)$ contains a continuua of orbits (has moduli). The problem is that the data needed to describe a curve germ (as opposed to a singularity class) is by its very nature, infinite, involving the full power series expansion, while the data needed to describe a point of the monster at any fixed level is finite. To make both data finite, we use finite jets of Legendrian curves. *In what follows we will show that the problem of classification of points in the Monster can be completely reduced to the problem of RL-contact classification of Legendrian curve jets.* This reduction, given below, is based on Theorem 2.19 and the analysis of the structure of the set $\mathrm{Leg}(p)$ for a fixed point p in the Monster.

2.5. From points to Legendrian curve jets

NOTATIONS AND DEFINITIONS.

1. Given a germ $\gamma = \gamma(t) \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we denote by $j^r\gamma$ the r-jet of γ at the point $t = 0$.
2. Given a set $S \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ we denote by j^rS the set $\{j^r\gamma,\ \gamma \in S\}$.
3. A reparameterization of an r-jet $\xi \in j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is the r-jet $j^r(\gamma(\phi(t)))$, where $\gamma(t)$ is a Legendrian curve germ representing the jet ξ and $\phi : (\mathbb{R}, 00 \to (\mathbb{R}, 0)$ is a local diffeomorphism.
4. Two r-jets in $j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ are RL-contact equivalent if the following holds for some (and then any) Legendrian curve germs $\gamma, \tilde{\gamma}$ representing these jets: $\tilde{\gamma}$ is RL-contact equivalent to a curve whose r-jet coincides with $j^r\gamma$.

To check that definition (3) is a good definition, i.e. that the choice of representative γ there is irrelevant, and to check the validity of the claim "for some implies for any " within definition (4), it suffices to note that any local diffeomorphism Φ and any reparameterization ϕ "respect" the filtration by jets, i.e. $j^r(\Phi \circ \gamma \circ \phi) = j^r(\Phi \circ \tilde{\gamma} \circ \phi)$ if $j^r\gamma = j^r\tilde{\gamma}$.

Fix a point p in the Monster. What is the structure of the set $\mathrm{Leg}(p) \subset \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$? We know (Theorem 2.17, (2)) that it is closed with respect to the reparameterization of the curves. What else? One can hope that along with any

Legendrian curve germ that $\mathrm{Leg}(p)$ also contains all Legendrian curve germs sharing the same r-jet, for r is sufficiently big. Assume that this is so for all $r \geq r_0$, but not for $r < r_0$. What can be said about the set $j^{r_0}\,\mathrm{Leg}(p)$? The best one could hope for is that it consists of a single r_0-jet up to reparameterization! In this case p can be identified with this r_0-jet.

Definition 2.21 (the jet-identification number). Fix a point p in the Monster. Assume that there exists an integer r satisfying the following two conditions:

(1) If $\gamma \in \mathrm{Leg}(p)$ then $\tilde{\gamma} \in \mathrm{Leg}(p)$ for any germ $\tilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ such that $j^r\tilde{\gamma} = j^r\gamma$.

(2) The set $j^r\,\mathrm{Leg}(p)$ consists of a single r-jet up to reparameterization.

In this case we will say that

- the point p can be identified with a single r-jet (the r-jet in item (2))
- r is the *jet-identification number* of the point p.

We will see later that at any level $i \geq 3$ that there are points p for which the jet-identification number is not defined, and other points for which the number is defined. Before attacking the question of distinguishing these points we fix several properties of points for which the number is defined.

Proposition 2.22. *If the jet-identification number of a point is defined, then that number is unique.*

Proof. Assume, by way of contradiction, that $r_1 < r_2$ are jet-identification numbers for the same point p. Let $\gamma \in \mathrm{Leg}(p)$. Since $r_1 < r_2$ we can find another curve $\tilde{\gamma}$ such that $j^{r_1}\tilde{\gamma} = j^{r_1}\gamma$ but with $j^{r_2}\tilde{\gamma} \neq j^{r_2}(\gamma \circ \phi)$ for any reparameterization ϕ. Indeed, for any positive integer r, the space of Legendrian $(r+1)$-jets having a fixed r-jet is 2-dimensional while the space of reparameterizations of an $(r+1)$-jet which do not change its r-jet is 1-dimensional. By item (1) in Definition 2.21 for $r = r_1$ one has $\tilde{\gamma} \in \mathrm{Leg}(p)$. This contradicts item (2) in Definition 2.21 for $r = r_2$. □

The next property is not obvious. It follows from the second statement of Theorem 2.19 which has not been used so far.

Proposition 2.23. *Assume that a point p has a jet-identification number r. Then any point $\tilde{p}$ in the same level which is equivalent to p has the same jet-identification number r.*

Proof. We have to check (1) and (2) in Definition 2.21. By Theorem 2.19, (2) the equivalence of p and $\tilde{p}$ implies the existence of a local contactomorphism Φ such that $\Phi \circ \gamma \in \mathrm{Leg}(p)$. Fix such a contactomorphism Φ.

Proof of (1). Let $\gamma \in \mathrm{Leg}(\tilde{p})$. Let $\tilde{\gamma} \in \mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ and $j^r\tilde{\gamma} = j^r\gamma$. We have to show that $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. Since $j^r\tilde{\gamma} = j^r\gamma$ then $j^r(\Phi \circ \tilde{\gamma}) = j^r(\Phi \circ \gamma)$. The number r is the jet-identification number for p, the curve $\Phi \circ \tilde{\gamma}$ is Legendrian, therefore $\Phi \circ \tilde{\gamma} \in \mathrm{Leg}(p)$. Applying again Theorem 2.19, (2) we obtain $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$.

Proof of (2). Let $\gamma, \tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. We have to show that $j^r\tilde{\gamma}$ is a reparameterization of $j^r\gamma$. Then $\Phi \circ \gamma,\ \Phi \circ \tilde{\gamma} \in \mathrm{Leg}(p)$. Since r is the jet-identification number for p then the r-jets $j^r(\Phi \circ \gamma),\ j^r(\Phi \circ \tilde{\gamma})$ are the same up to reparameterization. Consequently the r-jets $j^r\gamma,\ j^r\tilde{\gamma}$ are also the same up to reparameterization. □

Combining the *two* statements of Theorem 2.19 we obtain the following crucial property: *the equivalence problem for points of the Monster which have jet-identification numbers reduces to the RL-contact equivalence problem for Legendrian curve germs.* Namely, the following theorem holds.

THEOREM 2.24. *Let p and $\tilde{p}$ be points in $\mathbb{P}^i\mathbb{R}^2$ whose jet-identification numbers r and $\tilde{r}$ are defined. Take* **any** *germ $\gamma \in \mathrm{Leg}(p)$ and* **any** *germ $\tilde{\gamma} \in \mathrm{Leg}(\tilde{p})$. The points p and $\tilde{p}$ are equivalent if and only if $r = \tilde{r}$ and the r-jets $j^r\gamma, j^r\tilde{\gamma}$ are RL-contact equivalent.*

PROOF. Assume that $r = \tilde{r}$ and the r-jets $j^r\gamma, j^r\tilde{\gamma}$ are RL-contact equivalent. This means that $\tilde{\gamma}$ is RL-equivalent to a Legendrian curve germ μ such that $j^r\mu = j^r\gamma$. By (1) in Definition 2.21 one has $\mu \in \mathrm{Leg}(p)$. By Theorem 2.19, (1) the points p and $\tilde{p}$ are equivalent.

Assume now that the points p and $\tilde{p}$ are equivalent. By Proposition 2.23 one has $r = \tilde{r}$. By Theorem 2.19, (2) the germ $\tilde{\gamma}$ is RL-contact equivalent to some germ $\mu \in \mathrm{Leg}(p)$. The equivalence of curve germs implies the equivalence of their r-jets, therefore the r-jets $j^r\tilde{\gamma}$ and $j^r\mu$ are RL-contact equivalent. By (2) in Definition 2.21 the r-jets $j^r\mu$ and $j^r\gamma$ are the same up to reparameterization. Therefore the r-jets $j^r\tilde{\gamma}$ and $j^r\gamma$ are RL-contact equivalent. $\square$

In view of Theorem 2.24 the principal questions are as follows:

- For which points of the Monster are the jet-identification numbers defined?
- How do we compute the jet-identification number for these points?

The answer to the first question is given in the section 2.6, Theorem 2.29. It requires the decomposition of the Monster onto regular and critical points. The answer to the second question is given in section 2.8. It requires the degree of regularity of a regular point (section 2.7) and the parameterization number of a point (section 2.8).

2.6. Regular and critical directions and points.

We use critical curves to define critical and regular directions. Recall (Definition 2.14) that a critical curve in the kth level of the Monster is a curve whose projection to the first level is a constant curve.

DEFINITION 2.25. Let m be a point in the kth level.

1. A line $\ell \subset \Delta^k(m)$ is called critical if it is tangent to an immersed critical curve, otherwise the line is called regular.

2. A point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called regular or critical depending on whether or not the line $\ell \subset \Delta^k(m)$ is regular or critical.

REMARK 2.26. All lines in $\Delta^0 = T\mathbb{R}^2$ are regular. All lines in Δ^1 are regular. In fact, a curve in $\mathbb{P}^1\mathbb{R}^2$ is critical if and only if this curve is constant.

This remark implies the following claim, a claim corresponding to the classical Darboux and Engel theorems:

CLAIM 2.27 (follows from the definitions). All points in the first and the second level of the Monster, $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{P}^2\mathbb{R}^2$, are regular.

In Chapter 3 we will show that critical points exist starting from the third level, and within critical points we will distinguish vertical and tangency points, which allows us to decompose the Monster onto RVT classes.

REMARK 2.28. Directions and points are partitioned into two sets "critical" and "regular": every point and every direction is either critical or it is regular. For integral curves this strict partitioning is not valid: there are integral curves which are neither critical nor regular. See Remark 2.15.

THEOREM 2.29. *A point in the Monster has a jet-identification number, i.e. can be identified with a single r-jet of a Legendrian curve for some r, if and only if this point is regular.*

This theorem is proved in Chapter 6. To evaluate the jet-identification number of a regular point we need two ingredients. The first one is the degree of regularity of such point defined in the next section. It also allows to characterize the singular points in the Monster.

2.7. The degree of regularity. Critical and singular points

The reader should not confuse the partitioning of the monster into critical or regular points with its partitioning into singular or non-singular points. The Definition 2.9 of a non-singular point (= Cartan point) looks quite different from the definition of a regular point. Theorem 2.31 here asserts that

Any non-singular point is regular. At level $i \geq 3$ a regular point might be either non-singular or singular.

Given a regular point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$, $k \geq 1$, consider its projections

$$p_0 = p,\ p_1 = \pi_{k+1,k}(p),\ p_2 = \pi_{k+1,k-1}(p),\ \cdots,\ p_{k-1} = \pi_{k+1,2}(p). \tag{2.1}$$

DEFINITION 2.30. Let $p \in \mathbb{P}^{1+k}\mathbb{R}^2$, $k \geq 1$, be a regular point. The degree of regularity of p is the number q such that the projections $p_0, \ldots, p_{q-1}$ are regular points and p_q is not regular. If all points of sequence (2.1) are regular then the degree of regularity of p is k, which is the number of points in this sequence.

Any point in the second level is regular and has degree of regularity 1. The maximal possible value of the degree of regularity of a regular point at the $(k+1)$st level is k.

THEOREM 2.31. *A point $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is non-singular if and only if it is regular and has maximal possible degree of regularity k.*

This theorem provides one more definition of non-singular point. We will see that it is equivalent to Definition 2.9. This theorem is part of Theorem 3.18 in section 3.8 which one further equivalent definition.

2.8. The parameterization number

The jet-identification number in Theorem 2.29 depends on the degree of regularity and one more characteristic number, the *the parameterization number*. This number is defined below as the order of a good parameterization of a curve γ in the Legendrization of p.

DEFINITION 2.32. Let $c : (\mathbb{R}, 0) \to \mathbb{M}^n$ be the germ at $t = 0 \in \mathbb{R}$ of a parameterized analytic curve in an n-manifold M^n.

1. The curve germ c is called "bad" if it can be factored as $c(t) = \mu\big(\phi(t)\big)$ with $\mu : (\mathbb{R}, 0) \to M^n$ and $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ analytic germs such that $d\phi/dt(0) = 0$. If c is not bad we call it "good".

2. Assume that the germ c is good. The order of a good parameterization of c is the minimal integer d satisfying the following condition: any curve germ $\tilde{c} : (\mathbb{R}, 0) \to \mathbb{M}^n$ such that $j^d\tilde{c} = j^d c$ is also good.

EXAMPLE 2.33. The curve (t^4, t^6) is a bad parameterization of the plane curve $x^3 = y^2$ while (t^2, t^3) is a good parameterization of the same curve. The order of the good parameterization is 3.

It is well-known that any good analytic parameterized curve germ has a finite order of good parameterization. See for example [**W**].

THEOREM 2.34. *Let p be any point at any level of the Monster.*

(1) *Any Legendrian curve germ $\gamma \in \mathrm{Leg}(p)$ is good.*
(2) *The order of good parameterization is the same for all $\gamma \in \mathrm{Leg}(p)$.*

DEFINITION 2.35 (parameterization number). Let p be any point at any level. The parameterization number of p is the order of good parameterization of some (and hence any) curve germ $\gamma \in \mathrm{Leg}(p)$.

Note that the parameterization number is defined for *all* points of the Monster, whereas the jet-identification number is defined for regular points only.

THEOREM 2.36. *Let p be a regular point. The jet-identification number of p is equal to $d + q - 1$, where d is the parameterization number of p and q is the degree of regularity of p.*

Theorems 2.34 is a part of the stronger Theorem 3.42 (section 3.12). Theorem 2.36 is proved in Chapter 6.

The simplest particular case of Theorem 2.36 is the case of a non-singular point at the level $(k+1)$. Our Definition 2.9 implies that the Legendrization of a Cartan point consists of immersed curves and consequently the characteristic number d is equal to 1. The degree of regularity is $q = k$, see Theorem 2.31. Therefore

• *the jet-identification number of a non-singular point at the level $(k + 1)$ is equal to k.*

2.9. Things to Come

Theorems 2.29 and 2.36 completely reduce the equivalence problem for regular points in the Monster to an RL-contact equivalence problem for Legendrian curves. In the next chapter, after decomposing the monster onto natural RVT classes, we will show the following:

• the equivalence problem for *critical* points at any level of the Monster straightforwardly reduces to the equivalence problem for regular points. Consequently the equivalence problem for *any* points of the Monster reduces to the RL-contact equivalence problem for finite jets of Legendrian curves.

• The classification problem (constructing final normal forms) for any RVT class straightforwardly reduces to that for regular RVT classes – classes consisting of regular points. The order of regularity, the parameterization number, and consequently the jet-identification number of all points of any regular RVT class are the same. This allows to reduce the classification problem for any RVT class to the RL-contact classification problem for certain classes of finite jets of Legendrian curves.

• It follows (see section 1.5) that both the equivalence and classification problems for Goursat flags reduce to these same problems for finite jets of Legendrian curves.

CHAPTER 3

Critical curves. RVT classes

3.1. Critical, vertical, singular, and rigid curves

Recall (section 2.3) a curve in $\mathbb{P}^{1+k}\mathbb{R}^2$ is called critical if it is an integral curve whose k-step-projection is a constant curve in $\mathbb{P}^1\mathbb{R}^2$. Critical curves play a central role in our subsequent constructions. The simplest immersed critical curves are the fibers of the circle bundle projection $\mathbb{P}^{i+1}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2, i \geq 1$. Following usual bundle terminology, we call these "vertical curves".

Definition 3.1. A curve in $\mathbb{P}^{1+k}\mathbb{R}^2, k \geq 1$ is called "vertical" if it is an integral curve whose 1-step projection is constant.

In other words, a curve $\gamma : (a, b) \to \mathbb{P}^{1+k}\mathbb{R}^2$ is vertical if it has the form $\gamma(t) = \big(m(t), \ell(t)\big)$ with $m(t) \equiv const$. For each $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ there is an immersed vertical curve passing through p, and unique up to parameterization. It is the fiber through p.

Theorem 3.2. *Let γ be an immersed curve in $\mathbb{P}^i\mathbb{R}^2, i \geq 2$. The following properties are equivalent:*

(i) *γ is critical*
(ii) *γ is either vertical or the s-step prolongation of a vertical curve in $\mathbb{P}^{i-s}\mathbb{R}^2$*
(iii) *γ is singular (= abnormal) in the sense of sub-Riemannian geometry*
(iv) *γ is locally C^1-rigid*

The equivalence of (i) and (ii) is proved in section 3.2 as part of the proof of Proposition 3.4) which analyzes prolongations of critical curves. The equivalence of (i),(iii) and (iv) is proved in Appendix C where we also recall the definition of singular (=abnormal) and C^1-rigid curves.

3.2. Vertical directions and points. Prolongations of vertical curves

Recall (section 2.6) that we define a line $\ell \subset \Delta^k(m)$ to be critical if it is tangent to an immersed critical curve, and that a point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called critical if the line ℓ is critical. Within the class of critical curves we have the vertical curves. It follows that within the set critical directions or points we have subset of vertical directions or points.

Definition 3.3. Let $m \in \mathbb{P}^k\mathbb{R}^2$.

(1) A line in $\Delta^k(m)$ is called vertical if it is tangent to the immersed vertical curve through m. Equivalently, it is the line $ker(d\pi_{k,k-1})(p)$.
(2) A point $p = (m, \ell) \in \mathbb{P}^{1+k}\mathbb{R}^2$ is called vertical if the line ℓ is vertical.

In this section we analyze prolongations of immersed vertical curves. It is clear that any prolongation of such curves consists entirely of critical points and that the one-step prolongation consist entirely of vertical points. We will show that the higher prolongations contain no vertical points.

NOTATIONS. Let $m \in \mathbb{P}^i\mathbb{R}^2$.

a) Write V_m for the germ at the point $t = 0$ of an immersed vertical curve at level i passing throught m at time $t = 0$. (This germ is unique up to reparameterization.)
b) We will use the notation p_k for the k-step projection of the point p: $p_k = \pi_{i,i-k}(p)$. Here $k \geq 0$ and $p_0 = p$. (Note that this notation is similar to the notation of the projections of curves).

The equivalence of (i) and (ii) in Theorem 3.2 is a part of the following proposition.

PROPOSITION 3.4.

(i) *Let $m \in \mathbb{P}^i\mathbb{R}^2, i \geq 1$ and let $k \geq 1$. The prolonged vertical curves $(V_m)^k$, $k \geq 1$ are immersed critical curves. $(V_m)^1$ consists entirely of vertical points. The higher prolongations $(V_m)^j, j > 1$ contain no vertical points.*
(ii) *Let $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^i\mathbb{R}^2, m)$ be the germ of an immersed critical non-vertical curve. Then Γ is a reparameterization of the curve germ $(V_{m_k})^k$ for some $k \geq 1$.*

PROOF. (i) The fact that the curve $(V_m)^k$ is an immersed critical curve follows immediately from the definitions of critical and vertical curves. For each t, the vector $V_m'(t)$ spans the vertical line $\ell_{vert}(t)$. Consequently the first prolongation of V_m, namely $V_m^1(t) = \big(V_m(t), \ell_{vert}(t)\big)$, consists entirely of vertical points. This prolongation is immersed since the curve $t \to V_m(t)$ is immersed. The higher prolongations V_m^j, $j > 1$ of V_m have the form $V_m^j(t) = \big(V_m^{j-1}(t), r(t)\big)$ where $r(t)$ is spanned by $(V_m^{j-1})'(t)$. To check that V_m^j is nowhere vertical we must check that $r(t)$ is not vertical. This non-verticality follows from the relations

$$d\pi_{i+j-1,i+1}\big((V_m^{j-1})'(t)\big) = (V_m^1)'(t) \neq 0, \;\; d\pi_{i+j-1,i+1} = d\pi_{i+j-1,i+j-2} \circ d\pi_{i+j-2,i+1}.$$

(ii) The projection Γ_{i-1} is a constant curve in $\mathbb{P}^1\mathbb{R}^2$. Let k be the minimal number such that the projection Γ_{k+1} is a constant curve. Then $\Psi = \Gamma_k$ is a vertical non-constant curve and $\Gamma = \Psi^k$. Since $\Gamma(0) = m$ we have that $\Psi(0) = m_k$. To prove (ii) we have to show that Ψ is a reparameterization of the curve germ V_{m_k}. The curve Ψ is vertical, therefore $\Psi(t) = V_{m_k}(\phi(t))$ for some map $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$. It only remains to prove that $\phi'(0) \neq 0$. The curve Ψ is analytic and not constant. Therefore $\Psi'(t) \neq 0$ for any $t \neq 0$ sufficiently close to 0. It follows that $\Gamma(t) = \Psi^k(t) = V_{m_k}^k(\phi(t))$ for any $t \neq 0$. Differentiating this relation and taking the limit as $t \to 0$ we obtain $\Gamma'(0) = (V_{m_k}^k)'(0) \cdot \phi'(0)$. Since Γ is, by assumption, immersed, we have that $\Gamma'(0) \neq 0$ and so $\phi'(0) \neq 0$. $\square$

COROLLARY 3.5. *The k-fold-prolongation Γ^k of any immersed critical curve Γ is an immersed critical curve. If $k \geq 2$ then the curve Γ^k contains no vertical points.*

This statement is a logical corollary of Proposition 3.4. Another corollary is:

LEMMA 3.6. *Let $\Gamma, \widetilde{\Gamma} : (\mathbb{R}, 0) \to (\mathbb{P}^i\mathbb{R}^2, p)$ be germs of immersed critical curves passing through the same point p. If these curves are not vertical then they are reparameterizations of each other.*

PROOF. By Proposition 3.4 $\Gamma = (V_{p_r})^r$ and $\widetilde{\Gamma} = (V_{p_s})^s$ for some $r, s \geq 1$, up to reparametrization. We must show that $r = s$. Suppose not, so that, without loss of generality, $r < s$. Consider the curves $\Psi = V_{p_r}$ and $\Upsilon = (V_{p_s})^{s-r}$. They live at the same level $(i - r)$ and their r-fold prolongations pass through p: $\Psi^r(0) = \Gamma(0) = p$ and $\Upsilon^r(0) = \widetilde{\Gamma}(0) = p$. Since $1 \leq r$ we must also have that $\Psi^1(0) = \Upsilon^1(0)$. But, by Proposition 3.4 the point $\Psi^1(0)$ is vertical while $\Upsilon^1(0) = (V_{p_s})^{s-r+1}(0)$ is not vertical, since $s - r + 1 \geq 2$. This shows that $\Psi^1(0) = \Upsilon^1(0)$ is impossible, and this contradiction implies that we must have had $r = s$. □

3.3. Regular, vertical, and tangency points

In the previous section we defined the set of vertical directions and vertical points as subsets of the set of critical directions and critical points.

DEFINITION 3.7. A critical line in Δ^k which is not the vertical line will be called a tangency line. A critical point in the Monster which is not a vertical point will be called a tangency point.

The terminology "tangency" is explained in section 3.7 in Theorem 3.17.

Thus each point of the Monster falls into one of three disjoint classes: regular (R), vertical (V) or tangency (T). The set of vertical and tangency points together form the set of critical points. Recall (section 2.6) that there are no critical directions in Δ^0 and Δ^1 and consequently all points of the first two levels of the Monster, $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{P}^2\mathbb{R}^2$ are regular.

The following theorem relates the classes at one level to that of the next.

THEOREM 3.8 (See Figures). *Let $m \in \mathbb{P}^i\mathbb{R}^2, i \geq 2$. Let $\pi^{-1}_{i+1,i}(m) \subset \mathbb{P}^{i+1}\mathbb{R}^2$ be the fiber above m, i.e. the set $\{(m, \ell) \in \mathbb{P}^{i+1}\mathbb{R}^2\}$, where m is fixed and ℓ varies over $\Delta^i(m)$.*

(i) *If m is a regular point then there is exactly one critical point in the fiber over m, namely the vertical point of that fiber.*
(ii) *If m is a vertical or tangency point then there are exactly two critical points in the fiber over m: the vertical point, and a tangency point.*

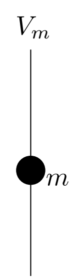

Figure 1. If m is regular, there is only critical line in $\Delta^i(m)$, the vertical line.

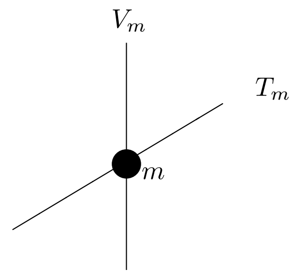

Figure 2. If m is critical there are exactly two critical lines in $\Delta^i(m)$.

Proof. To prove the first statement we must show that if the fiber over $m \in \mathbb{P}^i\mathbb{R}^2$ contains a tangency point then m a critical point. Write $m = (q, \ell)$, where q is a point at level $(i-1)$ and ℓ is a line in $\Delta^{i-1}(q)$. To say that the fiber over m contains a tangency point means that there exists an immersed critical curve Γ in $\mathbb{P}^i\mathbb{R}^2$ passing through m at $t = 0$ and which is not tangent to the vertical line in Δ^i. Its one-step-projection Γ_1 is an immersed critical curve passing through $q \in \mathbb{P}^{i-1}\mathbb{R}^2$ and tangent to the line ℓ at $t = 0$. It is immersed since γ is not tangent to the vertical line and it is critical since γ is critical. Then, by definition, the line ℓ is critical, and so the point $m = (q, \ell)$ is critical.

Now we prove the second statement. Let $m \in \mathbb{P}^i\mathbb{R}^2$ be a critical point. The uniqueness of a tangency point in the fiber over m follows from Lemma 3.6. We prove existence. Write $m = (q, \ell)$ with $q \in \mathbb{P}^{i-1}\mathbb{R}^2$. Since m is a critical point there exists an immersed critical curve germ $\Gamma : (\mathbb{R}, 0) \to (\mathbb{P}^{i-1}\mathbb{R}^2, q)$ with $\ell = span(\Gamma'(0))$. Then the prolongation of Γ passes through m: $m = \Gamma^1(0)$. Consider the point $p = (m, \ell_1)$ over m where $\ell_1 = span(\Gamma^1)'(0))$. The curve Γ^1 is critical, therefore the point p is critical. On the other hand $p = \Gamma^2(0)$ and by Corollary 3.5 the point p is not vertical. Consequently p is a tangency point. □

3.4. RVT classes

Definition and Notation. Associate to a point $p \in \mathbb{P}^i\mathbb{R}^2$ the $(i-2)$-tuple $(\alpha) = (\alpha_1, ..., \alpha_{i-2})$, where each $\alpha_j = R$, V, or T according to the type, regular, vertical, or tangency, of the point $\pi_{i,j+2}(p)$. We call (α) the RVT-code of p. If $i < 3$ the RVT-code is empty. Letters V and T will be called critical letters.

Example 3.9. Let $p \in \mathbb{P}^5\mathbb{R}^2$ be a regular point. Assume that the projection $\pi_{5,3}(p) \in \mathbb{P}^3\mathbb{R}^2$ is a vertical point. Then the RVT-code of the point p is either VRR, or VVR, or VTR depending on the type of the point $\pi_{5,4}(p) \in \mathbb{P}^4\mathbb{R}^2$.

Theorem 3.8 implies that a tuple $(\alpha) = (\alpha_1, ..., \alpha_{i-2})$ consisting of letters R, V, T is the RVT-code of some point $p \in \mathbb{P}^i\mathbb{R}^2$, if and only if it satisfies both of the following two conditions:

(1) (α) does not start with T: $\alpha_1 \neq T$;
(2) T does not follow R: if $\alpha_j = R$ then $\alpha_{j+1} \neq T$.

Definition 3.10. Let $(\alpha) = (\alpha_1, ..., \alpha_{i-2})$ be a realizable RVT-code, i.e. a tuple of letters R, V, T satisfying conditions (1) and (2). The set of points $p \in \mathbb{P}^i\mathbb{R}^2, i \geq 3$ with the code (α) will be called the RVT class and will be denoted by the same tuple (α), or, when confusion might arise, class (α).

THEOREM 3.11.

(1) *Each* RVT *class is a singularity class.*
(2) *Each* RVT *class* $(\alpha) = (\alpha_1, ...\alpha_{i-2})$ *is a non-singular analytic submanifold of* $\mathbb{P}^i\mathbb{R}^2$ *whose codimension is equal to the number of critical letters occurring in* (α).
(3) *The realizable class* $(\alpha) \subset \mathbb{P}^i\mathbb{R}^2$ *adjoins the realizable class* $(\beta) \subset \mathbb{P}^i\mathbb{R}^2$, *that is, the class* (α) *is in the closure of the class* (β), *if and only if the tuple* (β) *can be obtained from the tuple* (α) *by replacing some critical letters (V's or T's) by R's. Moreover, the closure of* β *is equal to the union of* β *together with the RVT classes* α *which adjoin* β.
(4) *The number of* RVT *classes for* $\mathbb{P}^i\mathbb{R}^2, i \geq 3$ *is the* $(2i-3)$*-d Fibonacci number.*

Theorem 3.11 is proved in section 3.6. The proof is based on the results in sections 3.2 and 3.3 and the explicit construction of tangency points given in the next section.

3.5. Constructing tangency points

DEFINITION 3.12. An RVT class is called critical if it ends with a V or T. It is called regular if it ends in R.

Thus, a critical class consists of critical points and a regular class consists of regular points. Fix a critical RVT class (α). By Theorem 3.8 there is a well-defined map

$$T : \text{class } (\alpha) \to \mathbb{P}^{s+3}\mathbb{R}^2, m \to (m, \ell) = \text{the tangency point in the fiber over } m \tag{3.1}$$

(Theorem 3.8 asserts that the tangency point in the fiber over a critical point $m \in (\alpha)$ is unique.) In this subsection we construct the map T more explicitly.

The first statement of Theorem 3.8 implies that at least one of the letters α_i in (α) is a V. Therefore the following number is well-defined.

NOTATION For a critical class $(\alpha) = (\alpha_1 \dots \alpha_s)$ denote by $r = r((\alpha))$ the minimal $r \in \{1, ..., s\}$ such that $\alpha_{s-r+1} = V$.

Note that $r = 1$ if $\alpha_s = V$ and $r \geq 2$ if $\alpha_s = T$.

EXAMPLE 3.13. Take $s = 5$ and let $*$ stand for any letter R, V or T. Then

$$r(****V) = 1,\ r(***VT) = 2,\ r(**VTT) = 3,$$
$$r(*VTTT) = 4,\ \text{while } r(VTTTT) = 5.$$

LEMMA 3.14. *The map (3.1) is* $T(m) = V_{m_r}^{r+1}(0),\ \ r = r((\alpha))$.

EXAMPLE 3.15. The class $RVVTT$ in the 7th level of the Monster has $s = 5$ and $r = 4$. For this class the map (3.1) is the composition of the projection $m \to m_4 = \pi_{7,3}(m)$ and the map $q \to V^4(q), q \in \mathbb{P}^3\mathbb{R}^2$.

PROOF OF LEMMA 3.14. Let $p = (m, \ell)$ be a tangency point in the fiber over $m \in (\alpha)$. Then there exists an immersed critical curve Γ at level $(s+2)$ passing through m at $t = 0$ in direction ℓ. The point p is not vertical, therefore the curve Γ is not vertical. By Proposition 3.4 $\Gamma = (V_{m_k})^k$ for some $k \geq 1$. It follows that

$$p = \Gamma^1(0) = (V_{m_k})^{k+1}(0).$$

It remains to prove that $k = r$, where $r = r((\alpha))$. By Proposition 3.4 the point $m_{k-1} = (V_{m_k})^1(0)$ is vertical. By the same Proposition, part (ii) the points $(V_{m_k})^j(0)$ for $j > 1$ are not vertical. But $V^j_{m_k}(0) = m_{k-j}$. The definitions of the RVT code and the number r imply that $k = r$.

3.6. Proof of Theorem 3.11

The first statement of Theorem 3.11 is a corollary of Proposition 2.6, (ii) which implies that symmetries at level $i \geq 2$ maps critical curves to critical curves and fibers to fibers.

The second statement of Theorem 3.11 is proved by induction on the number of letters in the RVT-code. Fix an RVT class $(\alpha) = (\alpha_1, ..., \alpha_{s+1}) \subset \mathbb{P}^{s+3}\mathbb{R}^2$ and assume that we have already proved the second statement of Theorem 3.11 for the class $(\widetilde{\alpha}) = (\alpha_1, ..., \alpha_s) \subset \mathbb{P}^{s+2}\mathbb{R}^2$. Consider the map

$$V^j_i : \ \mathbb{P}^i\mathbb{R}^2 \to \mathbb{P}^{i+j}\mathbb{R}^2, \ m \to V^j_m(0),$$

see the notation in the beginning of section 3.2. By Theorem 3.8 and Lemma 3.14 one has the following relations:

(a) if $\alpha_{s+1} = V$ then $(\alpha) = V^1_{s+2}((\widetilde{\alpha}))$;
(b) if $\alpha_{s+1} = T$ (and consequently $\alpha_s \in \{V, T\}$) then

$$(\alpha) = (V^{s-r+1}_{r+2} \circ \pi_{s+2,r+2})(\widetilde{\alpha}), \ \text{where } r = r((\widetilde{\alpha}))$$

(see the definition of $r((\alpha))$ in section 3.5).
(c) if $\alpha_{s+1} = R$ then $(\alpha) = \pi^{-1}_{s+3,s+2}((\widetilde{\alpha})) - (\widetilde{\alpha}, V) - (\widetilde{\alpha}, T)$.

It is easy to check that for fixed i, j the map V^j_i is an *analytic* section of the torus fibration $\mathbb{P}^{i+j}\mathbb{R}^2 \to \mathbb{P}^i\mathbb{R}^2$. Therefore relations (a)-(c) imply the second statement of Theorem 3.11 for the class (α).

The third statement of Theorem 3.11 is also proved by induction using Theorem 3.8. It suffices to prove that for any $\alpha_j, \beta_j \in \{R, V, T\}$ none of the classes $(\alpha_1, ..., \alpha_p, T), (\beta_1, ..., \beta_p, V)$ adjoins the other one. This follows from the continuity of the map $m \to T(m)$ in (3.1), by Lemma 3.14.

The last statement of Theorem 3.11 is an exercise in induction.

3.7. Why *tangency* points?

We explain the rationale behind the terminology "tangency line".

Notation. Given an RVT-class $(\alpha) = (\alpha_1, ..., \alpha_s)$ such that $\alpha_s \in \{V, T\}$ we denote by (α^*) an RVT class obtained from (α) by replacing by R all letters before the **last** letter V.

Example 3.16. If the last letter is $\alpha_s = V$ then $(\alpha^*) = (R, R, ..., R, V)$. If the last letter $\alpha_s = T$ then by Theorem 3.8 there exists a largest q, $q < s$ such that $\alpha_q = V$ while $\alpha_{q+1}, \alpha_{q+2}, ..., \alpha_s = T$. In this case $(\alpha^*) = (R, R, .., R, V, T, T, .., T)$ with V being in the q-th place. For example, if $(\alpha) = (RVTVRVRVT)$ then $(\alpha^*) = (RRRRRRRVT)$ while if $(\alpha) = (RVTVRVRVV)$ then $(\alpha^*) = (RRRRRRRRV)$.

Theorem 3.17. *Let $m \in \mathbb{P}^i\mathbb{R}^2$ be a vertical or tangency point, and let (α) be the RVT class of m. Let ℓ_{tan} be the tangency line in $\Delta^i(m)$ and let $L(m)$ be the tangent space at m to the closure $(\bar{\alpha}^*)$ of the class $(\alpha^*) \subset \mathbb{P}^i\mathbb{R}^2$. Then $\ell_{tan} = \Delta^i(m) \cap L(m)$.*

Remark. Theorem 3.17 includes the statement that $dim\big(\Delta^i(m) \cap L(m)\big) = 1$. If m is a vertical point then $codim(\alpha^*) = 1$ and consequently the 2-plane $\Delta^i(m)$ is transversal to the hypersurface $(\bar{\alpha}^*)$. If m is a tangency point then $(\alpha^*) = (R, ..., R, V, T, ..., T)$ with T being repeated $c \geq 1$ times. In this case $codim(\alpha^*) = codim L(m) = c + 1 \geq 2$, and the couple $\Delta^i(m), L(m)$ is not generic, since the intersection of a generic codimension ≥ 2 subspace with a 2-plane is $\{0\}$ whereas the intersection of $L(m)$ with $\Delta^i(m)$ is a line.

Proof. We have to prove two statements:

(a) $\Delta^i(m)$ is not a subset of the tangent space $L(m) = T_m(\bar{\alpha}^*)$, i.e. $dim \Delta^i(m) \cap L(m) \leq 1$;

(b) the tangency line ℓ_{tan} in $\Delta^i(m)$ does belong to $L(m)$.

Proof of (a). Let $(\alpha^*) = (R...RVT^q) \subset \mathbb{P}^i\mathbb{R}^2, q \geq 0$. Consider the following submanifold $Q \subseteq \mathbb{P}^i\mathbb{R}^2$:

- If $q \geq 1$ then Q is equal to the closure of the class $R, ..., R, V, T^{q-1}, R$;
- If $q = 0$ then $Q = \mathbb{P}^i\mathbb{R}^2$.

In either case $(\bar{\alpha}^*)$ is a hypersurface in Q, and $m \in (\bar{\alpha}^*) \subset Q$. The immersed vertical curve $\Gamma(t)$ through $m = (q_0, \ell_0)$ lies in Q, but there are only discrete values of t such that the point $\Gamma(t)$ is vertical or tangency, since there are precisely two critical points along a given fiber. Therefore Γ is not tangent to $(\bar{\alpha}^*)$ but is tangent to $\Delta^i(m)$, showing that the vertical line in $\Delta^i(m)$ does not belong to $L(m)$.

Proof of (b). Let Γ be the tangency curve through m at $t = 0$, i.e. the critical non-vertical curve through m. By Lemma 3.6 this curve is unique up to reparameterization. And $\Gamma'(0)$ spans ℓ_{tan}. By Proposition 3.4 $\Gamma = V^k_{m_k}$ for some $k \geq 1$. Note that $(\alpha^*) = (R...RVT^q)$ and $\Gamma(0) \in (\bar{\alpha}^*)$. Now Proposition 3.4 (i) implies that Γ lies wholly in the closure of the class (α^*): $\Gamma(t) \in (\bar{\alpha}^*)$ for all t. Therefore the line ℓ_{tan} is tangent to $(\bar{\alpha}^*)$, proving (b). $\square$

3.8. Two more definitions of a singular point

The open RVT class of Cartan's Theorem 1.4 from section 1.5.1 is the unique class of codimension 0. According to Theorem 3.11 this is the class $RR, ..., R$.

Theorem 3.18. *Let $p \in \mathbb{P}^i\mathbb{R}^2, i \geq 3$. The following conditions are equivalent:*

a) *p is non-singular (as per the Definition 2.9 from section 2.2.1);*

b) *p belongs to the class $RR...R$;*

c) *none of the points $\pi_{i,j}(p) \in \mathbb{P}^j\mathbb{R}^2$ is vertical for $3 \leq j \leq i$.*

Proof. The equivalence of (b) and (c) follows from the definition of 'regular' together with statement 1 of Theorem 3.8 which eliminates the possibility of any tangency points among the $\pi_{i,j}(p)$. We will prove the equivalence of (a) and (c).

Proof $(a) \implies (c)$. Assume that $p \in \mathbb{P}^{1+k}\mathbb{R}^2$ is non-singular, so that by definition $p = \gamma^k(0)$, where γ is immersed in $\mathbb{P}^1\mathbb{R}^2$. Its prolongation γ^1 is immersed in $\mathbb{P}^2\mathbb{R}^2$ and has the form $\gamma^1(t) = \big(\gamma(t), \ell(t)\big)$. Since $\gamma'(0) \neq 0$ the line spanned by the vector $(\gamma^1)'(0)$ is not vertical. Consequently the point $\gamma^2(0) \in \mathbb{P}^3\mathbb{R}^2$ is not a vertical point. The curve γ^2 has the form $\gamma^2(t) = \big(\gamma^1(t), \ell(t)\big)$. Since $(\gamma^1)'(0) \neq 0$ the line spanned by the vector $(\gamma^2)'(0)$ is not vertical. It follows that the point $\gamma^3(0) \in \mathbb{P}^4\mathbb{R}^2$ is not vertical . Continuing in this manner, we obtain that the sequence of points $\gamma^k(0), k \geq 2$ is never vertical.

PROOF $(c) \implies (a)$. Let $p = (m, \ell)$ be a non-vertical point at level i none of whose projections $\pi_{i,j}(p)$ for $j \geq 1$ are vertical points. Let Γ be an immersed integral curve at level $(i-1)$ through m and tangent to ℓ. Thus $\Gamma^1(0) = p$. Since point p is not vertical, the line ℓ is not the vertical line through m, and so the one-step-projection Γ_1 of Γ is an immersed integral curve at level $i-2$ passing through $q = \pi_{i,i-2}(p)$. Because m is not vertical, the line spanned by $\Gamma_1'(0)$ is not the vertical line through q and so $(\Gamma_1)_1 = \Gamma_2$ is an immersed integral curve at level $i-3$ passing through q. Continuing in this manner we are led to the fact that $\gamma = \Gamma_{i-1}$ is an immersed Legendrian curve at level 1 passing through $\pi_{i,1}(p)$. But $p = \gamma^i(0)$ and consequently p is a non-singular point. □

3.9. Classification problem: reduction to regular RVT classes

Recall (see definition 3.12 above) that an RVT class is called critical if it ends with a critical letter (a V or a T) and regular if it ends in an R.

DEFINITION 3.19. An RVT class is called entirely critical if it consists entirely of critical letters.

A critical RVT class (α) is either entirely critical, or it has the form

$$(\alpha) = (\widehat{\alpha}\omega), \text{ with } \omega \text{ entirely critical and } (\widehat{\alpha}) \text{ regular.} \tag{3.2}$$

The following theorem reduces the classification of points within an arbitrary RVT classes to the classification of points within some regular RVT class.

THEOREM 3.20. *[Method of Critical sections] All the points of an entirely critical RVT class are equivalent. Two points $p, \tilde{p}$ of a critical RVT class of the form (3.2) are equivalent if and only if their m-step projections $p_m, \tilde{p}_m$ in$(\widehat{\alpha})$ are equivalent, where m is the length of ω.*

PROOF. By Theorem 3.8 the vertical point in the fiber over any fixed point of the Monster is unique. The tangency point in the fiber over any critical point is unique. These facts imply that two points of a critical RVT class are equivalent if and only if their one-step projections are equivalent. Theorem 3.20 follows by iteration. □

REMARK 3.21. We have also called this theorem the "method of critical sections" because V defines a section of the bundle projection $\pi_{i+1,i}$ and T defines a section of the same projection restricted to critical points. As such, V and T define equivariant inverses to the projections induced by restricting $\pi_{i+1,i}$ to RVT classes, i.e. to $(\alpha V) \to (\alpha)$ for any (α) or $(\beta T) \to (\beta)$ when (β) is critical.

3.10. Legendrization: reduction to critical RVT classes

DEFINITION 3.22. Fix an RVT class $(\alpha) = (\alpha_1, ..., \alpha_{k-2})$. Any class of the form (αR^s) with any $s \geq 0$ will be called a regular prolongation of (α).

The Legendrization of the class $(RR, ..., R)$ consists of immersed Legendrian curve germs. Any other RVT class is a regular prolongation of a critical RVT class.

THEOREM 3.23. *The Legendrization of any regular prolongation of any fixed RVT class (α) coincides with the Legendrization of (α).*

PROOF. Let (α) be an RVT class in the kth level of the Monster and let $s \geq 1$.

1. Let $\gamma \in \mathrm{Leg}(\alpha)$. Let us show that $\gamma \in \mathrm{Leg}(\alpha R^s)$.

By definition of Legendrization there exists a regular curve germ $\Gamma : (\mathbb{R}, 0) \to \mathbb{P}^k\mathbb{R}^2$ such that $\Gamma(0) \in (\alpha)$ and γ is the projection of Γ to the first level. Consider the i-fold prolongations Γ^i, for $1 \leq i \leq s$. Each Γ^i is an immersed integral curve germ. We claim Γ^i is regular. To see this assume that Γ^i is tangent to a critical curve. Then $\Gamma = (\Gamma^i)_i$ is tangent to the i-step projection of this critical curve which is itself a critical curve. This contradicts the regularity of Γ. The fact that the curves $(\Gamma^i)'(0)$, $1 \leq i \leq s$, are regular implies that the points $\Gamma^i(0)$, $i \geq 1$, are regular. Therefore Γ^s is a regular curve such that $\Gamma^s(0)$ belongs to the class (αR^s). The projection of Γ^s to the first level is exactly the curve γ. Therefore $\gamma \in \mathrm{Leg}(\alpha R^s)$.

2. Let $\gamma \in \mathrm{Leg}(\alpha R^s)$. Let us show that $\gamma \in \mathrm{Leg}(\alpha)$.

Take a regular curve germ $\Gamma : (\mathbb{R}, 0) \to \mathbb{P}^{k+s}\mathbb{R}^2$ such that $\Gamma(0) \in (\alpha R^s)$ and γ is the projection of Γ to the first level. Consider the s-step projection $\widetilde{\Gamma} = \Gamma_s$. It is an integral curve and $\widetilde{\Gamma}(0) \in (\alpha)$. To prove that $\gamma \in \mathrm{Leg}(\alpha)$ it suffices to show that $\widetilde{\Gamma}$ is a regular curve germ. Since we know that $\widetilde{\Gamma}^1(0)$ is a regular point it suffices to show that the curve $\widetilde{\Gamma}$ is immersed. We will need:

LEMMA 3.24. *The one-step prolongation of a non-immersed integral curve germ is either non-immersed curve or an immersed curve which is tangent to the vertical direction.*

PROOF. The one-step prolongation of such a curve has the form $(m(t), \ell(t))$ with $m(t) \in \mathbb{P}^k\mathbb{R}^2$, $\ell(t)$ a line in $\Delta^k(m(t))$, and $m'(0) = 0$. $\square$

We now use this lemma to prove that $\widetilde{\Gamma}$ is an immersed curve. Assume, to get contradiction, that this is not so. We know that the points $\widetilde{\Gamma}^1(0), ..., \widetilde{\Gamma}^s(0)$ are regular. Therefore the directions of the vectors $\widetilde{\Gamma}'(0), (\widetilde{\Gamma}^1)'(0), \ldots, (\widetilde{\Gamma}^{s-1})'(0)$ cannot be vertical. Along with Lemma 3.24 this implies that each of the curves $\widetilde{\Gamma}$, $\widetilde{\Gamma}^1, \ldots, \widetilde{\Gamma}^{s-1}$ is not immersed. Then, using one more time Lemma 3.24 we obtain that the curve $\widetilde{\Gamma}^s$ is either non-immersed or is tangent to the vertical direction. Since $\widetilde{\Gamma}^s = \Gamma$ is a regular curve this is a contradiction. $\square$

3.11. Legendrization algorithm

In this section we give an explicit algorithm for calculating the Legendrization of any RVT class. This algorithm is based on Theorems 3.23, 3.26, 3.33, and 3.36 below. These theorems involve integers related to the Euclidean algorithm. We postpone explaining and proving these theorems until Chapter 5 since the proofs require local 'KR' coordinates on the Monster introduced in that chapter.

3.11.1. The map $E_\omega : \mathbb{N}^2 \to \mathbb{N}^2$ associated to an entirely critical class ω. Recall that an RVT class $(\omega)$is entirely critical if it consists entirely of critical letters, V or T. We need the following map E_ω sending a pair of non-negative integers to another pair of non-negative integers. This map is built up from the two maps

$$E_T : \ (n_1, n_2) \ \to \ (n_1, n_1 + n_2),$$
$$E_V : \ (n_1, n_2) \ \to \ (n_2, n_1 + n_2),$$

where n_1, n_2 are non-negative integers. Now, given an entirely critical RVT class

$$(\omega) = (\omega_1, ..., \omega_m), \quad \omega_i \in \{V, T\}$$

define the composition

$$E_\omega = E_{\omega_1} \circ E_{\omega_2} \circ \cdots \circ E_{\omega_m}.$$

Examples. Let $(\omega) = (VT)$. Since $E_T(1,2) = (1,3)$ and $E_V(1,3) = (3,4)$ we have $E_{VT}(1,2) = (3,4)$. Next consider $(\omega) = (VV)$. Since $E_V(1,2) = (2,3)$ and $E_V(2,3) = (3,5)$ we have $E_{VV}(1,2) = (3,5)$. One more example:

$$(\omega) = (VVTVTTV).$$

In this case one has $E_\omega(1,2) = (23,39)$ since the successive maps have the effect

$$(1,2) \xrightarrow{E_V} (2,3) \xrightarrow{E_T} (2,5) \xrightarrow{E_T} (2,7) \xrightarrow{E_V}$$
$$\to (7,9) \xrightarrow{E_T} (7,16) \xrightarrow{E_V} (16,23) \xrightarrow{E_V} (23,39).$$

Since both E_V and E_T map relatively prime pairs to relatively prime pairs it is clear that the following claim holds:

Claim 3.25. If n_1 and n_2 are relatively prime numbers and ω is any entirely critical RVT class (ω) then the integers $(\tilde{n}_1, \tilde{n}_2) = E_\omega(n_1, n_2)$ are also relatively prime. In particular the numbers $(a,b) = E_\omega(1,2)$ are relatively prime.

3.11.2. Legendrization of RVT classes $(R^s\omega)$, (ω) entirely critical.

Theorem 3.26. *For (ω) an entirely critical RVT class and $s \geq 0$ an integer, set $(a,b) = E_\omega(1,2)$, and form the pair of relatively prime integers:*

$$\lambda_0 = a, \quad \lambda = sa + a + b. \tag{3.3}$$

Then the Legendrization of the class $(R^s\omega)$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of the plane curve germs

$$x = t^{\lambda_0}, \quad y = t^\lambda f(t), \quad f(0) \neq 0,$$

where $f(t)$ is an arbitrary function germ at the point $t = 0$ such that $f(0) \neq 0$.

3.11.3. Examples. Now we will give few examples combining Theorem 3.26 and Theorem 3.23 with results of section 2.4.

Example 3.27 (The A-singularities). Since $E_V(1,2) = (2,3)$ then by Theorem 3.26 and Theorem 3.23 the Legendrization of any RVT class of the form R^sVR^m $(s, m \geq 0)$ is the class of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs $c : (t^2, t^{2s+5}f(t)), f(0) \neq 0$. This is the classical A-singularity. Any plane curve germ of the given form is RL-equivalent to (t^2, t^{2s+5}). Therefore the one-step prolongation of c is RL-contact equivalent to that of (t^2, t^{2s+5}). Consequently the Legendrization of R^sVR^m consists of a single orbit with respect to the RL-contact equivalence. By Theorem 2.20 we see that *all points of the class R^sVR^m are equivalent.*

Example 3.28. Since $E_{VT}(1,2) = (3,4)$ by Theorem 3.26 and Theorem 3.23 the Legendrization of the class VTR^m $(m > 0)$ is the class of Legendrian curve germs equivalent to the one-step prolongations of the plane curve germs $(t^3, t^7f(t))$, $f(0) \neq 0$. Though not all such plane curve germs are RL-equivalent their one-step prolongations are all RL-contact equivalent. See appendix B. Therefore, as in the previous example, the Legendrization of VTR^m consists of a single orbit. Theorem 2.20 now implies that *all points of the class VTR^m are equivalent.*

EXAMPLE 3.29. Since $E_{VV}(1,2) = (3,5)$ by Theorem 3.26 and Theorem 3.23 the Legendrization of VVR^m is the class of Legendrian curve germs equivalent to the one-step prolongations of the plane curve germs $c : (t^3, t^8 f(t)), f(0) \neq 0$. As in the previous example, not all such plane curve germs are RL-equivalent, but their one-step prolongations are all RL-contact equivalent. See appendix B. Therefore Theorem 2.20 implies that *all points of the class VVR^m are equivalent.*

EXAMPLE 3.30. By Theorem 3.26 and Theorem 3.23 the Legendrization of the class R^sVTR^m, respectively R^sVVR^m are the classes of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs $c : (t^3, t^{3s+7} f(t))$, $f(0) \neq 0$, respectively $c : (t^3, t^{3s+8} f(t))$, $f(0) \neq 0$. If $s \geq 1$ then these classes of Legendrian curve germs consist of a finite number $d > 1$ of orbits with respect to RL-contact equivalence. See Appendix B. Therefore Theorem 2.20 implies that the RVT classes R^sVTR^m or R^sVVR^m each consist of a finite number $d_1 \leq d$ of orbits. To determine the number d_1 and to provide a complete classificication we need more that Theorem 2.20 and will have to use the results of sections 3.12, 3.13 below.

EXAMPLE 3.31. Consider an RVT class of the form $(\alpha) = VVVR^{\geq 0}$. Calculating $E_{VVV}(1,2) = (5,8)$ and using Theorems 3.26 and 3.23 we obtain that the Legendrization of (α) is the class of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs $c : (t^5, t^{13} f(t)), f(0) \neq 0$. This class of Legendrian curve germs consists of continuum distinct orbits with respect to the RL-contact equivalence, see Appendix B. This does not mean that the class (α) consists of continuum orbits, see section 2.4.1. In Chapter 4 we will show that the class $VVVR^q$ consists of a finite number of orbits if $q \leq 4$ and of continuum distinct orbits if $q \geq 5$.

EXAMPLE 3.32. Let $(\omega) = VTVTTV$. Calculate $E_\omega(1,2) = (16,23)$. By Theorems 3.26 and 3.23 we obtain that the Legendrization of an RVT class of the form $R^s\omega R^m$ is the class of Legendrian curve germs equivalent to the one-step prolongations of the plane curve germs $(t^{16}, t^{16s+23} f(t))$, $f(0) \neq 0$.

3.11.4. Recursion formulae. Now we give a recursion formulae for computing the Legendrization of any RVT class. We begin with the following theorem.

THEOREM 3.33. *Let (β) be any RVT class. There exists relatively prime integers λ_0 and λ and a set $\mathcal{G}$ of polynomials of one variable such that the following holds:*

1. *The set $\mathcal{G}$ contains the zero function germ.*
2. *For any $g(t) \in \mathcal{G}$ the plane curve germ $(t^{\lambda_0}, g(t))$ is NOT well-parameterized.*
3. *The Legendrization of (β) consists of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of plane curve germs*

$$P = \left\{t^{\lambda_0},\ g(t) + t^\lambda f(t),\quad f(0) \neq 0,\ g(t) \in \mathcal{G}, \right\} \tag{3.4}$$

where $f(t)$ is an arbitrary function germ at the point $t = 0$ such that $f(0) \neq 0$.

EXAMPLE 3.34. In the simplest case when $(\beta) = R^s\omega R^{\geq 0}$, where (ω) is entirely critical class Theorem 3.33 is a corollary of Theorem 3.26. One has $\mathcal{G} = \{0\}$, and λ_0 and λ are given by (3.3).

DEFINITION 3.35. The set P in Theorem 3.33 will be called the set of plane curve germs describing the Legendrization of (β).

Consider now a class (α) of the form

$$(\alpha) = (\beta R^s \omega R^{\geq 0}), \tag{3.5}$$

where (β) is a critical class and (ω) is entirely critical class.

THEOREM 3.36. *Consider an RVT class α of the form (3.5). Let P be the set of plane curve germ describing the Legendrization of (β). Set*

$$(a, b) = E_\omega(1, 2).$$

Then the Legendrization of (α) is described by the set of plane curve germs

$$\widetilde{P} = \big\{c(t^a) + \big(0,\ t^{\widetilde{\lambda}} f(t)\big),\ f(0) \neq 0,\quad c \in P\big\}, \tag{3.6}$$

where

$$\widetilde{\lambda} = a(\lambda + s - 1) + b - a.$$

and $f(t)$ is an arbitrary function germ at the point $t = 0$ such that $f(0) \neq 0$.

REMARK 3.37. The integer $\widetilde{\lambda}$ occuring in theorem 3.36 is the order of good parameterization of the curve c there.

REMARK 3.38. The set $\widetilde{P}$ in Theorem 3.36 has the form (3.4) with λ_0 replaced by $a \cdot \lambda_0$, with λ replaced by $\widetilde{\lambda}$, and with the set $\mathcal{G}$ replaced by the set of $(\widetilde{\lambda} - 1)$-jets of the polynomials $g(t^a), g \in \mathcal{G}$.

Theorems 3.33 and 3.36 give a recursion algorithm for Legendrizing any RVT class. In fact, any RVT class (α) containing at least one critical letter, V or T, has the form

$$(\alpha) = R^{s_1}\omega^{(1)} R^{s_2}\omega^{(2)} \cdots R^{s_m}\omega^{(m)} R^{\geq 0}, \quad \omega^{(i)} - \text{entirely critical classes.} \tag{3.7}$$

Here $s_i \geq 0$. In the case $m = 1$ we know the Legendrization of (α) (or equivalently the set of plane curve germs describing this Legendrization) by Theorem 3.26. We can express (α) in the form (3.4), where $(\beta) = R^{s_1}\omega^{(1)} R^{s_2}\omega^{(2)} \cdots R^{s_{m-1}}\omega^{(m-1)}$ and $s = s_m, (\omega) = (\omega^{s_m})$. Then β is a critical class containing $(m-1)$ entirely critical classes. If we know the Legendrization of (β) then Theorem 3.36 gives us the Legendrization of (α). Therefore the parameter m in (3.7) is the recursion parameter, Theorem 3.26 is the recursion base, and Theorems 3.33 and 3.36 are the recursion transitions.

3.11.5. Further examples of Legendrization. In subsection 3.11.3 we used Theorem 3.26 to compute the Legendrization of the simplest RVT classes – classes of the form $R^s \omega R^{\geq 0}$, where (ω) is an entirely critical class. Now we will use Theorem 3.36 to compute the Legendrizations of more complicated RVT classes.

EXAMPLE 3.39. Consider an RVT class of the form

$$(\alpha) = R^s V R^m V R^{\geq 0},\ s \geq 0,\ m \geq 1.$$

We showed (Example 3.27) that the Legendrization of the class $(R^s V)$ is described by the set of plane curve germs $(t^2, t^{2s+5} f(t))$, $f(0) \neq 0$, i.e. by a set P of the form (3.4) with $\lambda_0 = 2, \lambda = 2s + 5$. By Theorem 3.36 the Legendrization of (α) is described by the set of plane curve germs (3.6) with

$$a = 2,\ \widetilde{\lambda} = 2(2s + 5 + m - 1) + 3 - 2 = 4s + 2m + 9.$$

Therefore the Legendrization of (α) consists of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs

$$\big(t^4,\ t^{4s+10}f_1(t^2)+t^{4s+2m+9}f_2(t)\big),\quad f_1(0)\neq 0, f_2(0)\neq 0, \tag{3.8}$$

where $f_1(t)$ and $f_2(t)$ are arbitrary function germs such that $f_1(0)\neq 0, f_2(0)\neq 0$.

Example 3.40. Consider an RVT class of the form

$$(\alpha)=R^sVVR^mVTR^{\geq 0},\ s\geq 0,\ m\geq 1.$$

We showed (Example 3.29) that the Legendrization of the class (R^sVV) is described by the set of plane curve germs $(t^3,t^{3s+8}f(t))$, $f(0)\neq 0$, i.e. by a set P of the form (3.4) with $\lambda_0=3,\ \lambda=3s+8$. One has $E_{VT}=(3,4)$. By Theorem 3.36 the Legendrization of (α) is described by the set of plane curve germs (3.6) with

$$a=3, \widetilde{\lambda}=3(3s+8+m-1)+4-3=9s+3m+22.$$

Therefore the Legendrization of (α) consists of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs

$$\big(t^9,\ t^{9s+24}f_1(t^3)+t^{9s+3m+22}f_2(t)\big),\quad f_1(0)\neq 0, f_2(0)\neq 0.$$

Example 3.41. Consider an RVT class of the form

$$(\alpha)=R^sVR^mVR^nVR^{\geq 0},\ s\geq 0,\ m_1,m_2\geq 1.$$

We showed (Example 3.39) that the Legendrization of the class $(R^sVR^{m_1}V)$ is described by the set of plane curve germs of the form (3.8). Since $E_V(1,2)=(2,3)$ then by Theorem 3.36 the Legendrization of (α) consists of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of the plane curve germs

$$\Big(t^8,\ t^{8s+20}f_1(t^4)+t^{8s+4m+18}f_2(t^2)+t^{8s+4m+2n+17}f_3(t)\Big),$$
$$f_1(0)\neq 0, f_2(0)\neq 0, f_3(0)\neq 0.$$

3.12. The parameterization and the jet-identification numbers

In section 2.8 we defined the parameterization number of a point p of the Monster to be the order of the good parameterization of a curve $\gamma\in \mathrm{Leg}(p)$. We also stated that this order is the same for any two such curves. See Definition 2.32 and Theorem 2.34. Now we will prove a stronger statement.

Theorem 3.42. *All the curves in the Legendrization of any point of a fixed RVT class have the same order of good parameterization which depends on the class only.*

Proof. Theorems 3.33 and 3.36 imply that the Legendrization of any RVT class (α) consists of Legendrian curve germs which are RL-contact equivalent to the one-step prolongations of plane curve germs of the form

$$\big(t^{\lambda_0},\ a_1t^{\lambda_1}+\cdots+a_mt^{\lambda_m}+t^{\lambda}f(t)\big),\ f(0)\neq 0, \tag{3.9}$$

where the integers $\lambda_0,...,\lambda_m$ have a common factor greater than 1, the integers $\lambda_0,...,\lambda_m,\lambda$ do not have such a common factor, and the numbers λ_0 and λ depends on the RVT code of the class (α) only. It follows that all plane curve germs of set (3.9) have the same order of good parameterization, it is equal to p. Consequently, the one-step prolongations of plane curve germs (3.9) also have the same order of good parameterization, it is equal to $p-\lambda_0$. Therefore $p-\lambda_0$ is the parameterization number of any curve in the Legendrization of any point of the class (α). $\square$

Definition 3.43. The parameterization number of an RVT class (α) is the order of good parameterization of some (and hence any) Legendrian curve germ in the set $\mathrm{Leg}(\alpha)$.

Remark 3.44. Equivalently, the parameterization number of (α) is the parameterization number of some (and hence any) point in (α). The given proof of Theorem 3.42 includes the proof of a weaker Theorem 2.34.

Recall from section 2.6 that any regular point p has a jet-identification number $r = r(p)$, and that this means that the point can be identified with a single r-jet of a Legendrian curve germ. See Definition 2.21 and Theorem 2.29.

Theorem 3.45. *All points of a fixed regular RVT class have the same jet-identification number.*

Definition 3.46. The jet-identification number of a regular RVT class is the jet-identification number of some (and hence any) point of this class.

The jet-identification number of the open class R^{k-1} in the $(k+1)$st level is equal to k, see section 2.8. Any other regular RVT class (α) has the form

$$(\alpha) = (\beta R^q), \quad \beta \text{ critical}, \quad q \geq 1. \tag{3.10}$$

Theorem 3.47. *The jet-identification number of an RVT class (α) of form (3.10) is equal to $d+q-1$, where d is the parameterization number of the class (β).*

Proof of Theorems 3.45 and 3.47. Let (α) be an RVT class of form (3.10). Then q is exactly the degree of regularity of any point of (α), see Definition 2.30. By Theorem 2.36 any point of (α) has the same jet-identification number $d(\alpha) + q - 1$, where $d(\alpha)$ is the parameterization number of (α). By Theorem 3.23 one has $\mathrm{Leg}(\alpha) = \mathrm{Leg}(\beta)$, therefore the parameterization numbers of (α) and (β) are the same.

Recall from section 2.3 that the Monsterization is left inverse to the Legendrization: $\mathrm{Monster}^k(\mathrm{Leg}(Q)) = Q$ for any singularity class $Q \in \mathbb{P}^{1+k}\mathbb{R}^2$. We conclude this section with the following statement on the continuity of the Monsterization within a fixed RVT class (in general the Monsterization is not a continuous operation, see section 7.1).

Proposition 3.48. *Fix a regular RVT class (α). Let U be an open set in the level of (α). Let r be the jet-identification number of (α). The set $j^r \,\mathrm{Leg}((\alpha) \cap U)$ is the intersection of the set $j^r \,\mathrm{Leg}(\alpha)$ with an open set in the space of r-jets of Legendrian curves on $\mathbb{P}^1\mathbb{R}^2$.*

Let us explain that this statement means the continuity of the Monsterization within a fixed RVT class. Given an r-jet $\xi \in j^r \,\mathrm{Leg}(\alpha)$ take any Legendrian curve curve γ representing this jet and evaluate the point $p = \gamma^k(0) \in (\alpha)$, where $(k+1)$ is the level of the class (α). By the definition of the jet-identification number the point p does not depend on the choice of γ, so we have a well-defined map from $j^r \,\mathrm{Leg}(\alpha)$ to (α) which sends the r-jet ξ to the point p. Propsoition 3.48 is equivalent to the continuity of this map.

3.13. Reduction algorithms for the equivalence and classification problems

Let S be a set with an equivalence relation. The equivalence problem for S is the problem of determining when two fixed points in S are equivalent. A “singularity class” is a union of equivalence classes. The classification problem for a singularity class S is the problem of constructing a subset $N \subset S$ such that any point in S is equivalent to one and only one point in N. In this case N is called an exact normal form for S. If instead any point of S is equivalent to some point of N, but not necessarily a unique such point, then N is called preliminary normal form for S.

The results of Chapters 2 and 3 reduce the equivalence and classification problems for the Monster (or for specific RVT classes within the monster) to the RL-contact equivalence and RL-contact classification problems for the set of jets of Legendrian curves.

3.13.1. Reduction algorithm for equivalence problem. Let p and $\tilde{p}$ be points in the same level k of the Monster. We are to determine if p and $\tilde{p}$ are equivalent. We show here how to reduce this equivalence problem to the (well-studied) problem of determining whether or not two r-jets of Legendrian curves are RL-contact equivalent.

Step 1. Is $k \leq 2$ or is $k > 2$? If $k \leq 2$ then p and $\tilde{p}$ are equivalent. (See Theorem 1.3.) If $k > 3$ and $p, \tilde{p}$ belong to different RVT classes then they are not equivalent. See the first statement of Theorem 3.11.

In what follows we assume that p and $\tilde{p}$ belong to the same RVT class (α) in the kth level, $k \geq 3$.

Step 2. Is the class critical or regular? If the class (α) is critical (i.e. it ends with a V or T) then, as in section 3.9 we have two mutually exclusive alternatives:

Case 2.1. The code contains is entirely critical.

Case 2.2. The code contains at least one letter R.

In case 2.1 the points p and $\tilde{p}$ are equivalent by the first statement of Theorem 3.20.

In case 2.2 we can express the class (α) in the form of equation 3.2 as $(\alpha) = (\widehat{\alpha}\omega)$ with ω entirely critical and $(\widehat{\alpha})$ regular, i.e. ending in R. The class $(\widehat{\alpha})$ belongs to the level k_1 where $k_1 < k$. Consider the projections

$$\pi_{k,k_1}(p),\ \pi_{k,k_1}(\tilde{p}) \in (\widehat{\alpha}). \tag{3.11}$$

The points $p, \tilde{p}$ are equivalent if and only if their projections (3.11) are equivalent, according to the second statement of Theorem 3.20.

Cases 2.1 and 2.2 above immediately reduce us to the case where the points $p, \tilde{p}$ are in the same **regular** RVT class $(\widehat{\alpha}) \subset \mathbb{P}^k\mathbb{R}^2$, $k \geq 3$. If the RVT code of $(\widehat{\alpha})$ consists of letters R only then p and $\tilde{p}$ are non-singular points and then they are equivalent. Therefore in what follows we consider the case that $(\widehat{\alpha}) \subset \mathbb{P}^k\mathbb{R}^2, k \geq 3$ is a regular RVT class whose code contains at least one letter V or T. In this case $(\widehat{\alpha})$ has the form

$$(\widehat{\alpha}) = (\beta R^q), \quad \beta \text{ critical}, \quad q \geq 1. \tag{3.12}$$

Step 3. Take regular curve germs $\Gamma, \tilde{\Gamma} : (\mathbb{R}, 0) \to \mathbb{P}^k\mathbb{R}^2$ such that $\Gamma(0) = p, \tilde{\Gamma}(0) = \tilde{p}$. Project Γ and $\tilde{\Gamma}$ to $\mathbb{R}^2$ so as to obtain plane curve germs

$$c = \pi_{k,0}\Gamma, \quad \tilde{c} = \pi_{k,0}\tilde{\Gamma}. \tag{3.13}$$

The one-step prolongations of these plane curves are Legendrian curve germs in $\mathrm{Leg}(p)$ and $\mathrm{Leg}(\tilde{p})$:

$$\gamma = c^1 \in \mathrm{Leg}(p), \quad \tilde{\gamma} \in \tilde{c}^1 \in \mathrm{Leg}(\tilde{p}). \tag{3.14}$$

Step 4. Take one of the plane curve germs (3.13), say c. Reparameterize it to have one of the two forms

$$c: \ x(t) = \pm t^{\lambda_0}, \quad y(t) = a_1 t^{\lambda_1} + a_2 t^{\lambda_2} + \cdots, \quad \lambda_0 \le \lambda_1 < \lambda_2 < \cdots, a_i \neq 0. \tag{3.15}$$

$$c: \ y(t) = \pm t^{\lambda_0}, \quad x(t) = a_1 t^{\lambda_1} + a_2 t^{\lambda_2} + \cdots, \quad \lambda_0 \le \lambda_1 < \lambda_2 < \cdots, \quad a_i \neq 0. \tag{3.16}$$

By the first statement of Theorem 2.34 the curve $\gamma = c^1$ is well-parameterized. It follows that the curve c is also well-parameterized and therefore there exists i such that $\lambda_0, \lambda_1, \ldots, \lambda_i$ have no common factor greater than 1. Let i^* be the minimal i satisfying this requirement. Then λ_{i^*} is the order of good parameterization of c. It follows that

$$d = \lambda_{i^*} - \lambda_0$$

is the order of good parameterization of the curve germ $\gamma = c^1$. Consequently, d is the parameterization number of the point p. By Theorem 3.47 the number $d+q-1$ is the jet-identification number of the class (α), so that any point of this class can be identified with a single $(d+q-1)$-jet of a Legendrian curve.

Step 5. By Theorem 2.24 the points p and $\tilde{p}$ are equivalent if and only if the $(d+q-1)$-jets of Legendrian curves $\gamma = c^1$ and $\tilde{\gamma} = \tilde{c}^1$ are RL-contact equivalent.

3.13.2. Reduction algorithm for classification problem. Fix an RVT class (α) in the kth level of the Monster, $k \ge 3$. The following algorithm reduces the problem of finding an exact normal form for class(α) to the problem of finding an exact normal form for a certain singularity class of Legendrian curve r-jets.

Step 1. Is the code (α) entirely critical (no Rs) or not? If it is entirely critical then all points of its singularity class are equivalent. (See the first statement of Theorem 3.20). Any single point of class (α) provides an exact normal form for class (α).

Step 2. Having dispensed with the case of an entirely critical class, is the class (α) critical or regular? If it is critical then it has form of equation (3.2): $(\alpha) = (\widehat{\alpha}\omega)$ with ω entirely critical and $(\widehat{\alpha})$ regular. The class $(\widehat{\alpha})$ is a regular class in level $k_1, k_1 < k$. Assume that we know an exact normal form $N_{\widehat{\alpha}}$ for the class $(\widehat{\alpha})$. The second statement of Theorem 3.20 implies that the set

$$N_\alpha = \{p \in (\alpha) : \quad \pi_{k,k_1}(p) \in N_{\widehat{\alpha}}\}$$

is an exact normal form for the class (α).

Steps 1 and 2 reduce the classification problem for the class (α) to the case of (α) regular. Then (α) has form of equation (3.12): $(\alpha) = (\beta R^q)$ with β critical. In what follows we assume that (α) is of that form.

Step 3. Calculate the singularity class $\mathrm{Leg}(\beta)$. The algorithm for calculating this class is given in section 3.11.

Step 4. Take any Legendrian curve germ $\gamma \in \mathrm{Leg}(\beta)$. Write $\gamma = c^1$, where c is a plane curve germ and parameterize c so that it has the form (3.15) or (3.16). Find the number d as in step 4 of subsection 3.13.1. This d is the parameterization number of the class (β). By Theorem 3.47 the class (α) has the jet-identification number $d + q - 1$.

Step 5. Find an exact normal form $\mathcal{N}$ for the class $j^{d+q-1}\,\mathrm{Leg}(\beta)$. Express this form as

$$\mathcal{N} = \{j^{d+q-1}c^1,\ c \in P\},$$

where P is a certain finite-dimensional set of plane curve germs. Theorem 3.23, stating that $\mathrm{Leg}(\alpha) = \mathrm{Leg}(\beta)$, Theorem 2.24 and the basic properties of the Legendrization and the Monsterization in the beginning of Chapter 2 imply that the Monsterization of the set P,

$$\mathrm{Monster}^k(P) = \{c^k(0),\ c \in P\}$$

is an exact normal form for the class (α).

Remark 3.49. Finding the P of step 5 is a finite-dimensional classificiation problem. Moduli for class(α) can occur here.

CHAPTER 4

Determination and classification of simple points

4.1. Tower-simple and stage-simple points

Recall that a point is called *simple* if it is contained in a neighborhood which is covered by a finite number of equivalence classes. The notion of "neighborhood" can be either taken within the level of the Monster which contains that point, or taken in the entire Monster tower. These two choices lead to two definitions of "simple".

DEFINITION 4.1. A tower-neighborhood of the point p is an infinite sequence $U_0, U_1, ...$, where U_j is an open set in $\mathbb{P}^j\mathbb{R}^2$, $\pi_{j+1,j}U_{j+1} = U_j$ and $p \in U_i$.

DEFINITION 4.2. A point $p \in \mathbb{P}^i\mathbb{R}^2$ is called simple if there is a stage neighborhood U of p in $\mathbb{P}^i\mathbb{R}^2$ covered by a finite number of orbits. The point p is called tower-simple if there is a tower neighborhood $(U_0, U_1, ...)$ of p such that each of the open sets $U_j, j = 0, 1, ...$ is covered by a finite number of orbits.

We will sometimes refer to simple points as being "stage-simple" to differentiate the idea from that of being "tower-simple". Any tower-simple point is simple, but the converse is not true.

The results of this chapter imply that simplicity is a property of RVT classes.

THEOREM 4.3. *A point is tower-simple simple (resp. stage-simple) if and only if every point of its RVT class is tower-simple (resp. stage-simple).*

This statement is a direct corollary of Theorems 4.4 and 4.5 on the determination of tower-simple and stage-simple points. These theorems and their corollaries are formulated in section 4.2 and are proved in sections 4.3 - 4.8. We have not succseed to give a direct proof of Theorem 4.3.

In sections 4.9 and 4.10 we illustrate the reduction algorithm given in section 3.13 which reduces the classification of points of any RVT class to the RL-contact classification of a certain class of Legendrian curve jets. Using this algorithm, in section 4.9 we present a complete classification of all tower-simple points in the Monster, and in section 4.10 we present a complete classification of points of some RVT classes consisting of (a) stage-simple but not tower-simple points and (b) points which are not stage-simple. Note that in the case (b) and in this case only the classification involves moduli.

4.2. Determination theorems

The Darboux and Engel Theorems (Theorem 1.3) imply that any two points at level 1 are equivalent, and any two points at level 2 are equivalent. (i.e. $\mathbb{P}^1\mathbb{R}^2$ and $\mathbb{P}^2\mathbb{R}^2$) are equivalent. Therefore *all points at level 1 and 2 are simple.* All these points are also tower simple as is seen by noting that the regular prolongation of any such point lies in the dense open RVT class RR...R which forms a single open orbit. (See Theorem 3.18 and the Cartan Theorem, Theorem 2.10). Therefore points at level 1 and 2 are tower-simple. The simple and tower-simple points at the higher levels can be characterized as follows.

THEOREM 4.4. *A point of* $\mathbb{P}^i\mathbb{R}^2, i > 2$ *is tower simple if and only if its RVT class is a regular prolongation of one of the classes*

$$R,\ R^sV,\ R^sVT,\ R^sVV,\ VR^mV,\ VTT,\ VVT,\quad s \geq 0,\ m \geq 1. \tag{4.1}$$

THEOREM 4.5. *A point of* $\mathbb{P}^i\mathbb{R}^2, i > 2$ *is simple if and only if its RVT class does* **not** *adjoin any of the classes listed in Tables 4.1 and 4.2.*

TABLE 4.1. Fencing RVT classes starting with R

Class	codim	dim
$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	2	≥ 11
$R^{\geq 1}VTTR^{\geq 3}$	3	≥ 11
$R^{\geq 1}VVTR^{\geq 4}$	3	≥ 12
$R^{\geq 1}VTR^{\geq 2}VR^{\geq 1}$	3	≥ 11
$R^{\geq 1}VTRVR^{\geq 2}$	3	≥ 11
$R^{\geq 1}VVR^{\geq 3}VR^{\geq 1}$	3	≥ 12
$R^{\geq 1}VVR^2VR^{\geq 2}$	3	≥ 12

TABLE 4.2. Fencing RVT classes starting with V

Class	codim	dim
$VR^{\geq 1}VR^{\geq 1}VR^{\geq 1}$	3	≥ 10
$VR^{\geq 1}VTR^{\geq 2}$	3	≥ 10
$VR^{\geq 1}VVR^{\geq 3}$	3	≥ 11
$VTR^{\geq 1}VR^{\geq 3}$	3	≥ 11
$VTVR^{\geq 5}$	3	≥ 12
$VVVR^{\geq 5}$	3	≥ 12
$VTTTR^{\geq 4}$	4	≥ 12
$VTTR^{\geq 3}VR^{\geq 1}$	4	≥ 12
$VTTR^2VR^{\geq 2}$	4	≥ 12

Theorems 4.5 and 4.4 are proved in sections and using results from Chapters 2 and 3 and the notion of a fence *fencing RVT classes* introduced in section 4.5.

REMARK 4.6. An RVT class is a regular prolongation of itself and adjoins itself. Accordingly, the points of the classes (4.1) are tower simple and the points of the classes in Tables 4.1, 4.2 are not stage simple.

The set of simple points of the Monster is much larger than the set of tower-simple points. By Theorem 4.4 if an RVT class has codimension greater than 3 than none of its points are tower-simple. On the other hand, by Theorem 4.5 there are RVT classes of codimension ≤ 7 consisting of stage-simple points. Examples of such classes are VT^i for $0 < i < 7$. The class VT^7 is not stage-simple because it adjoins the fencing class VT^3R^4 in Table 4.2. Codimension 7 is the maximal codimension of a class made up of simple points. Using Theorem 4.5 it is easy to prove the following

COROLLARY 4.7.

(1) *No point of an* RVT *classes of codimension* ≥ 8 *is stage-simple. There are* RVT *classes of every codimension* ≤ 7 *consisting of stage-simple points.*
(2) *No point of an* RVT *class of codimension* ≥ 7 *which begins with* R *is stage-simple. There are* RVT *classes of every codimension* ≤ 6 *starting with* R *and consisting of stage-simple points.*

The third column in Tables 4.1 and 4.2 is the minimal dimension of the Monster manifold containing the corresponding RVT classes. Note that an RVT class coded by r letters lives in the Monster manifold of dimension $r + 4$. The minimal number in the third column of Tables 4.1 and 4.2 is 10. Therefore Theorem 4.5 implies the following corollary.

COROLLARY 4.8 (results obtained in the works described in Section 1.5.3). *All points of the Monster manifolds of dimension* ≤ 9 *are stage-simple.*

Theorem 4.5 allows us to determine stage-simple points in the Monster manifold of any dimension ≥ 10. For example, analyzing Tables 4.1 and 4.2 we see that a point in the Monster manifold of dimension 10 is stage-simple if and only if its RVT class is not one of the classes $V * VTR^2$, $V * VTRV$, $V * VTV*$, $V * VTT*$, $VRVRVR$, where * denotes any of the letters R, V, T.

4.3. Table of Legendrizations and jet-determiantion numbers

The first ingredient for the proof of Theorems 4.4 and 4.5 are Tables 4.3 and 4.4 below containing the Legendrization of all critical RVT classes of codimension ≤ 3 and some critical classes of codimension 4. The second column of the tables is the codimension of the classes. The Legendrization is given in the third column. The fourth column is the parameterization number, see section 3.12. The last column plays a crucial role in the proof of Theorems 4.4 and 4.5 and is explained in section 4.4. See Theorem 4.15.

PROPOSITION 4.9. *The Legendrization of an RVT class* (α) *from Table 4.3 or 4.4 consists of Legendrian curve germs RL-contact equivalent to the one-step prolongation of the plane curve germ from the third column in the row of* (α)*, where* $f_i(t)$ *are arbitrary function germs of one variable such that* $f_i(0) \neq 0$.

PROOF. The proof is an application of the Legendrization algorithm of section 3.11. First, consider the Legendrization of the classes

$$R^sV,\ R^sVT,\ R^sVV,\ R^sVTT,\ R^sVVT,\ VTTT \tag{4.2}$$

TABLE 4.3. Legendrization of RVT classes of codimension ≤ 3.

Class	codim.	Legendrization (see Prop. 4.9) $f_i(0) \neq 0$	param. number d	q^* (see Th. 4.15)
$R^sV, \quad s \geq 0$	1	$\big(t^2, \quad t^{2s+5}f_1(t)\big)$	$2s+3$	∞
$R^sVT, \quad s \geq 0$	2	$t^3, \quad \big(t^{3s+7}f_1(t)\big)$	$3s+4$	∞
$R^sVV, \quad s \geq 0$	2	$\big(t^3, \quad t^{3s+8}f_1(t)\big)$	$3s+5$	∞
R^sVR^mV $s \geq 0, \ m \geq 1$	2	$\Big(t^4, \quad t^{4s+10}f_1(t^2)+$ $t^{4s+9+2m}f_2(t)\Big)$	$4s+5+2m$	$s=0: \quad \infty$ $s \geq 1: \quad 2$
$R^sVTT, \ s \geq 0$	3	$\big(t^4, \quad t^{9+4s}f_1(t)\big)$	$4s+5$	$s=0: \quad \infty$ $s \geq 1: \quad 2$
$R^sVVT, \ s \geq 0$	3	$\big(t^4, \quad t^{11+4s}f_1(t)\big)$	$4s+7$	$s=0: \quad \infty$ $s \geq 1: \quad 3$
$R^sVTV, \ s \geq 0$	3	$\big(t^5, \quad t^{12+5s}f_1(t)\big)$	$5s+7$	$s=0: \quad 4$ $s \geq 1: \quad 2$
$R^sVVV, \ s \geq 0$	3	$\big(t^5, \quad t^{13+5s}f_1(t)\big)$	$5s+8$	$s=0: \quad 4$ $s \geq 1: \quad 3$
R^sVTR^mV $s \geq 0, \ m \geq 1$	3	$\Big(t^6, \quad t^{6s+14}f_1(t^2)+$ $t^{6s+13+2m}f_2(t)\Big)$	$6s+7+2m$	$s=0: \ 2$ $s \geq 1, m=1: \ 1$ $s \geq 1, m \geq 2: \ 2$
R^sVR^mVT $s \geq 0, \ m \geq 1$	3	$\Big(t^6, \quad t^{6s+15}f_1(t^3)+$ $t^{6s+13+3m}f_2(t)\Big)$	$6s+7+3m$	1
R^sVR^mVV $s \geq 0, \ m \geq 1$	3	$\Big(t^6, \quad t^{6s+15}f_1(t^3)+$ $t^{6s+14+3m}f_2(t)\Big)$	$6s+8+3m$	2
R^sVVR^mV $s \geq 0, \ m \geq 1$	3	$\Big(t^6, \quad t^{6s+16}f_1(t^2)+$ $t^{6s+15+2m}f_2(t)\Big)$	$6s+9+2m$	$s=0: \ 2$ $s \geq 1, m=1: \ 1$ $s \geq 1, m=2: \ 2$ $s \geq 1, m \geq 3: \ 0$
$R^sVR^mVR^nV$ $s \geq 0, m, n \geq 1$	3	$\Big(t^8, \quad t^{8s+20}f_1(t^4)+$ $t^{8s+18+4m}f_2(t^4)+$ $t^{8s+17+4m+2n}f_3(t)\Big)$	$8s+9+$ $4m+2n$	0 unless $s \geq 1, n \geq 3$ see Remark 4.16

appearing in Table 4.3. The Legendrizations of the first three classes (the codimension 1 and 2 classes) were already computed in the examples following Theorem

TABLE 4.4. Legendrization of some RVT classes of codimension 4

Class	codim.	Legendrization (see Prop. 4.9) $f_i(0) \neq 0$	param. number d	q^* (see Th. 4.15)
$VTTT$	4	$\left(t^5,\ \ t^{11}f_1(t)\right)$	6	3
$VTTR^mV$ $m \geq 1$	4	$\left(t^8,\ \ t^{18}f_1(t^2)+ t^{17+2m}f_2(t)\right)$	$9+2m$	$m=1:\ 2$ $m=2:\ 1$ $m \geq 3:\ 0$

3.26, namely Examples, 3.27, 3.28, 3.29, and 3.30. The Legendrizations of the rest of the classes in (4.2) are computed using the same method. We recall that the relevant exponents for the Legendrizations as stated in the table arise out of the computations

$$E_V(1,2) = (2,3),\ E_{VT}(1,2) = (3,4),\ E_{VV}(1,2) = (3,5),$$
$$E_{VTT}(1,2) = (4,5),\ E_{VVT}(1,2) = (4,7),\ E_{VTTT}(1,2) = (5,6).$$

To get the Legendrization of the classes

$$R^sVTR^mV,\ R^sVR^mV,\ R^sVR^mVT,\ R^sVR^mVV,\ R^sVVR^mV,\ VTTR^mV$$

apply Theorem 3.36 to the results from (4.2). To get the Legendrization of the remaining class $R^sVR^mVR^nV$ apply Theorem 3.36 to the Legendrization of the class R^sVR^mV. □

The plane curve germs in Tables 4.3 and 4.4 have the form $c = (x(t), y(t))$ with $x = t^{\lambda_0}$, $y = a_1t^{\lambda_1} + a_2t^{\lambda_2} + \cdots$, where $\lambda_0 < \lambda_1 < \cdots$. It is clear that the order of good parameterization of c is equal to λ_{i^*}, where i^* is the minimal of numbers i such that the numbers $\lambda_1, ..., \lambda_i$ have no common factor. It is also clear that the order of good parameterization of the one-step prolongation of c is equal to $\lambda_{i^*} - \lambda_0$. These observations and Proposition 4.9 imply the following theorem.

THEOREM 4.10. *The number d given in the fourth column of Tables 4.3 and 4.4 is the parameterization number of the RVT class for that row.*

4.4. Local simplicity of RVT classes

In this section we explain the number q^* in the last column of Tables 4.3, 4.4. We need the following definition.

DEFINITION 4.11. A singularity class $Q \subset \mathbb{P}^k\mathbb{R}^2$ is locally simple at a point $p \in Q$ if there exists a neighbourhood U of p in $\mathbb{P}^k\mathbb{R}^2$ such that the set $U \cap Q$ is covered by a finite number of orbits. Similarly, a singularity class (or a subset) $S \subset j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ is locally simple at a point $\xi \in S$ if there exists a neighbourhood U of ξ in $j^r\,\mathrm{Leg}(\mathbb{P}^1\mathbb{R}^2)$ such that the set $U \cap S$ is covered by a finite number of orbits.

In terms of local simplicity of an RVT classes we can give the following necessary condition for the stage-simplicity of a point in the Monster.

PROPOSITION 4.12. *If a point p in the Monster belongs to the closure of a non-empty singularity class Q which is not locally simple at any its point then p is not simple.*

PROOF. Any neighbourhood U of p contains a point $\tilde{p} \in Q$. Take a neighbourhood $\widetilde{U}$ of $\tilde{p}$ such that $\widetilde{U} \subset U$. The neighbourhood $\widetilde{U} \cap Q$ and consequently the neighbourhood U is not covered by a finite number of orbits. □

REMARK 4.13. A singularity class can be locally simple at every one of its points, yet each of these points can fail to be simple. Indeed this property holds for the classes $Q = (\omega)$ where (ω) is any entirely critical class having 8 or more letters. For every entirely critical class consists of a single orbit (Theorem 3.20), so that its points are trivially locally simple. But since (ω) has 8 or more letters it has codimension greater than 7 and so none of its points are simple (Corollary 4.7). Proposition 4.12 is the reason underlying the failure of simplicity of (ω) (and underlying Corollary 4.7): (ω) adjoins a non-empty RVT class (one of our fencing classes) which is not locally simple at any its point.

REMARK 4.14. If a singularity class in the Monster consists of a finite number of orbits then obviously it is locally simple at any its point. On the other hand, there exist singularity classes which are locally simple at some points, but contain a piece which is comprised of a continuum of orbits. For example, this is situation for the singularity class consisting of the entire monster at level 12. Conjecturally, such "mixed structures" cannot occur within a single RVT class, i.e. an RVT class either is locally simple at any its point or every relative neighbhorhood of every one of its point intersects a continuum of orbits. This statement will not be used in this chapter.

THEOREM 4.15. *Let (α) be one of the RVT classes in the first column of Tables 4.3, 4.4. Let q^* be the number in the last column, the row of (α).*

(1) *If $q^* < \infty$ and $0 \leq q \leq q^*$ then the class (αR^q) consists of a finite number of orbits. If $q^* = \infty$ then (αR^q) consists of a finite number of orbits for any $q \geq 0$.*

(2) *If $q^* < \infty$ and $q > q^*$ then the class (αR^q) is not locally simple at any its points.*

REMARK 4.16. If $\alpha = R^s V R^m V R^n V$ (the last row of Table 4.3) and $s \geq 1$, $m \geq 1, n \geq 3$ then already the class (α) is not locally simple at any its point. To see this, look at the 4th row of the same Table. From Theorem 4.15 it follows that $R^s V R^m V R^n$, with s, m, n in the same range, is not locally simple at any of its points. Now apply Theorem 3.20, i.e the method of critical sections.

To prove Theorem 4.15 we need the following statement which is proved in Appendix B where we present results on RL-contact classification of Legendrian curves.

THEOREM 4.17. *Fix a row in one of the Tables 4.3, 4.4. Let P be the set of plane curve germs given in the third column and let P^1 be the one-step prolongation of P. Let d and q^* be the integers in the last two columns.*

(1) *If $q^* < \infty$ and $0 \leq q \leq q^*$ then the set $j^{d+q-1}(P^1)$ intersects a finite number of orbits with respect to RL-contact equivalence. If $q^* = \infty$ then*

this set intersects a finite number of orbits with respect to RL-contact equivalence for any $q \geq 0$.

(2) *If* $q^* < \infty$ *and* $q > q^*$ *then the set* $j^{d+q-1}(P^1)$ *is not locally simple at any its point.*

Theorem 4.15 follows from Theorem 4.17 and the following corollaries of our results in Chapters 2 and 3.

THEOREM 4.18.

(1) *Let* (α) *be a regular RVT class with the jet-identification number* r. *If the class* $j^r\,\mathrm{Leg}(\alpha)$ *consists of a finite number of orbits with respect to the RL-contact equivalence then the class* (α) *also consists of a finite number of orbits.*

(2) *Let* (α) *be as in the previous statement. If the class* $j^r\,\mathrm{Leg}(\alpha)$ *is not locally simple at any of its point then the class* (α) *is not locally simple at any of its point.*

(3) *Let* $(\alpha) = (\widehat{\alpha}\omega)$ *be a critical RVT class where* $\widehat{\alpha}$ *is a regular class and* (ω) *is an entirely critical class. (See Definition 3.19). Let* r *be the jet-identification number of the class* $\widehat{\alpha}$. *If the class* $j^r\,\mathrm{Leg}(\widehat{\alpha})$ *consists of a finite number of orbits with respect to RL-contact equivalence then the class* (α) *consists of a finite number of orbits.*

PROOF. The first statement is a direct corollary of Theorem 2.24. The second statement is a direct corollary of Theorem 2.24 and Theorem 3.48 on the continuity of the Monsterization within a fixed RVT class. The third statement is a direct corollary of Theorems 2.24 and 3.20. □

PROOF OF THEOREM 4.15. By Theorem 4.10 and Theorem 3.47 the class (αR^q), $q \geq 1$, has jet-determination number $d + q - 1$, so for the case $q \geq 1$ Theorem 4.15 is a direct corollary of Theorem 4.17 and the first two statements of Theorem 4.18. The case $q = 0$ concerns only the first statement of Theorem 4.15 . To deal with this case, note that the classes of the form $R^s\omega$, ω entirely critical, which occur in the table consist of a single orbit according to the method of critical sections (Theorem 3.20.) All but one of the remaining classes have the form $R^s\omega R^q\tilde{\omega}$ with $\omega, \tilde{\omega}$ entirely critical. For these classes we can use the method of critical sections to reduce the case $q = 0$ in the first statement of our Theorem, Theorem 4.15, to the case $q > 0$ in the last statement of the Theorem , but now applied to the class $R^s\omega$. Finally, for the last row, $R^sVR^mVR^nV$, use the same trick, with n playing the role of q. (See also Remark 4.16 regarding this last row.)

4.5. Fences of RVT classes

The final ingredient for the proofs of Theorems 4.4 and 4.5 is the notion of a fence. Throughout this section the letters α, β will denote RVT classes. Recall the definition of “adjoin”: α adjoins the class β if $\alpha \subset \bar{\beta}$. A fence is a collection of RVT classes with the property that every class *not* adjoining the fence consists of stage-simple points.

TERMINOLOGY. If $\mathcal{F}$ is a collection of non-empty singularity classes, and α is another singularity class, we will say that α adjoins $\mathcal{F}$ if α adjoins at least one set in $\mathcal{F}$; equivalently, if $\alpha \subset \cup_{F\in\mathcal{F}}\bar{F}$.

DEFINITION 4.19. A fence $\mathcal{F}$ is a collection of regular RVT classes (not necessarily at the same level) satisfying the following two conditions:

(1) Each $F \in \mathcal{F}$ is **not** locally simple at **any** of its point.

(2) If (α) is an RVT class which does not adjoin $\mathcal{F}$ then (α) consists of a finite number of orbits.

If $\mathcal{F}$ satisfies the additional requirement that there are no adjaciences between its classes then we will say that it is a minimal fence. The elements of a fence will be called "fencing classes".

PROPOSITION 4.20. *Let $\mathcal{F}$ be a fence and (α) an RVT class.*

(1) *If (α) adjoins $\mathcal{F}$ then no point of (α) is simple.*

(2) *If (α) does not adjoin $\mathcal{F}$ then every point of (α) is simple.*

PROOF OF THE FIRST STATEMENT OF PROPOSITION 4.20. This follows directly from the first requirement in Definition 4.19 and Proposition 4.12.

To prove the second statement we use the following lemmas.

LEMMA 4.21. *If (α) does not adjoin (β) then $(\alpha) \cap (\bar{\beta}) = \emptyset$, that is, no point of (α) adjoins (β).*

PROOF. The lemma follows immediately from the third statement in Theorem 3.11. $\square$

LEMMA 4.22. *Assume that every RVT class which (α) adjoins consists of a finite number of orbits. Then every point of (α) is simple.*

PROOF. The lemma follows from Lemma 4.21, the fact that each level is the union of a *finite* number of RVT classes. $\square$

PROOF OF THE SECOND STATEMENT OF PROPOSITION 4.20. Denote by $\mathcal{A}$ the finite collection of RVT classes adjoined by (α). Observe that 'adjoins' defines a partial order on the set of RVT classes. It follows that if (α) does not adjoin $\mathcal{F}$ then no class in $\mathcal{A}$ can adjoin $\mathcal{F}$. By the second requirement in Definition 4.19 every class in $\mathcal{A}$ consists of a finite number of orbits. The second statement of Proposition 4.20 now follows from Lemma 4.22.

Now we define a tower fence in order to 'fence off" the tower-simple points. This definition requires first extending the definition of the adjaciency to the full tower setting.

DEFINITION 4.23. Let (α) and $(\tilde{\alpha})$ be RVT classes in the level k and $\tilde{k}$ respectively. We will say that (α) adjoins-by-prolongation $(\tilde{\alpha})$ if the regular prolongation of (α) to the level $s = max(k, \tilde{k})$ adjoins the regular prolongation to the same level of the class $(\tilde{\alpha})$.

In analogy with the terminology introduced at the beginning of this section we can also speak of (α) adjoining-by-prolongation a collection of RVT classes.

Note that in this definition the regular prolongation to the level s of one of the classes (α), $(\tilde{\alpha})$ is the 0-step prolongation, i.e. it coincides with this class. If the classes (α) and $(\tilde{\alpha})$ belong to the same level then "adjoins-by-prolongation" is the same property as "adjoins".

Example. The class VTT in the 8th level of the Monster adjoins-by-prolongation the classes R, V, VR, VT which belong to lower levels. It also adjoins-by-prolongation the classes $VR^{q_1}, VTR^{q_2}, VTTR^{q_3}$ with arbitrary $q_1, q_2, q_3 \geq 0$ which belong to higher level if $q_1 \geq 3, q_2 \geq 2, q_3 \geq 1$. The class VTT does *not* adjoin-by-prolongation any of the classes $RVR^q, VVR^q, q \geq 0$.

Definition 4.24. A tower fence is a collection of critical RVT classes (not necessarily at the same level of the Monster) satisfying :

(1) Every sufficiently long regular prolongation of any class in $\mathcal{F}$ is **not** locally simple at **any** of its point.

(2) If (α) is a critical RVT class which does not adjoin-by-prolongation $\mathcal{F}$ then any regular prolongation of (α) consists of a finite number of orbits.

Proposition 4.25. *Suppose that $\mathcal{F}$ is a tower fence and (α) is a critical RVT class.*

(1) *If (α) adjoins-by-prolongation one of the classes of the tuple $\mathcal{F}$ then no point of any regular prolongation of (α) is tower-simple.*

(2) *If (α) does not adjoin-by-prolongation $\mathcal{F}$ then any point of any regular prolongation of (α) is tower-simple.*

Proof. Assume that (α) is a critical RVT class at level k which adjons-by-prolongation the class (β) at level $\tilde{k}$ of $\mathcal{F}$. Let $m \geq max(k, \tilde{k})$ and $(\hat{\alpha})$, $(\hat{\beta})$ the regular prolongations of (α), (β) to level m so that $(\hat{\alpha})$ adjoins $(\hat{\beta})$. If m is sufficiently big then, according to the first requirement of Definition 4.24, the class $(\hat{\beta})$ is not locally simple at any of its point. By Proposition 4.12 no point $p \in (\hat{\alpha})$ is simple. Therefore every sufficiently long regular prolongation of (α) contains no simple points, and the first statement follows.

Now we prove the second statement. Assume that (α) does not adjoin-by-prolongation $\mathcal{F}$. We have to prove that any point of the class (αR^q) is stage-simple, for any $q \geq 0$. Fix q. Denote by $\mathcal{A}$ the finite collection of RVT classes which (αR^q) adjoins. Any class of $\mathcal{A}$ has the form $(\tilde{\alpha} R^q)$, where (α) adjoins $(\tilde{\alpha})$. The assumption that (α) does not adjoin-by-prolongation $\mathcal{F}$ implies that $(\tilde{\alpha})$ does not adjoin-by-prolongation $\mathcal{F}$. By the second requirement in Definition 4.24 the class $(\tilde{\alpha} R^q)$ consists of a finite number of orbits. We have proved that any RVT class in $\mathcal{A}$ consists of a finite number of orbits. By Lemma 4.22 any point of (αR^q) is simple. □

4.6. Proof of Theorem 4.4

The proof is based on the construction of a tower fence.

Theorem 4.26. *Let $\mathcal{F}$ be the collection of RVT classes consisting of the class $VTTT$ and all critical RVT classes of codimension 3 except VTT and VVT. Then $\mathcal{F}$ is a tower fence.*

To prove Theorem 4.26 and to show that this theorem implies Theorem 4.4 we need the following lemma.

Lemma 4.27. *Any critical RVT class not in the list (4.1) adjoins-by-prolongation $\mathcal{F}$ of Theorem 4.26.*

PROOF. The list (4.1) of Theorem 4.4 includes all RVT classes of codimension ≤ 2. Therefore we have to show that any critical RVT class (α) of codimension ≥ 4 adjoins-by-prolongation either the class $VTTT$ or one of the critical RVT classes of codimension 3 besides VTT, VVT. The proof is by cases. If the code of (α) starts with R then (α) adjoins-by-prolongation a critical codimension 3 class whose code starts with R. The same holds if the code of (α) starts with VR or VV. If the code of (α) starts with VT, but not with VTT then (α) adjoins-by-prolongation a critical codimension 3 class whose code starts with VR. If the code of (α) starts with VTT, but not with $VTTT$ then (α) adjoins-by-prolongation a critical codimension 3 class whose code starts with VTR. In the remaining case that the code of (α) starts with $VTTT$ the class (α) adjoins-by-prolongation the class $VTTT$. □

FROM THEOREM 4.26 TO THEOREM 4.4. Lemma 4.27 can be reformulated as follows: if (α) is a critical RVT class which does not adjoin-by-prolongation $\mathcal{F}$ of Theorem 4.26 then (α) belongs to the list (4.1) of Theorem 4.4. Theorem 4.4 follows from this statement, Theorem 4.26, and Proposition 4.25.

PROOF OF THEOREM 4.26. In view of Lemma 4.27, to prove that $\mathcal{F}$ is a tower fence we must prove the following two statements:

1. Let $(\alpha) = VTTT$ or let (α) be a critical RVT class of codimension 3 except VTT or VVT. Then for sufficiently big q the class (αR^q) is not locally simple at any of its points.

2. Let (α) be any of the classes of the list (4.1). Then any regular prolongation of (α) consists of a finite number of orbits.

These statements are direct corollaries of Theorem 4.15, Table 4.3 (listing all critical classes of codimension ≤ 3), and the first row of Table 4.4 concerning the class $VTTT$. Indeed for (α) a class from Table 4.3 or Table 4.4 and $q^* = q^*(\alpha)$ the number in the last column of the row of (α), we have the fact that $q^*(\alpha) = \infty$ if and only if (α) is one of the critical classes of our list (4.1) of Theorem 4.4 .

4.7. Proof of Theorem 4.5

Theorem 4.5 is a direct corollary of Proposition 4.20 and the following statement.

THEOREM 4.28. *The classes listed in Tables 4.1 and 4.2 form a fence.*

REMARK 4.29. This fence is not a minimal fence. For example, the class $RVTRRVRRR$ (the 4th row of Table 4.1) adjoins the class $RVRRRVRRR$ (the first row of Table 4.1). It is easy to remove some from the fence to obtain a minimal fence, but using a minimal fence instead our fence would complicate the proof of Theorem 4.28.

Theorem 4.28 follows immediately from the definition of a fence and the following two statements.

THEOREM 4.30. *The RVT class from Table 4.1 or Table 4.2 are not locally simple at any of their point.*

THEOREM 4.31. *If the RVT class (α) does not adjoin any of the classes listed in Tables 4.1 and 4.2 then (α) consists of finitely many orbits.*

Proof of Theorem 4.30. Theorem 4.30 is a direct corollary of Theorem 4.15, and and the fact that every class (α) in Table 4.1 and 4.2 is a regular prolongation $(\alpha) = (\beta R^q)$ of a critical class (β) from Table 4.3 or Table 4.4 for which q is bigger than the number q^* in the last column of the table, the row of (β). To see the validity of this relation between the classes of the two sets of tables, first note that Table 4.3 is a complete list critical classes of codimension ≤ 3 and that all but 3 of the classes listed in Tables 4.1 and 4.2 have codimension ≤ 3. The 3 classes not so covered occur at the bottom of Table 4.2, have codimension 4 and are regular prolongations (βR^q) of classes (β) from Table 4.4 with q bigger than the number q^* from the last column of that Table.

The rest of this section is devoted to the proof of Theorem 4.31.

Claim 4.32. It suffices to prove Theorem 4.31 for the case that the class (α) is regular.

Proof. Take a crtical RVT class (α). If its code contains no R then (α) consists of a single orbit (the first statement of Theorem 3.20). If (α) contains at least one R then it has the form $(\alpha) = (\widehat{\alpha}\omega)$, where ω is entirely critical class and $(\widehat{\alpha})$ is a regular class. Assume that (α) does not adjoin any of the classes of Tables 4.1, 4.2. Since these tables contain along with an RVT class any its regular prolongation then $\widehat{\alpha}$ does not adjoin any of the classes in Tables 4.1, 4.2. Assume that Theorem 4.31 is proved for regular classes. Then the class $(\widehat{\alpha})$ consists of a finite number of orbits. By Theorem 3.20 the class (α) consists of exactly the same number of orbits. $\square$

Claim 4.33. Let (α) be a regular RVT class of codimension ≤ 3. If (α) does not adjoin any of the classes in Tables 4.1 or 4.2 then (α) consists of a finite number of orbits.

Proof. If (α) is a regular RVT class of codimension ≤ 3 and $(\alpha) \neq (RR...R)$ then $(\alpha) = (\beta R^q)$, where (β) is a critical class of Table 4.3. It is easy to check that if (α) does not adjoin any of the classes in Tables 4.1, 4.2 then q is not bigger than the number q^* in the last column of Table 4.3, the row of β. Now the claim follows from Theorem 4.15. $\square$

Claims 4.32 and 4.33 reduce Theorem 4.31 to the following statement.

Theorem 4.34. *Let (α) be a regular RVT class of codimension ≥ 4. If (α) does not adjoin any of the classes of Tables 4.1, 4.2 then (α) consists of a finite number of orbits.*

The proof of Theorem 4.34 is based on the analysis of the structure RVT classes satisfying the assumption of this theorems.

Proposition 4.35. *Let (α) be a regular RVT class of codimension ≥ 4. If (α) does not adjoin any of the classes of Tables 4.1, 4.2 then (α) has one of the forms below, where ω and $\widetilde{\omega}$ denote arbitrary entirely critical classes.*

$$R^{\geq 0}\omega R; \tag{4.3}$$

$$R^{\geq 0}\omega RR; \tag{4.4}$$

$$R^{\geq 0}\omega R\widetilde{\omega}R; \tag{4.5}$$

$$R^{\geq 0}VR^{\geq 1}\omega R,\ R^{\geq 1}VVR^{\leq 3}\omega R; \tag{4.6}$$

$$VTR^{\geq 1}\omega R,\quad VVR^{\geq 1}\omega R,\ VTTR^{\leq 3}\omega R,\ VVTR^{\leq 2}\omega R; \tag{4.7}$$

$$VTTRVRR,\quad VTTTRRR. \tag{4.8}$$

REMARK 4.36. In Proposition 4.35 "if" cannot be replaced by "if and only if". Some of the classes (4.3)-(4.8) do adjoin classes from Tables 4.1 or 4.2.

The proof of Proposition 4.35 is based on a straightforward case-by-case study involving the grammar of the letters R, V, T. It is given in the next section.

CLAIM 4.37. The classes (4.8) consist of a finite number of orbits.

PROOF. Follows from Theorem 4.15 and Table 4.4. In fact, the number q^* in the last column of Table 4.4, the row of the class VTTRV (resp. VTTT) equals 2 (resp. 3). □

LEMMA 4.38. *Any RVT class of form (4.3) consists of a single orbit. Any RVT class of form (4.4) consists of ≤ 2 orbits.*

PROOF. Fix s and fix an entirely critical class ω. By Theorem 3.26 the Legendrization of the class $R^s\omega$ consists of Legendrian curve germs equivalent to the one-step prolongation of plane curve germs of the form

$$(t^{\lambda_0}, t^{\lambda}f(t)), f(0) \neq 0, \tag{4.9}$$

with certain relatively prime numbers λ_0, λ_1 such that $\lambda_0 > 1$ and $\lambda > 2\lambda_0$. The order of good parameterization of such plane curve germs is equal to λ, and the order of good parameterization of their one-step prolongation is equal to $\lambda - \lambda_0$. Therefore the parameterization number of the class $R^s\omega$ is equal to $\lambda - \lambda_0$. By Theorem 3.47 the jet-identification number of the class $R^s\omega R$ is equal to $\lambda - \lambda_0$ and the jet-identification number of the class $R^s\omega RR$ is equal to $\lambda - \lambda_0 + 1$. The $(\lambda - \lambda_0)$-jet of the one-step prolongations of curves of form (4.9) are represented by the space curve of the form

$$x = t^{\lambda_0},\ y = 0,\ u = \kappa t^{\lambda - \lambda_0},\ \kappa \in \mathbb{R},\ \kappa \neq 0, \tag{4.10}$$

where the contact structure on $\mathbb{P}^1\mathbb{R}^2$ is described by the 1-form $dy - udx$. The $(\lambda - \lambda_0 + 1)$-jet of the one-step prolongations of curves of form (4.9) are represented by the space curve of the form

$$x = t^{\lambda_0},\ y = 0,\ u = \kappa_1 t^{\lambda - \lambda_0} + \kappa_2 t^{\lambda - \lambda_0 + 1},\ \kappa_1, \kappa_2 \in \mathbb{R},\ \kappa_1 \neq 0. \tag{4.11}$$

Now Lemma 4.38 follows from Theorem 2.24 and the following observation: a scale contactomorphism

$$x \to k_1 x,\quad y \to k_2 y,\quad u \to (k_2/k_1)u,\quad t \to k_3 t \tag{4.12}$$

with suitable k_1, k_2, k_3 reduces the parameter κ in (4.10) to 1 and the couple of parameters (κ_1, κ_2) in (4.11) to $(1, 1)$ or to $(1, 0)$. In fact, κ reduces to 1 by the scale contactomorphism (4.12) with $k_1 = k_3 = 1,\ k_2 = \kappa$. If in (4.11) one has $\kappa_2 = 0$ then the couple $(\kappa_1, \kappa_2) = (\kappa_1, 0)$ reduces to $(1, 0)$ by the same scale contactomorphism. If $\kappa_2 \neq 0$ then the couple (κ_1, κ_2) reduces to $(1, 1)$ by the scale contactomorphism (4.12) with

$$k_1 = \left(\frac{\kappa_1}{\kappa_2}\right)^{\lambda_0},\quad k_2 = \kappa_1 \cdot \left(\frac{\kappa_1}{\kappa_2}\right)^{\lambda},\quad k_3 = \frac{\kappa_1}{\kappa_2}.$$

□

Now, to prove Theorem 4.5 it remains to prove

LEMMA 4.39. *Each of the classes (4.5)-(4.7) consists of a finite number of orbits.*

This lemma follows from the following more general statement.

PROPOSITION 4.40. *Let* $(\widehat{\alpha})$ *be an RVT class of the form*

$$(\widehat{\alpha}) = R^s\omega R^q\widetilde{\omega}R, \quad s \geq 0, \ q \geq 1, \ (\omega), (\widetilde{\omega}) \text{ are entirely critical classes.}$$

If the class $(R^s\omega R^q)$ *consists of a single orbit then the class* $(\widehat{\alpha})$ *consists of* ≤ 2 *orbits.*

FROM PROPOSITION 4.40 TO LEMMA 4.39. For classes (4.5) Lemma 4.39 is a direct corollary of Proposition 4.40 and Lemma 4.38 for classes (4.3). For classes (4.6) and (4.7) Lemma 4.39 is a corollary of Proposition 4.40 and the following statement:

each of the classes $R^{\geq 0}VR^{\geq 1}, \ R^{\geq 1}VVR^{\leq 3}, VTR^{\geq 1}, \ VVR^{\geq 1}, \ VTTR^{\leq 3}, \ VVTR^{\leq 2}$ *consists of a single orbit.*

This statement is a part of Theorem 4.41 on the classification of tower-simple points which is formulated and proved in the next section independently of any of the results in this section. Note that the classes in the list above consists of tower-simple points, by Theorem 4.4.

PROOF OF PROPOSITION 4.40. By Theorem 3.26 the Legendrization of the class $(R^s\omega)$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongation of plane curve germs of the form (4.9) with certian relatively prime λ_0, λ_1 such that $\lambda_0 > 1$ and $\lambda > 2\lambda_0$. The parameterization number of the class $(R^s\omega)$ is equal to $\lambda - \lambda_0$, see the proof of Lemma 4.38. By Theorem 3.47 the jet-identification number of the class $(R^s\omega R^q)$ is equal to $\lambda - \lambda_0 + q - 1$. By the assumption that all points of the class $(R^s\omega R^q)$ are equivalent and by Theorem 2.24 the set of $(\lambda - \lambda_0 + q - 1)$-jets of one step prolongations of plane curves (4.9) is covered by a single orbit with respect to RL-contact equivalence. It follows that the Legendrization of the class $(R^s\omega)$ can be characterized as follows: it consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form

$$\left(t^{\lambda_0}, \ t^{\lambda} + t^{\lambda+q}h(t)\right),$$

where $h(t)$ is an arbitrary function germ. Now we use Theorem 3.36 to construct the Legendrization of the class $(R^s\omega R^s\widetilde{\omega})$. Let $(a,b) = E_{\widetilde{\omega}}(1,2)$. Then the Legendrization of $(R^s\omega R^s\widetilde{\omega})$ consists of Legendrian curve germs RL-contact equivalent to the one-step prolongations of plane curve germs of the form

$$x = t^{a\lambda_0}, \ \ y = t^{a\lambda} + t^{a(\lambda+q)}h(t^a) + t^{\tilde{\lambda}}g(t), \ \ \tilde{\lambda} = a(\lambda+q-1)+b-a, \ \ g(0) \neq 0. \tag{4.13}$$

The parameterization number of the one-step prolongations of curves (4.13) is equal to $\tilde{\lambda} - \lambda_0$, see Remark 3.37. By Theorem 3.47 the jet-identification number of the class $(\widehat{\alpha}) = (R^s\omega R^q\widehat{\omega}R)$, also equals $\tilde{\lambda} - \lambda_0$. Now Theorem 2.24 reduces Proposition 4.40 to the following statement:

the set of $(\tilde{\lambda} - \lambda_0)$-jets of the one-step prolongations of curves (4.13) intersects ≤ 2 orbits with respect to RL-contact equivalence.

To prove this statement observe that if ω is an entirely critical class and $(a,b) = E_\omega(1,2)$ then $b<2a$. This observation easily follows from the fact that ω starts with V, not with T. Therefore in (4.13) one has $b<2a$ and consequently $a(\lambda+q)>\tilde{\lambda}$. It follows that the part $t^{a(\lambda+q)}h(t^a)$ in (4.13) does not play role in calculation the set of $(\tilde{\lambda}-\lambda_0)$-jets of the one-step prolongations of curves (4.13) and this set of jets is represented by by the space curve of the form

$$x = t^{a\lambda_0}, \quad y = 0, \quad u = \frac{\lambda}{\lambda_0}\cdot t^{a(\lambda-\lambda_0)} + \kappa t^{\tilde{\lambda}}, \quad \kappa \in \mathbb{R}, \ \kappa \neq 0,$$

where the contact structure on $\mathbb{P}^1\mathbb{R}^2$ is described by the 1-form $dy - udx$. It remains to note that κ can be reduced to 1 or to -1 by a scale contactomorphism (4.12) with suitable k_1, k_2.

4.8. Proof of Proposition 4.35

4.8.1. Reduction to the case of ≤ 4 letters V. We begin by reducing to the case where α contains fewer than 5 letters V. If (α) contains 5 Vs or more, then it adjoins a fencing class. To see this, first suppose that such an (α) starts with R. Replace all Vs and Ts by an R except for the first and third occurrence of V. We get a class of the form $R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$ which is the first entry in the fencing class Table 4.1. And if (α) starts with V then by replacing all Vs and Ts except the first, third and fifth V by an R we see that α adjoins a fencing class of the form $VR^{\geq 1}VR^{\geq 1}VR^{\geq 1}$ which occurs as the first entry in the fencing class Table 4.2.

Therefore Proposition 4.35 has to be proved for classes (α) containing one, two, three, or four letters V.

4.8.2. The case of one V. In the case of one V an RVT class satisfying the assumptions of Proposition 4.35 then it has the form

$$(\alpha) = R^s VT^m R^q, \quad m \geq 3.$$

The proof for classes of such form follows from the following observations:

1. If $q=1$ or $q=2$ the class (α) has form (4.3) or (4.4).
2. If $q \geq 3$ and $s \neq 0$ then (α) adjoins a fencing class of the form $R^{\geq 1}VTTR^{\geq 3}$ which occurs as the second entry in the fencing class Table 4.1.
3. If $q \geq 4, s=0$ or $q=3, m\geq 4, s=0$ then (α) adjoins a fencing class of the form $VTTTR^{\geq 4}$ of Table 4.2.
4. In the remaining case $q=3, m=3, s=0$ the class (α) is one of the classes (4.8).

4.8.3. The case of 2,3,or 4 letters V: organization of the tables. We have reduced Proposition 4.35 to the case that the class (α) contains two, three, or four letters V. For such classes the case-by-case analysis becomes much more branched and the proof is given in Tables 4.5 - 4.10. The tables are organized as follows:

- If we see that a class (α) belongs to one of the groups (4.3)-(4.8) then this group is pointed out in the third column; the second column remains empty. This does *not* mean that (α) does not adjoin any of the fencing classes of Table 4.1 or 4.2.

- If we see that a class (α) adjoins one of the fencing classes of Table 4.1 or 4.2 then we point out this fencing class in the second column; the third column remains empty. This does *not* mean that (α) does not belong to one of the groups (4.3)-(4.8).

Tables 4.5 - 4.10 contain all regular RVT classes of codimension ≥ 4 with two, three, or four letters V. By these tables any such class (α) either belongs to one of the groups (4.3)-(4.8) or adjoins one of the fencing classes of Table 4.1 or 4.2. These two possibilities might hold simultaneously, but we do not care about that: Proposition 4.35 for class (α) is a logical corollary of the fact that at least one of these two possibilities holds.

4.8.4. The case of two Vs. A regular class of codimension ≥ 4 starting with R has one of the forms, depending on the first letter of its code, R or V:

$$R^{\geq 1}VT^{m_1}R^nVT^{m_2}R^{q\geq 1}, \quad m_1, m_2, n \geq 0,\ m_1 + m_2 \geq 2. \tag{4.14}$$

$$VT^{m_1}R^nVT^{m_2}R^{q\geq 1}, \quad m_1, m_2, n \geq 0,\ m_1 + m_2 \geq 2. \tag{4.15}$$

The proof of Proposition 4.35 for the classes of form (4.14) is contained in Table 4.5, and for the classes of form (4.15) - in Table 4.6.

Table 4.5. Proof of Proposition 4.35 for the classes of the form (4.14): $R^{\geq 1}VT^{m_1}R^nVT^{m_2}R^{q\geq 1}, \quad m_1, m_2, n \geq 0,\ m_1 + m_2 \geq 2$

Indeces	Adjoins a fencing class of the form	Belongs to the group
$q \geq 3,\ (m_1, n) \neq (0,0)$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$q \geq 3,\ m_1 = n = 0$	$R^{\geq 1}VVTR^{\geq 4}$	
$q = 2,\ m_2 \neq 0,\ (m_1, n) \neq (0,0)$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$q = 2,\ m_1 = n = 0$	$R^{\geq 1}VVR^{\geq 4}$	
$q = 2,\ m_2 = 0$	$R^{\geq 1}VTR^{\geq 1}VR^{\geq 2}$	
$q = 1,\ m_1 \geq 3$	$R^{\geq 1}VTR^{\geq 2}VR^{\geq 1}$	
$q = 1,\ m_1 = 2,\ n \neq 0$	$R^{\geq 1}VTR^{\geq 2}VR^{\geq 1}$	
$q = 1,\ m_1 = 2,\ n = 0$		(4.3)
$q = 1,\ m_1 = 1,\ n \geq 2$	$R^{\geq 1}VTR^{\geq 2}VR$	
$q = 1,\ m_1 = 1,\ n = 1$		(4.5)
$q = 1,\ m_1 = 1,\ n = 0$		(4.3)
$q = 1,\ m_1 = 0$		(4.6)

TABLE 4.6. Proof of Proposition 4.35 for the classes of the form (4.15): $VT^{m_1}R^nVT^{m_2}R^{q\geq 1}$, $m_1, m_2, n \geq 0,\ m_1+m_2 \geq 2$

Indeces	Adjoins a fencing class of the form	Belongs to the group
$q \geq 3,\ m_1 \neq 0,\ n \neq 0$	$VTR^{\geq 1}VR^{\geq 3}$	
$q \geq 3,\ m_1 \neq 0,\ n = 0,\ m_2 \neq 0$	$VR^{\geq 1}VTR^{\geq 2}$	
$q \geq 3,\ m_1 \neq 0,\ n = 0,\ m_2 = 0$	$VTR^{\geq 1}VR^{\geq 3}$	
$q \geq 3\ m_1 = 0,\ n \neq 0$	$VR^{\geq 1}VTR^{\geq 2}$	
$q \geq 3,\ m_1 = 0,\ n = 0$	$RVTTR^{\geq 3}$	
$q = 2,\ n \neq 0,\ m_1 \neq 0,\ m_2 \neq 0$	$VTR^{\geq 1}VR^{\geq 3}$	
$q = 2,\ n \neq 0,\ m_1 \geq 3,\ m_2 = 0$	$VTTR^{\geq 2}VR^{\geq 2}$	
$q = 2,\ n \geq 2,\ m_1 = 2,\ m_2 = 0$	$VTTR^{\geq 2}VR^{\geq 2}$	
$q = 2,\ n = 1,\ m_1 = 2,\ m_2 = 0$		(4.8)
$q = 2,\ n \neq 0,\ m_1 = 0$	$VR^{\geq 1}VTR^{\geq 2}$	
$q = 2,\ n = 0$		(4.4) or (4.3)
$q = 1,\ m_1 \geq 3,\ (n, m_2) \neq (0,0)$	$VTTTR^{\geq 4}$	
$q = 1,\ m_1 \geq 3,\ n = m_2 = 0$		(4.3)
$q = 1,\ m_1 = 2,\ n \geq 3$	$VTTR^{\geq 3}VR^{\geq 1}$	
$q = 1,\ m_1 = 2,\ n \leq 2$		(4.7)
$q = 1,\ m_1 = 1$		(4.7) if $n \neq 0$; (4.3) if $n = 0$
$q = 1,\ m_1 = 0$		(4.6) if $n \neq 0$; (4.3) if $n = 0$

4.8.5. The case of three Vs.

4.8.5.1. *The code starts with R.* A regular RVT class of codimension ≥ 4 containing three Vs and starting with R has the form

$$(\alpha) = R^{\geq 1}VT^{m_1}R^{n_1}VT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}, \quad (m_1, m_2, m_3) \neq (0,0,0).$$

If $(m_1, n_1) \neq (0,0)$ and $(m_2, n_2, m_3) \neq (0,0,0)$ then (α) adjoins a fencing class of the form $R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$. Therefore we have to consider the case $m_2 = n_2 = m_3 = 0$, when

$$(\alpha) = R^{\geq 1}VT^{m_1}R^{n_1}VVR^{q\geq 1}, \quad m_1 \neq 0, \tag{4.16}$$

and the case $m_1 = n_1 = 0$, when

$$(\alpha) = R^{\geq 1}VVT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}, \quad (m_2, m_3) \neq (0,0). \tag{4.17}$$

In the case (4.16) with $q \geq 2$ then (α) adjoins a fencing class of the form $R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$. If $q = 1, n_1 \neq 0$ then (α) adjoins a fencing class of the form $R^{\geq 1}VTR^{\geq 1}VR^{\geq 2}$. In the remaining case $q = 1, n_1 = 0$ the class (α) has form (4.3).

In the case (4.17) the proof of Proposition 4.35 is contained in Table 4.7.

TABLE 4.7. Proof of Proposition 4.35 for the classes of the form (4.17): $R^{\geq 1}VVT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}$, $(m_2, m_3) \neq (0, 0)$

Indeces	Adjoins a fencing class of the form	Belongs to the group
$q \geq 3$ or $q = 2, m_3 \neq 0$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$q = 2,\ m_3 = 0,\ n_2 \neq 0$	$R^{\geq 1}VVTR^{\geq 4}$	
$q = 2,\ m_3 = 0,\ n_2 = 0$		(4.4)
$q = 1,\ m_2 \geq 3$ or $q = 1,\ m_2 = 2,\ n_2 \neq 0$	$R^{\geq 1}VVTR^{\geq 4}$	
$q = 1,\ m_2 = 2,\ n_2 = 0$		(4.3)
$q = 1,\ m_2 = 1,\ n_2 + m_3 \geq 2$	$R^{\geq 1}VVTR^{\geq 4}$	
$q = 1,\ m_2 = 1,\ n_2 = 0$		(4.3)
$q = 1,\ m_2 = 1,\ n_2 = 1,\ m_3 = 0$		(4.5)
$q = 1,\ m_2 = 0,\ n_2 \geq 3$	$R^{\geq 1}VVR^{\geq 3}VR^{\geq 1}$	
$q = 1,\ m_2 = 0,\ n_2 \leq 2$		(4.6)

4.8.5.2. *The code starts with V.* A regular RVT class of codimension ≥ 4 containing three Vs and starting with V has the form

$$(\alpha) = VT^{m_1}R^{n_1}VT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}, \quad (m_1, m_2, m_3) \neq (0, 0, 0).$$

If $(m_1, n_1) \neq (0, 0)$ and $(m_2, n_2) \neq (0, 0)$ then (α) adjoins a fencing class of the form $VRVRVR^{\geq 1}$. Therefore we have to consider the case $m_2 = n_2 = 0$ when (α) has the form

$$VT^{m_1}R^{n_1}VVT^{m_3}R^{q\geq 1}, \quad (m_1, m_3) \neq (0, 0), \tag{4.18}$$

and the case $m_1 = n_1 = 0$ when (α) has the form

$$VVT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}, \quad (m_2, m_3) \neq (0, 0). \tag{4.19}$$

The proof of Proposition 4.35 for the case (4.18) is given in Table 4.8 and for the case (4.19) - in Table 4.9.

TABLE 4.8. Proof of Proposition 4.35 for the classes of the form (4.18): $VT^{m_1}R^{n_1}VVT^{m_3}R^{q\geq 1}$, $(m_1, m_3) \neq (0,0)$

Indeces	Adjoins a fencing class of the form	Belongs to the group
$q \geq 2,\ m_3 \geq 1$	$VR^{\geq 1}VTR^{\geq 2}$	
$q \geq 2,\ m_3 = 0,\ n_1 \neq 0$	$VTR^{\geq 1}VR^{\geq 3}$	
$q \geq 2,\ m_3 = 0,\ n_1 = 0,\ m_1 \geq 2$	$VTR^{\geq 1}VR^{\geq 3}$	
$q \geq 3,\ m_3 = 0,\ n_1 = 0,\ m_1 = 1$	$VTR^{\geq 1}VR^{\geq 3}$	
$q = 2,\ m_3 = 0,\ n_1 = 0,\ m_1 = 1$		(4.4)
$q = 1,\ m_1 \geq 3,\ n_1 \geq 1$	$VTTTR^{\geq 4}$	
$q = 1,\ m_1 \geq 3,\ n_1 = 0$		(4.3)
$q = 1,\ m_1 = 2,\ n_1 \geq 2$	$VTTR^{\geq 2}VR^{\geq 2}$	
$q = 1,\ m_1 = 2,\ n_1 = 1$		(4.5)
$q = 1,\ m_1 = 2,\ n_1 = 0$		(4.3)
$q = 1,\ m_1 = 1$		(4.7) or (4.3)
$q = 1,\ m_1 = 0$		(4.6) or (4.3)

TABLE 4.9. Proof of Proposition 4.35 for the classes of the form (4.19): $VVT^{m_2}R^{n_2}VT^{m_3}R^{q\geq 1}$, $(m_2, m_3) \neq (0,0)$

Indeces	Adjoins a fencing class of the form	Belongs to the group
$q \geq 3,\ (m_2, n_2) \neq (0,0)$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$q \geq 3,\ m_2 = n_2 = 0$	$VR^{\geq 1}VTR^{\geq 2}$	
$q = 2,\ (m_2, n_2) \neq (0,0),\ m_3 \neq 0$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$q = 2,\ m_2 = n_2 = 0$		(4.4)
$q = 1,\ m_2 \geq 3$	$RVTR^{\geq 2}VR^{\geq 1}$	
$q = 1,\ m_2 = 2,\ n_2 \neq 0$	$RVTR^{\geq 2}VR^{\geq 1}$	
$q = 1,\ m_2 = 2,\ n_2 = 0$		(4.3)
$q = 1,\ m_2 = 1,\ n_2 \geq 2$	$RVTR^{\geq 1}VR^{\geq 1}$	
$q = 1,\ m_2 = 1,\ n_2 \leq 1$		(4.7) or (4.3)
$q = 1,\ m_2 = 0$		(4.7) or (4.3)

4.8.6. The case of four Vs. A regular RVT class containing four Vs has the form

$$R^sVT^{m_1}R^{n_1}VT^{m_2}R^{n_2}VT^{m_3}R^{n_3}VT^{m_4}R^{q\geq 1}, \quad q_i, m_i \geq 0. \tag{4.20}$$

The proof of Proposition 4.35 for such classes is contained in Table 4.10.

TABLE 4.10. Proof of Proposition 4.35 for the classes of form (4.20)

Indeces	Adjoins a fencing class of the form	Belongs to the group
$s \neq 0,\ (m_1, n_1) \neq (0,0)$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$s \neq 0,\ m_1 = n_1 = 0,$ $(m_3, n_3, m_4, q) \neq (0,0,0,1)$	$R^{\geq 1}VR^{\geq 1}VR^{\geq 3}$	
$s \neq 0,\ m_1 = n_1 = 0,\ m_3 = n_3 = m_4 = 0;$ $q = 1;\ \ m_2 \geq 2$ or $m_2 = 1, n_2 \neq 0$	$R^{\geq 1}VVR^{\geq 2}VR^{\geq 2}$	
$s \neq 0,\ m_1 = n_1 = 0,\ m_3 = n_3 = m_4 = 0,$ $q = 1,\ m_2 = 1,\ n_2 = 0$		(4.3)
$s \neq 0,\ m_1 = n_1 = 0,\ m_3 = n_3 = m_4 = 0,$ $q = 1,\ m_2 = 0,\ n_2 \geq 3$	$R^{\geq 1}VVR^{\geq 3}VR^{\geq 1}$	
$s \neq 0,\ m_1 = n_1 = 0,\ m_3 = n_3 = m_4 = 0,$ $q = 1,\ m_2 = 0,\ n_2 \in \{1, 2\}$		(4.6)
$s \neq 0,\ m_1 = n_1 = 0,\ m_3 = n_3 = m_4 = 0,$ $q = 1,\ m_2 = 0,\ n_2 = 0$		(4.3)
$s = 0,\ (m_1, n_1, m_3, n_3) \neq (0,0,0,0)$	$VR^{\geq 1}VR^{\geq 1}VR^{\geq 1}$	
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,\ q \geq 3$	$RVR^{\geq 1}VR^{\geq 3}$	
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,\ q = 2;$ $m_4 \neq 0$ or $m_4 = 0,\ (m_2, n_2) \neq (0,0)$	$RVR^{\geq 1}VR^{\geq 3}$	
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,\ q = 2;$ $m_4 = 0,\ m_2 = n_2 = 0$		(4.4)
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,$ $q = 1,\ m_2 \geq 2$	$RVTR^{\geq 1}VR^{\geq 2}$	
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,$ $q = 1,\ m_2 = 1,\ n_2 \geq 2$	$RVTR^{\geq 2}VR^{\geq 1}$	
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,$ $q = 1,\ m_2 = 1,\ n_2 = 1$		(4.7)
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,$ $q = 1,\ m_2 = 1,\ n_2 = 0$		(4.3)
$s = 0,\ m_1 = n_1 = m_3 = n_3 = 0,$ $q = 1,\ m_2 = 0,\ n_2 \leq 1$		(4.7) or (4.3)

4.9. Classification of tower-simple points

By Theorem 4.4 a point in the Monster is tower-simple if and only if it belongs to a regular prolongation of one of the classes in Table 4.11, which is a part of Table 4.3.

TABLE 4.11. Determination of tower-simple points. A point is tower-simple if and only if its RVT class is a regular prolongation of a class in the first column. The third column describes a family of plane curves such that any curve in the Legendrization of the class of that row is RL-contact equivalent to the one step prolongation of that plane curve.

Class	codim	Legendrization (see Prop. 4.9), $f(0) \neq 0$	param. number d
R	0	$(t, 0)$	1
$R^sV, s \geq 0$	1	$\big(t^2,\ \ t^{2s+5}, f(t)\big)$	$2s+3$
$R^sVT, s \geq 0$	2	$\big(t^3,\ \ t^{3s+7}f(t)\big)$	$3s+4$
$R^sVV, s \geq 0$	2	$\big(t^3,\ \ t^{3s+8}f(t)\big)$	$3s+5$
$VR^mV, m \geq 1$	2	$\big(t^4, t^{10} + t^{9+2m}f(t)\big)$	$5+4m$
VTT	3	$\big(t^4,\ \ t^9 f(t)\big)$	5
VVT	3	$\big(t^4,\ \ t^{11} f(t)\big)$	7

By Theorem 1.5 any point in the i-th level of the Monster can be represented as the evaluation of the i-fold prolongation of a non-constant plane analytic curve. That curve can be thought of as a 'normal form' for the point. Any regular prolongation of any of the classes in Table 4.11 consists of a finite number of different orbits, and hence have a finite number of different normal form representatives. In Table 4.12 we summarize the normal forms for all tower-simple points. For example, the last entry of the last column specifies three normal form if $q \in \{3, 4, 5, 6\}$ and five normal forms if $q \geq 7$.

THEOREM 4.41. *A tower-simple point in $\mathbb{P}^k\mathbb{R}^2$, $k \geq 3$ is equivalent to one and only one of the points $c^k(0)$, where c is a plane curve germ in the last column of Table 4.12, k is the number given in the third column, with the row of those entries stating the RVT class of that point.*

The first row in Table 4.12 corresponds to Theorem 2.10 from section 2.2.1: any non-singular point in the Monster is realized by the appropriate length prolongation of an immersed plane curve. The second row in Table 4.12 is equivalent to the classification of codimension one singularities of Goursat 2-distributions discussed earlier and found in [**Mor4**].

COROLLARY 4.42. *A generic singular point in $\mathbb{P}^i\mathbb{R}^2$ is realized by the appropriate length prolongation of the plane curve ("A_{k+2}-singularity") (t^2, t^{2k+5}) as read off from the second row in Table 4.12. "Generic" here means a point of an open and dense set A in the codimension one variety S of all singular points. A is the* *union of the classes R^qVR^m with $q+m=i-3$. The set $S-A$ has codimension one in S, and so codimension 2 within $\mathbb{P}^i\mathbb{R}^2$.*

TABLE 4.12. Classification of tower-simple points. Any tower-simple point is equivalent to one and only one of the points $c^k(0)$ where c is the plane curve in the last column.

RVT class	codim	level k	Jet-deter. number r (if $q \neq 0$)	Exact normal form
$R^q,\ \ q \geq 1$	0	$q+2$	1	$(t,0)$
R^sVR^q $s,q \geq 0$	1	$s+q+3$	$2s+2+q$	(t^2,t^{2s+5})
R^sVTR^q $s,q \geq 0$	2	$s+q+4$	$3s+3+q$	$s=0$ or $q \in \{0,1\}:\ (t^3,t^7)$ $s \geq 1, q \geq 2:\ E_{q,j};\ (t^3,t^{3q+7})$, $j=0,...,\min(j^*,s-1)$ $j^*=[(q-2)/3]$
R^sVVR^q $s,q \geq 0$	2	$s+q+4$	$3s+4+q$	$s=0$ or $q \in \{0,1,2\}:\ (t^3,t^8)$ $s \geq 1, q \geq 4:\ E'_{q,j};\ (t^3,t^{3q+8})$ $j=0,...,\min(j^*,s-1)$ $j^*=[(q-3)/3]$
VR^mVR^q $m \geq 1, q \geq 0$	2	$m+q+4$	$6+3m+q$	$(t^4,t^{10}+t^{9+2m})$
$VTTR^q$ $q \geq 0$	3	$q+5$	$q+4$	$q \in \{0,1,2\}:\ (t^4,t^9)$ $q \geq 3:\ (t^4,t^9 \pm t^{11})$; (t^4,t^9)
$VVTR^q$ $q \geq 0$	3	$q+5$	$q+6$	$q \in \{0,1,2\}:\ (t^4,t^{11})$ $q \in \{3,4,5,6\}:$ $(t^4,t^{11} \pm t^{13});\ (t^4,t^{11})$ $q \geq 7:\ (t^3,t^{11} \pm t^{13})$; $(t^4,t^{11} \pm t^{17});\ (t^4,t^{11})$

The notation $E_{q,j}$ and $E'_{q,j}$ found in rows 3 and 4 of that table stand for the plane curves:

- $E_{q,j}=(t^3,t^{3q+7} \pm t^{3q+8+3j}),\ \ \pm \hookrightarrow +$ if j is even,
- $E'_{q,j}=(t^3,t^{3q+8} \pm t^{3q+10+3j}),\ \ \pm \hookrightarrow +$ if j is odd.

CLAIM 4.43. The number r given in the fourth column of Table 4.12 is the jet-identification number of the RVT class in the same row, provided that $q \neq 0$.

PROOF. Follows from Theorem 3.47 relating the jet-identification and the parameterization numbers and the last column of Table 4.11. □

PROOF OF THEOREM 4.41 FOR THE CASE $q \neq 0$. In this case Theorem 4.41 is a direct corollary of claim 4.43, Theorem 2.24 (see also the algorithm in section 3.13.2) and the following statement on on classification of certain Legendrian curve jets, continuing Proposition 4.17 and proved in Appendix B.

PROPOSITION 4.44. *Fix a row in Table 4.11 corresponding to a critical RVT class* (α). *Let* P *be the set of plane curve germs given in the third column and and let* P^1 *be the one-step prolongation of* P. *Let* d *the integer in the last column and* $r = d + q - 1, q \geq 1$. *The* r*-jet of any Legendrian curve of the set* $j^r(P^1)$ *is RL-contact equivalent to the one-step prolongation of one and only one plane curve germ given in the last column of Table 4.12, the row of the class* (αR^q).

PROOF OF THEOREM 4.41 FOR THE CASE $q = 0$. If $q = 0$ then the last column of Table 4.12 contains a single plane curve. The one-step prolongation of this curve belongs to the Legendrization of the corresponding RVT class. Therefore we have to prove that each of the classes $R^sVT, R^sVV, VR^mV, VTT, VVT$ consists of a single orbit. The classes R^sVT, R^sVV, VTT, VVT consist of a single orbit by Theorem 3.20. The class VR^mV consists of a single orbit by Theorem 3.20 and the second row of Table 4.12 with $s = 0, q \geq 1$.

4.10. Further examples of classification results

By the same method we can present a complete classification of **any** RVT class, not only the tower-simple classes. In Table 4.13 we give several examples. The notations in this table are the same as in Table 4.12.

THEOREM 4.45. *Fix an RVT class* (α) *in Table 4.13 in the kth level of the Monster (the number* k *is given in the third column). Any point of* (α) *is equivalent to one and only one of the points* $c^k(0)$, *where* c *is a plane curve germ in the last column of Table 4.13, with the row of that entry stating the class* (α). *The parameters* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *take any real values, and the parameter* $\mathbf{r}$ *takes any non-negative real values.*

The proof of this theorem is exactly the same as that of Theorem 4.41. The RVT classes in Table 4.13 have the form $(\alpha) = (\beta R^q)$, where β is a critical class. The jet-identification number r of (α), given in the fourth column of Table 4.13, is computed using the formulae $r = d + q - 1$ in Theorem 3.47, where d is the parameterization number of the class β. The number d can be found in Table 4.3. In the same table one can find the set $\mathrm{Leg}(\alpha) = \mathrm{Leg}(\beta)$. It remains to find (using techniques in Appendix B) an exact normal form for the class of Legendrian r-jets $j^r\,\mathrm{Leg}(\alpha)$ and to express it in the form $\{c^1, c \in P\}$, where P is a certain set of plane curve germs. The simplest set P with this property of is given in the last column of Table 4.13.

TABLE 4.13. Exact normal forms for some RVT classes of codimension 3. Any point of a given class is equivalent to one and only one of the points $c^k(0)$ where c is the plane curve in the last column.

RVT class	level k	Jet-identif. number r (if $q \neq 0$)	Exact normal form $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{r} \in \mathbb{R}, \quad \mathbf{r} \geq 0$
$VTVR^q$	$q+5$	$6+q$	$0 \leq q \leq 4: \quad (t^5,\ t^{12}+t^{13})$ $5 \leq q \leq 6: \quad (t^5,\ t^{12}+t^{13}+\mathbf{a}t^{16}),$ $q \geq 7: \quad (t^5,\ t^{12}+t^{13}+\mathbf{a}t^{16}+\mathbf{b}t^{18})$
$RVRVTR^q$	$q+5$	$15+q$	$0 \leq q \leq 1: \quad (t^6,\ t^{21}+t^{22})$ $2 \leq q \leq 3: \quad (t^6,\ t^{21}+t^{22}+\mathbf{a}t^{23})$ $q=4: \quad (t^6,\ t^{21}+t^{22}+\mathbf{a}t^{23}+\mathbf{b}t^{25})$ $q=5,6:$ $(t^6,\ t^{21}+t^{22}+\mathbf{a}t^{23}+\mathbf{b}t^{25}+\mathbf{c}t^{26})$
$RVRVVR^q$	$q+5$	$16+q$	$0 \leq q \leq 2: \quad (t^6,\ t^{21} \pm t^{23})$ $q=3: \quad (t^6,\ t^{21} \pm t^{23}+\mathbf{a}t^{25})$ $4 \leq q \leq 5: \quad (t^6,\ t^{21} \pm t^{23}+\mathbf{a}t^{25}+\mathbf{r}t^{26})$
$VVRVR^q$	$q+5$	$10+q$	$0 \leq q \leq 2: \quad (t^6,\ t^{16}+t^{17})$ $3 \leq q \leq 8: \quad (t^6,\ t^{16}+t^{17}+\mathbf{a}t^{19})$

CHAPTER 5

Local coordinate systems on the Monster

CHAPTER 6

Resolution via Prolongation

CHAPTER 7

Open questions

Appendix A. Classification of integral Engel curves

Appendix B. Contact classification of Legendrian curves

Appendix C. Singular and rigid curves. Proof of Theorem 3.3

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