MATH 235 - HW # 1

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[1] For $P(x) \in \mathbb{R}[x]$ we are interested in the differential equation $\dot{x} = P(x)$, an autonomous equation. In order to determine for which polynomials the flow is complete we first observe the linear case:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= ax + b \\ \frac{\mathrm{d}x}{ax+b} &= \mathrm{d}t \\ \int \frac{\mathrm{d}x}{ax+b} &= \int \mathrm{d}t \\ \frac{1}{a}\ln|ax+b| &= t + C_1 \\ ax + b &= C_2 e^{at} \\ x &= \frac{C_2}{a}e^{at} - \frac{b}{a} \end{aligned}$$

With the explicit solution solution above we know that $x(t)$ does not blow up for any finite time, hence the flow is complete. Now for all higher order polynomials we first mention the *Racetrack Principle*: Given two functions $f(x)$ and $g(x)$ s.t. $f(a) = g(a)$ and $f'(x) \geq g'(x)$ then $f(x) \geq g(x)$ for $x > a$. Thus, setting $f'(x) = P(x)$ a nth degree polynomial for $n > 1$ we can determine a value $x = a$ s.t. $f(a) = g(a)$ for $g'(x) = Kx^n$, where the value of $K \in \mathbb{R}$ is picked small enough so that $f'(x) \geq g'(x)$. It follows that for $\dot{x} = g'(x)$:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= Kx^n \\ \frac{\mathrm{d}x}{x^n} &= K\ \mathrm{d}t \\ \frac{1}{1-n}x^{1-n} &= Kt + C \\ x^{1-n} &= K(1-n)t + C(1-n) \\ x &= \frac{1}{(K(1-n)t + C(1-n))^{\frac{1}{n-1}}} \end{aligned}$$

there is a blow up in finite time. By the setup of the racetrack principle we know that the flow corresponding to $P(x)$ is greater than the flow above, thus this flow will also blow up in finite time. Hence, for $n > 1$ the flow is not complete.

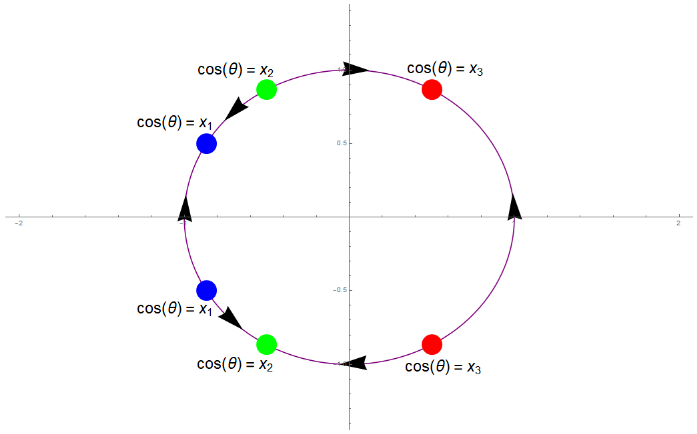
[2] We are given a nth degree polynomial $P(x)$ with distinct roots in the domain $[-1, 1]$ and the associated differential equation $\dot{\theta} = P(\cos(\theta))$ on $\mathbb{S}^1$.

(a) We begin with the explicit form $P(x) = a(x - x_1)(x - x_2)\ldots(x - x_n)$ where we may assume without loss of generality that $x_i < x_j$ for $i < j$. The replacement of x with $\cos(\theta)$ will provide two solutions of θ for each root, with the exceptions of $x_1 = -1$ and $x_n = 1$ giving only one value of θ. Thus, the number of equilibrium points is given by:

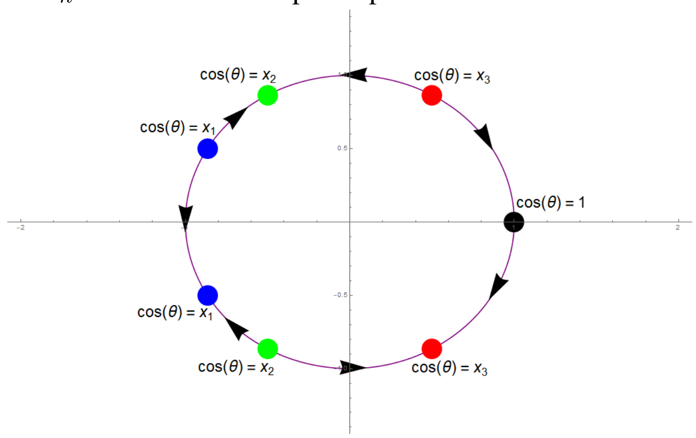
$$\text{\# of equilibrium points} = \begin{cases} 2n & x_1 \neq -1, x_n \neq 1 \\ 2n-1 & x_1 \neq -1, x_n = 1 \quad \text{or} \quad x_1 = -1, x_n \neq 1 \\ 2n-2 & x_1 = -1, x_n = 1 \end{cases}$$

Let us now consider a couple of base cases:

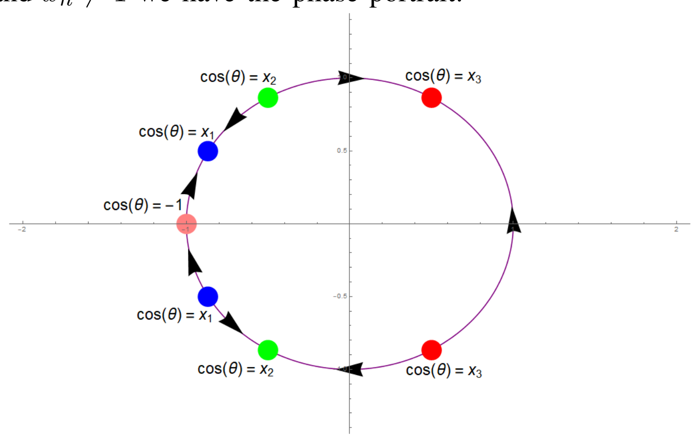
* If $n = 3$ with $x_1 \neq -1$ and $x_n \neq 1$ we have the phase portrait:



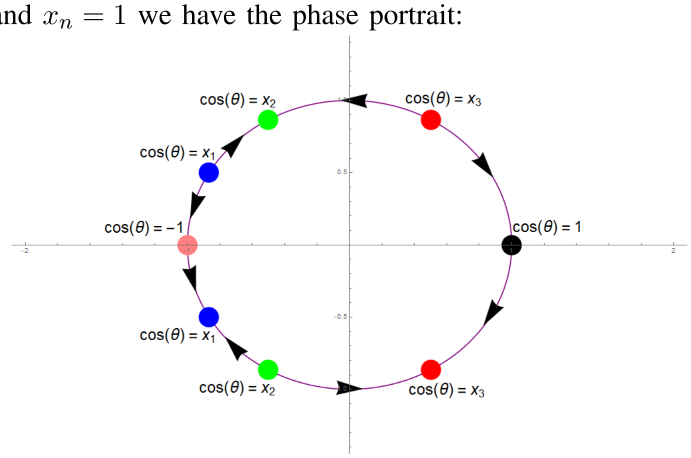
* If $n = 4$ with $x_1 \neq -1$ and $x_n = 1$ we have the phase portrait:



* If $n = 4$ with $x_1 = -1$ and $x_n \neq 1$ we have the phase portrait:



* If $n = 5$ with $x_1 = -1$ and $x_n = 1$ we have the phase portrait:



(b) Notice from the above phase portraits that between equilibrium points the direction changes in the arrows. This is obviously not a coincidence because it occurs whenever some factor $(x - x_i)$ flips sign as we pass by $x = x_i$. Furthermore, if $x_n \neq 1$ then the arrow between the two values of θ that give us $\cos(\theta) = x_n$ is always pointing in the positive direction. On the other hand, if $x_n = 1$ then the arrows between $\cos(\theta) = x_{n-1}$ and $\cos(\theta) = x_n$ are always pointing in the negative direction. Using these two possibilities as a starting point we generate the rest of the phase portrait by alternating the direction of the arrow as we move towards $\theta = \pi$ from $\theta = 0$ along the upper semicircle and similarly for $\theta = -\pi$ from $\theta = 0$ along the lower semicircle.

With the general pattern determined we can see that given some initial condition $\theta_* \in (x_i, x_{i+1})$ we will have $\theta(t) \to \theta_\infty$ where $\cos(\theta_\infty) = x_i$ or $\cos(\theta_\infty) = x_{i+1}$. To know whether the cosine of the angle tends to x_i or x_{i+1} we draw the associated phase portrait and follow the direction of the arrow.