HW 5

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1) a) Let $f \in L^2(\mathbb{T}^2)$ then by Fourier series expansion:

$$f = \sum_{p,q} a_{p,q} e^{2\pi ipx} e^{2\pi iqy}$$

Let A $= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ be the matrix associated with the cat map.
Then $Ue_{p,q} = e^{2\pi ip(2x+y)} e^{2\pi iq(x+y)} = e^{2\pi i(2p+q)x} e^{2\pi i(p+q)y} = e_{A(p,q)}$.

b) Now $Uf = \sum_{p,q} a_{p,q} e_{A(p,q)}$, (by linearity of U) .
Then $Uf = f$ if and only if the coefficients of the basis elements match i.e. when $a_{p,q} = a_{A(p,q)}$ for all $(p, q) \in \mathbb{Z}^2$.

c) If $Uf = f$ then $a_{p,q} = a_{A(p,q)} = a_{A^n(p,q)}$ for all $n \in \mathbb{N}$ i.e. the Fourier coefficients are constant on all the orbits of A.
Let us now try to understand how the orbits of A look like for $(p, q) \in \mathbb{Z}^2$. $A(p, q)$ are integer lattice points on $\mathbb{R}^2$, hence either they are bounded (in that case periodic since there are finite number of choices) or $||A^n(p, q)||$ is unbounded.

d) Let $(p, q) \in \mathbb{Z}^2$ such that $(p, q) \neq (0, 0)$. Suppose the sequence $A^n(p, q)$ is bounded, then since there are finite many choices of the elements in the orbit and A^{-1} exists therefore $\exists r \in \mathbb{N}$ such that $A^r(p, q) = (p, q)$. So, (p, q) is an eigenvector for A^r for eigen-value 1, this is impossible since all the eigen values of A are irrational and different from 1. Therefore the only possible bounded orbit is $(0, 0)$.

e) We have $\int f = \sum |a_{p,q}|^2 < \infty$. Suppose there exists a sequence $(p_k, q_k) \to \infty$ such that $|a_{(p_k,q_k)}| > c$ for some $c > 0$. Then $\sum |a_{p,q}|^2 > kc$ for all $k \in \mathbb{N}$ hence a contradiction. Therefore $a_{p,q} \to 0$ as $(p, q) \to \infty$.

f) Suppose $Uf = f$ then $a_{p,q} = a_{A^n(p,q)}$ for all $(p, q) \in \mathbb{Z}^2$. The sequence $A^n(p, q)$ is unbounded for non-zero (p, q), therefore since integral of f is bounded $a_{p,q} = 0$ for all non-zero (p, q). Therefore $f = a_{0,0} = const.$, this proves ergodicity.

2) a) Continuing the same notation from the previous problem we have, $U^n f = \sum_{p,q} a_{p,q} e_{A^n(p,q)}$. We recall from Fourier analysis that $e_{p,q}$ forms an orthogonal basis for $L^2(\mathbb{T}^2)$ with respect to the standard inner product. Let $g = \sum_{p',q'} b_{p',q'} e_{p',q'}$.

To prove $< U^n f, g > \to \int f \int g$ it is enough to show that for all basis elements $e_{p,q}, e_{p',q'}$ is due to two standard arguments from Fourier analysis:
i) In order to show this equation holds for all $f, g \in L^2(\mathbb{T}^2)$, it suffices to prove it for all f, g in some dense subset $V \subset L^2(\mathbb{T}^2)$(the limits works perfectly since U is unitary).

ii) From the computations above we observe both side are bilinear in $< f, q >$. Hence it is sufficient to show for pairs in $\{v_1, v_2, ...\}$ which spans V a dense subset in $L^2(\mathbb{T}^2)$.
For our situation we consider V= span$\{e_{p,q} : (p, q) \in \mathbb{Z}^2\}$ and result will follow.

b) For all non zero (p, q) we have $\int e_{p,q} d\mu = \int_{-1}^{1} \int_{-1}^{1} e^{2\pi ipx} e^{2\pi iqy} dxdy = \int_{-1}^{1} e^{2\pi ipx} dx \int_{-1}^{1} e^{2\pi iqy} dy = 0$.

c) Let $(p, q) \neq (0, 0)$ and $(r, s) \in (Z)^2$. Suppose there exist r_1, r_2 such that $A^{r_i}(p, q) = (s, t)$. Since A^{-1} exists therefore (p, q) is periodic point for A which is impossible by $1(d)$, therefore for all large n $A^n(p, q) \neq (r, s)$.

d) Let $e_{p,q}$ and $e_{p',q'}$ be two basis elements such that $(p, q) \neq (0, 0)$. $A^n(p, q) \neq (p', q')$ for all large n and by orthogonality of the basis elements we have $< e_{A^n(p,q)}, e_{p',q'} > \to 0$. By 2(a) the mixing of the cat map follows.