Dynamical Systems HW 1

Matthew Salinger

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Problem 1
Let $P(x)$ be a polynomial with real coefficients. Determine the conditions for the vector field

$$X = P(x)\frac{\partial}{\partial x}$$

on the real line to be complete.

Solution
First we show that it is sufficient for $\deg(P(x)) \leq 1$. When the degree is 0 we have a constant vector field so that the flow is linear, existing for all time. So then let $P(x) = ax + b,\ a \neq 0$. Then

$$\dot{x} = ax + b.$$

We can easily see that $x = -\frac{b}{a}$ is the unique equilibrium solution. Now after separation of variables we get

$$x(t) = Ce^{at} - \frac{b}{a}. \tag{1}$$

When $a < 0$ the solution curves converge to the equilibrium $-\frac{b}{a}$ and when $a > 0$ diverge away. Either way, $x(t)$ is defined for all $t \in \mathbb{R}$.

We start running into problems when $\deg P(x) > 1$. For example, let $P(x) = x(x-1)$. That is,

$$\dot{x} = x(x-1).$$

After separation of variables we obtain the relation

$$\frac{x-1}{x} = Ce^{t} \tag{2}$$

where C is an arbitrary constant. The equilibria are located at $x = 0$ (stable) and $x = 1$ (unstable). In the limit as $t \to \infty$, the right hand side approaches $\pm\infty$ (depending on C) which implies that $x(t) \to 0$ from the right when the limit is

$-\infty$ and from the left when the limit is $+\infty$. Of course this tells us no more than we already know about solution curves with initial condition $x(0) < 1$ (drawing the phase plot makes this easy to see). However, we made no assumption about the initial condition in equation (2) when taking the limit $t \to \infty$. Consequently, this shows that *all* the solution curves with initial condition $x(0) > 1$ blowup in finite time, since if $\lim_{t\to\infty} x(t)$ existed, then such a solution curve would have to converge to 0 which requires passing through the equilibrium $x = 1$, which is impossible.
We will see that $\deg(P(x)) = 0, 1$ is also necessary for completeness.

Proposition 1 *The vector field* $X = x^{n+1}\frac{\partial}{\partial x}$, $n \geq -1$ *is complete if and only if n is -1 or 0.*

Proof
We already proved that $n = -1, 0$ is sufficient. To prove it is necessary we will prove a slightly modified statement: suppose that $n \geq 1$ and suppose that

$$\dot{x} \geq x^{n+1}.$$

Since this holds for all t,

$$\int_0^t \frac{\dot{x}}{x^{n+1}}\, dt \geq t,$$

or

$$-\frac{1}{n}x^{-n} + \frac{1}{n}x_0^{-n} \geq t$$

where $x_0 := x(0)$.
After some rearrangement we get

$$x^n \geq \frac{x_0^n}{1 - nx_0^n t}. \tag{3}$$

The right hand side blows up at $t = \frac{1}{nx_0^n}$, so $x(t)$ cannot be defined for all t. Note that the larger x_0 is, the quicker the blowup.
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This is almost enough to show that $\deg(P(x)) > 1$ has finite time blowups. To see why, suppose that $P(x)$ has degree $n > 1$ and with positive leading coefficient. Then for some $c \in (0, 1)$ and sufficiently large positive integer N, $x > N$ implies that $P(x) > cx^n$. Therefore if the initial condition is taken to be large enough, e.g. $x_0 > N$, it follows that for $t \geq 0$,

$$\dot{x}(t) > cx(t)^n$$

and apply the method in the proof of proposition 1. If the leading coefficient is negative, then an analogous argument holds with negative initial conditions.

Problem 2

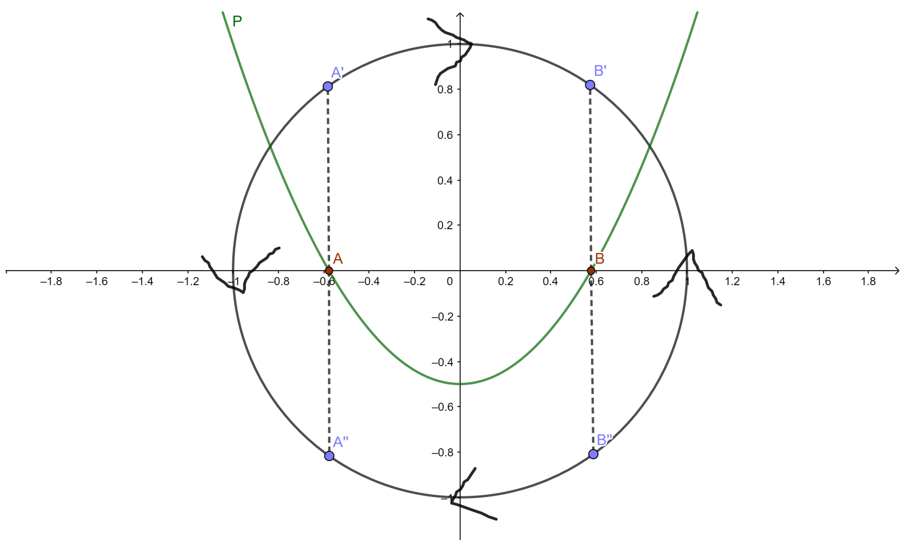
Suppose $P(x)$ is a polynomial of degree n all of whose roots lie in $[-1, 1]$ (Legendre polynomials, for example).

1. Sketch a phase portrait of the system $\dot{\theta} = P(\cos\theta)$ on the standard unit circle, parameterized by $\theta \in [0, 2\pi]$.

2. Suppose you start off with $\theta(0) = \theta_*$ such that $\cos(\theta_*)$ lies between consecutive zeroes of $P(x)$, say $x_i < \cos(\theta_*) < x_{i+1}$. Show that, as $t \to \infty$, we have $\theta(t) \to \theta_\infty$, where $\cos(\theta_\infty)$ equals either x_i or x_{i+1}. How can you know which angle $\theta(t)$ tends to, the angle corresponding to x_i or x_{i+1}.

   Solution

   Suppose $P(x) = \frac{1}{2}(3x^2 - 1)$. The phase portrait looks like



where the points A and B are the roots of $P(x)$ on the real line, $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, respectively; the points A', B', A'', B'' correspond to the zeros of $P(\cos\theta)$. To determine the direction of the arrows, one can use local stability criterion: let $Q(\theta) = P(\cos\theta)$. Then the differential equation can be written as

$$\dot{\theta} = Q(\theta).$$

By local stability criterion, a critical point θ is stable if $\frac{d}{d\theta}Q(\theta) < 0$ and unstable if $\frac{d}{d\theta}Q(\theta) > 0$. Now

$$\frac{d}{d\theta}Q(\cos\theta) = -\sin\theta P'(\cos\theta) = -3\sin\theta\cos\theta. \tag{4}$$

The sign of $-3\sin\theta\cos\theta$ alternates with the quadrants in the order -+-+. That is, going clockwise starting at $\theta = 0$, B' is stable, A' unstable, A'' stable, and B' unstable. Doing so tells us the direction of the flow off of the critical points. This gives a general procedure. And, the flows exist for all time since $\dot{\theta}$ is bounded above and below by a constant (in our example $-5/2 \leq \theta \leq 5/2$). This answers 2.