Math 235 - Assignment 2

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1. Let Σ and Σ' be two Poincaré sections to the same periodic orbit c and passing through the same point $p = c(0)$ of c. Let $\Psi : (\Sigma, p) \longrightarrow (\Sigma, p)$ and $\Psi' : (\Sigma', p) \longrightarrow (\Sigma', p)$ be the correspoding Poincaré maps respectively. Then the two maps are *locally conjugate*, i.e. there exists a smooth map $\Phi : (\Sigma, p) \longrightarrow (\Sigma', p)$ which is a local diffeomorphism at p and satisfies

$$\Phi \circ \Psi = \Psi' \circ \Phi.$$

Proof. Let $\mathbf{V} : M \longrightarrow TM$ be the vector field having the above periodic orbit c and $\{\varphi_t\}_{t \in I} \subset \mathrm{Diff}(M)$ be its associated flow. For each $q \in \Sigma$ and $q' \in \Sigma'$ sufficiently close to p, put

$$\begin{aligned} T(q) &:= \min\{t > 0 : \varphi_t(q) \in \Sigma\} \\ T'(q') &:= \min\{t > 0 : \varphi_t(q') \in \Sigma'\} \end{aligned}$$

so that

$$\Psi(q) = \varphi_{T(q)}(q) \quad \text{and} \quad \Psi'(q') = \varphi_{T'(q')}(q').$$

Note that $\mathbf{V}(p) = \dot{c}(0) \neq \mathbf{0}$ (in fact $\dot{c}$ is nonvanishing, since c is an orbit (that is an actual curve) and so cannot contain a critical point), and so by the Straightening Lemma, there exist local coordinates $(x^1, \ldots, x^n)$ for M about p such that

$$\mathbf{V} = \frac{\partial}{\partial x^1}$$

in a coordinate neighborhood of p. More precisely, if M has dimension n (so that Σ and Σ' are $(n-1)$-dimensional manifolds), there exists an open subset $\widetilde{U} \subset M$ containing p and a diffeomorphism $\widetilde{f} : \widetilde{U} \longrightarrow \widetilde{f}(\widetilde{U}) \subset \mathbb{R}^n$ such that for any $q = \widetilde{f}^{-1}(x^1, \ldots, x^n) \in \widetilde{U}$,

$$\mathbf{V}(q) = \frac{\partial}{\partial x^1}.$$

Of course this neighborhood $\widetilde{U}$ can be arranged to be small enough so that the times of return T and T', and hence the Poincaré maps Ψ and Ψ', are well-defined on $\Sigma \cap U$ and $\Sigma' \cap U$, respectively. Moreover, it can be arranged that it is "literally" small enough, i.e. $\widetilde{f}(\widetilde{U})$ has closure contained in some ball of finite radius in $\mathbb{R}^n$.

Meanwhile, since c is an embedded curve in M, by the Tubular Neighborhood Theorem, c has a tubular neighborhood. Moreover, since c is compact , for some small enough $\epsilon > 0$, c has a tubular neighborhood in M which is the diffeomorphic image of the (open) ϵ-neighborhood

$$\{(x, v) \in N(c) : \|v\| < \epsilon\}$$

of the normal bundle $N(c) \subset TM$ of c. Finally, let U be the (open) intersection of this tubular neighborhood with $\widetilde{U}$ and $f : U \longrightarrow f(U)$ be the diffeomorphism which is the restriction of $\widetilde{f}$ on U.

Since $\Sigma \pitchfork c$ and $\Sigma' \pitchfork c$, we still have $\mathcal{O} := \Sigma \cap U$ and $\mathcal{O}' := \Sigma' \cap U$ transverse to $\dfrac{\partial}{\partial x^1}$ in U. Thus, by the Implicit Function Theorem, one can arrange $\mathcal{O}$ and $\mathcal{O}'$ small enough so that there exist functions σ and σ' (with appropriate domains $\subset \{x^1 = 0\}$, as will be implied as follows) such that $\mathcal{O}$ and $\mathcal{O}'$ are the graphs of σ and σ', respectively, i.e.

$$\begin{aligned} f(\mathcal{O}) &= \{(x^1, x^2, \ldots, x^n) \in U : x^1 = \sigma(x^2, \ldots, x^n)\} \\ f(\mathcal{O}') &= \{(x^1, x^2, \ldots, x^n) \in U : x^1 = \sigma'(x^2, \ldots, x^n)\} \end{aligned}$$

Finally, define the vector field $\mathbf{W} : U \longrightarrow TU$ as follows: If $q \in U$ has coordinates $f(q) = (x_q^1, x_q^2, \ldots, x_q^n) \in \mathbb{R}^n$,

$$\mathbf{W}(q) = (\sigma' - \sigma)(x_q^2, \ldots, x_q^n) \frac{\partial}{\partial x^1}$$

Just to be clear, the function $\sigma' - \sigma$ is well-defined; this subtraction occurs in $\mathbb{R}$, since σ, σ' have images both subsets of $\mathbb{R}$ (the "x^1-axis" in $\mathbb{R}^n$, i.e. the time axis of the flow of $\mathbf{V}$). Essentially, this expression gives the time (i.e. value of $t = x^1$) that it takes for a point on $\mathcal{O}$ to reach a (unique, since they're both graphs) point on $\mathcal{O}'$ when following the flow of $\mathbf{V}$ (along exactly the direction of increasing x^1). Indeed, the "absolute" times for being at the points $q = f(x_q^1, x_0^2, \ldots, x_0^n) \in \mathcal{O}$ and $q' = f(x_{q'}^1, x_0^2, \ldots, x_0^n) \in \mathcal{O}'$ on the same integral curve $\{x^2 = x_0^2, \ldots, x^n = x_0^n\}$ (it's a "line" when viewed using the straightening coordinates) of $\mathbf{V}$ are precisely

$$x^1 = \sigma(x_0^2, \ldots, x_0^n) \text{ and } x^1 = \sigma'(x_0^2, \ldots, x_0^n),$$

respectively. Thus, the time it takes to go from q to q' along the aforementioned integral curve of $\mathbf{V}$ is exactly the difference of these two absolute times.

<u>Claim</u>: The time-1 flow of $\mathbf{W}$ gives a diffeomorphism $\mathcal{O} \longrightarrow \mathcal{O}'$.

Indeed, first let $q \in U$ and $f(q) = (x_q^1, x_q^2, \ldots, x_q^n)$. Let us solve the initial-value problem $\dot{x} = \mathbf{W}(f^{-1}(x))$ (happening in $\mathbb{R}^n$) with initial condition $\dot{x}(0) = (x_q^1, x_q^2, \ldots, x_q^n)$. The ODE is equivalent to

$$\begin{array}{rcl} \dot{x}^1 & = & (\sigma' - \sigma)(x^2, \ldots, x^n) \\ \dot{x}^2 & = & 0 \\ & \vdots & \\ \dot{x}^n & = & 0 \end{array}$$

Of course the solutions of the second up to the n-th equations are constant functions; invoking the initial condition gives

$$\begin{array}{rcl} x^2(t) & = & x_q^2 \\ & \vdots & \\ x^n(t) & = & x_q^n \end{array}$$

which also consequently leads to

$$\dot{x}^1 = (\sigma' - \sigma)(x_q^2, \ldots, x_q^n).$$

Note that the right hand side here is constant. The solution to this last (actually, first) equation, satisfying the initial condition, is then clearly

$$x^1(t) = x_q^1 + t(\sigma' - \sigma)(x_q^2, \ldots, x_q^n),$$

That is, if $\{\vartheta_t\}_{t \in J} \subset \mathrm{Diff}(M)$ is the flow of $\mathbf{W}$, then

$$\vartheta_t(q) = f^{-1}\left(x_q^1 + t(\sigma' - \sigma)(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\right) \tag{1}$$

Now, if $q \in \mathcal{O}$, then $x_q^1 = \sigma(x_q^2, \ldots, x_q^n)$, and so we can write

$$x^1(t) = (1-t)\sigma(x_q^2, \ldots, x_q^n) + t\sigma'(x_q^2, \ldots, x_q^n)$$

That is, the unique solution to our initial value problem is

$$x(t) = \left((1-t)\sigma(x_q^2, \ldots, x_q^n) + t\sigma'(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\right)$$

That is,

$$f(\vartheta_t(q)) = \big((1-t)\sigma(x_q^2, \ldots, x_q^n) + t\sigma'(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\big)$$

In particular, we have

$$\vartheta_0(q) = f^{-1}\left(\sigma(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\right) = f^{-1}\left(x_q^1, x_q^2, \ldots, x_q^n\right) = q,$$

as of course is expected, and more importantly,

$$\vartheta_1(q) = f^{-1}\left(\sigma'(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\right) \in \mathcal{O}'.$$

Thus, indeed, the time-1 flow is a map $\vartheta_1 : \mathcal{O} \longrightarrow \mathcal{O}'$. Moreover, it is smooth and injective, as flows are diffeomorphisms. Finally, it is surjective and hence is an actual diffeomorphism of $\mathcal{O}$ with $\mathcal{O}'$: Indeed, given any $q' \in \mathcal{O}'$, if

$$f(q') = (x_{q'}^1, x_{q'}^2, \ldots, x_{q'}^n) = (\sigma'(x_{q'}^2, \ldots, x_{q'}^n), x_{q'}^2, \ldots, x_{q'}^n),$$

choose $q = f^{-1}(\sigma(x_{q'}^2, \ldots, x_{q'}^n), x_{q'}^2, \ldots, x_{q'}^n) \in \mathcal{O}$. Then,

$$\vartheta_1(q) = f^{-1}\left(\sigma'(x_{q'}^2, \ldots, x_{q'}^n), x_{q'}^2, \ldots, x_{q'}^n\right) = q'$$

For sanity's sake, we explain in words what ϑ_1 does: At the level of the straightening coordinates, ϑ_1 simply replaces the x^1-coordinate, which originally makes the point lie in $\mathcal{O} \subset \Sigma$, by the unique proper value such that the image lies in $\mathcal{O}' \subset \Sigma'$. Visually, every integral curve of $\mathbf{W}$ intersects $\mathcal{O}$ and $\mathcal{O}'$ uniquely at some points q and q', respectively. Along this integral curve, the coordinates $x^2, \ldots, x^n$ are constant. Now, applying ϑ_1 to q is tantamount to replacing the x^1-coordinate of q (which is $\sigma(x^2, \ldots, x^n)$ on the integral curve) by the x^1-coordinate of q' (which is $\sigma'(x^2, \ldots, x^n)$ on the integral curve). The inverse does the reverse replacement. As a special important case, since p lies on both $\mathcal{O}$ and $\mathcal{O}'$, the values of σ and σ' at p are the same, and hence $\vartheta_1(p) = p$.

Therefore, ϑ_1 is a diffeomorphism of a neighborhood of p in Σ with a neighborhood of p in Σ'. Finally, define

$$\Phi = \vartheta_1 : \mathcal{O} \longrightarrow \mathcal{O}'.$$

Before finally showing local conjugacy, we make an important observation: How are the flows φ_t and ϑ_t of the vector fields $\mathbf{V}$ and $\mathbf{W}$ related? We observed earlier that the x^1-coordinate can be thought of as the time-coordinate, the coordinate along the flow of the original vector field. Since $\mathbf{W}$ is just a rescaling of $\mathbf{V} = \dfrac{\partial}{\partial x^1}$ (not constantly over U), the time coordinate, we expect that the orbits (as actual goemetric curves, i.e. images of parametrizations, not as parametrized curves, i.e. the parametrizations themselves) of the two vector fields are the same, except when the rescaling is by a factor of 0 ($\mathbf{W}$ has "more" critical points than $\mathbf{V}$; precisely, it has the extra critical points $\mathcal{O} \cap \mathcal{O}'$ where the graphs intersect, i.e. σ and σ' coincide as functions). One of them is just a "faster" (or reverse) flow of the other. Indeed, if $q \in U$, with $f(q) = (x_q^1, x_q^2, \ldots, x_q^n)$, we clearly have (since, in the straightening coordinates, $\mathbf{V}$ is just the constant vector field $(1, 0, \ldots, 0)$)

$$\varphi_t(q) = f^{-1}(x_q^1 + t, x_q^2, \ldots, x_q^n)$$

and hence, from (1),

$$\begin{aligned} \vartheta_t(q) &= f^{-1}\left(x_q^1 + t(\sigma' - \sigma)(x_q^2, \ldots, x_q^n), x_q^2, \ldots, x_q^n\right) \\ &= \varphi_{t(\sigma'-\sigma)(x_q^2,\ldots,x_q^n)}(q) \end{aligned}$$

Now we show local conjugacy. First, let

$$q = f^{-1}(x_q^1, x_q^2, \ldots, x_q^n) \in \mathcal{O} \text{ and } \Psi(q) = f^{-1}(x_{\Psi(q)}^1, x_{\Psi(q)}^2, \ldots, x_{\Psi(q)}^n) \in \mathcal{O}'.$$

Then $\Phi(q) = \vartheta_1(q)$ is the first point on $\mathcal{O}'$ hit by the orbit of the flow of $\pm\mathbf{V}$ when started at q. (Here, the sign coincides with the sign of $(\sigma' - \sigma)(x_q^2, \ldots, x_q^n)$. If it is positive, we follow the direction of $\mathbf{V}$; if it is negative, we go in the opposite direction along the orbit. If it turns out to be zero, we have $q \in \mathcal{O} \cap \mathcal{O}'$, and we simply don't move. In any case, we have to follow the direction which brings us from Σ to Σ', along the orbit.) And then, $\Psi'(\Phi(q))$ is the first point on $\mathcal{O}'$ hit by the orbit of the flow of $\mathbf{V}$ when started at $\Phi(q)$. The crucial point is that these two orbits are exactly the same, for otherwise there would be two integral curves of $\mathbf{V}$ passing through the common point $\Phi(q)$, contradicting uniqueness of the integral curve through a given point. Call this orbit γ_1; arrange it so that $\gamma_1(0) = q$. Meanwhile, $\Psi(q)$ is the first distinct point on $\mathcal{O}$ hit by the orbit of the flow of $\mathbf{V}$ when started at q. And then, $\Phi((\Psi(q))$ is the first point on $\mathcal{O}'$ hit by the orbit of the flow of $\pm\mathbf{V}$ when started at $\Psi(q)$ (similarly, the sign coincides with the sign of $(\sigma' - \sigma)(x_{\Psi(q)}^2, \ldots, x_{\Psi(q)}^n)$). By the same argument previously, the two orbits coincide; call it γ_2, and arrange it so that $\gamma_2(0) = q$. Now both γ_1 and γ_2 are integral curves of the same vector field $\mathbf{V}$ with the same initial condition, and hence again by uniqueness, $\gamma_1 = \gamma_2$. So, in summary, the point $\Psi'(\Phi(q))$ is obtained by moving along γ_1, starting at q in the direction that brings us fastest to a point on $\mathcal{O}'$ (to get $\Phi(q)$), and then from there move along γ_1 in the (original) direction of $\mathbf{V}$ until we return to $\mathcal{O}'$. Now, we have chosen the $\mathcal{O}$ and $\mathcal{O}'$ small enough so that the latter return process will always result in hitting both of them. In particular, it will hit $\mathcal{O}$, and this will be the first time $\gamma_1 = \gamma_2$ hits $\mathcal{O}$ again, i.e. it is the point $\Psi(q)$. Finally then, the last point $\Psi'(\Phi(q))$ must be the first point on $\mathcal{O}'$ reached by moving from $\Psi(q)$ along $\gamma_1 = \gamma_2$ (whether forward or backward in time). That is, $\Psi'(\Phi(q)) = \Phi(\Psi(q))$.

More precisely, we can see in coordinates that $\gamma_1 = \gamma_2$ as follows:

$$\begin{aligned}
(\Psi' \circ \Phi)(q) &= (\Psi' \circ \vartheta_1)(q) \\
&= \varphi_{T'(\vartheta_1(q))} \circ \varphi_{(\sigma'-\sigma)(x_q^2,\ldots,x_q^n)}(q) \\
&= \varphi_{T'(\vartheta_1(q))+(\sigma'-\sigma)(x_q^2,\ldots,x_q^n)}(q) \\
&= f^{-1}(x_q^1 + T'(\vartheta_1(q)) + (\sigma'-\sigma)(x_q^2,\ldots,x_q^n), x_q^2,\ldots,x_q^n)
\end{aligned}$$

and

$$\begin{aligned}
(\Phi \circ \Psi)(q) &= (\vartheta_1 \circ \Psi)(q) \\
&= \varphi_{(\sigma'-\sigma)(x_{\Psi(q)}^2,\ldots,x_{\Psi(q)}^n)} \circ \varphi_{T(q)}(q) \\
&= \varphi_{(\sigma'-\sigma)(x_{\Psi(q)}^2,\ldots,x_{\Psi(q)}^n)+T(q)}(q) \\
&= f^{-1}(x_q^1 + (\sigma'-\sigma)(x_{\Psi(q)}^2,\ldots,x_{\Psi(q)}^n) + T(q), x_q^2,\ldots,x_q^n)
\end{aligned}$$

And so we see that the two points are on the same integral curve indeed. Now, the fact that they are the same point is then equivalent to the equation

$$T'(\vartheta_1(q)) + (\sigma'-\sigma)(x_q^2,\ldots,x_q^n) = (\sigma'-\sigma)(x_{\Psi(q)}^2,\ldots,x_{\Psi(q)}^n) + T(q).$$

which, in words, simply says that, using the flow φ_t, it takes the same time t to map q to $(\Psi' \circ \Phi)(q)$ and to $(\Phi \circ \Psi)(q)$.

$\square$

2. Suppose that the unit circle is a periodic orbit for some smooth vector field $\mathbf{V}$ on $\mathbb{R}^2$. The line segment Σ joining $(1-\epsilon, 0)$ to $(1+\epsilon, 0)$ is a slice of this orbit, and so defines a Poincaré section. Use the interval $(-\epsilon, \epsilon)$ to coordinatize this slice by sending $x \mapsto (1+x, 0)$. Prove that it is not possible for the map $x \mapsto -x$ to represent the Poincaré map associated to Σ.

Proof. Let $\{\phi_t\}_{t\in I} \subset \mathrm{Diff}(\mathbb{R}^2)$ be the flow of the vector field $\mathbf{V} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, for some interval I containing 0. Let $q = 1 + x \in \Sigma$ be sufficiently close to (and distinct from) 1, with $0 < |x| < \epsilon$. Then q is in either the interior or the exterior of the unit circle. Let $\gamma : [0, b) \longrightarrow \mathbb{R}^2$ $(0 < b \leqslant +\infty)$ be the orbit of the flow starting at q, i.e.

$$\gamma(t) = \phi_t(q), \quad 0 \leqslant t < b.$$

If $\Psi : (\Sigma, p) \longrightarrow (\Sigma, p)$ denotes the Poincaré map associated to the section Σ, then

$$\Psi(q) = \phi_{\tau(q)}(q) = \gamma(\tau(q)),$$

where $\tau(q)$ is the minimal $t > 0$ such that $\phi_t(q) \in \Sigma$. Note that Image(γ) cannot intersect the orbit c; otherwise, there would be at least two distinct integral curves through the point of intersection (c and γ), contradicting the uniqueness of solutions to $\dot{x} = \mathbf{V}(x)$. Thus, Image$(\gamma) \subset \mathbb{R}^2 - c$.

Moreover γ, being an integral curve of $\mathbf{V}$, is at least continuous. Thus, since $[0, b)$ is connected, the actual curve Image$(\gamma) \subset \mathbb{R}^2$ must be connected. Note that the unit circle c is a Jordan (i.e. simple closed) curve, and so by the Jordan Separation Theorem, $\mathbb{R}^2 - c$ is disconnected, with the interior and exterior of the unit circle being the two connected components. Thus, if $q = \gamma(0)$ is in the interior of c, i.e. $x < 0$, then Image(γ) is entirely contained in the interior of c. Likewise, if $q = \gamma(0)$ is in the exterior of c, i.e. $x > 0$, then Image(γ) is entirely contained in the exterior of c.

Finally, aiming for a contradiction, assume instead that the map (at the level of coordinates)

$$\begin{array}{ccc} (-\epsilon, \epsilon) & \longrightarrow & (-\epsilon, \epsilon) \\ x & \longmapsto & -x \end{array}$$

represents the Poincaré map Ψ. More precisely, $\Psi : (\Sigma, p) \longrightarrow (\Sigma, p)$ factors as

$$\begin{array}{ccccccc} (\Sigma, p) & \longrightarrow & (-\epsilon, \epsilon) & \longrightarrow & (-\epsilon, \epsilon) & \longrightarrow & (\Sigma, p) \\ 1 + x & \longmapsto & x & \longmapsto & -x & \longmapsto & 1 - x \end{array}$$

Then, if $q = 1 + x$ is in the exterior of c, $x > 0$ and so $\Psi(q) = 1 - x$ is in the interior of c. Likewise, if $q = 1 + x$ is in the interior of c, $x > 0$ and so $\Psi(q) = 1 - x$ is in the exterior of c. In either case, Image(γ) is not entirely contained in the interior nor the exterior of c. Contradiction.

$\therefore$ the map $x \longmapsto -x$ cannot represent the Poincaré map for (Σ, p). $\square$