Dynamical Systems HW 1

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Problem 1
Let $P(x)$ be a polynomial with real coefficients. Determine the conditions for the vector field

$$X = P(x)\frac{\partial}{\partial x}$$

on the real line to be complete.

Solution
First we show that it is sufficient for $\deg(P(x)) \leq 1$. When the degree is 0 we have a constant vector field so that the flow is linear, existing for all time. So then let $P(x) = ax + b,\ a \neq 0$. Then

$$\dot{x} = ax + b.$$

We can easily see that $x = -\frac{b}{a}$ is the unique equilibrium solution. Now after separation of variables we get

$$x(t) = Ce^{at} - \frac{b}{a}. \tag{1}$$

When $a < 0$ the solution curves converge to the equilibrium $-\frac{b}{a}$ and when $a > 0$ diverge away. Either way, $x(t)$ is defined for all $t \in \mathbb{R}$.

We start running into problems when $\deg P(x) > 1$. For example, let $P(x) = x(x-1)$. That is,

$$\dot{x} = x(x-1).$$

After separation of variables we obtain the relation

$$\frac{x-1}{x} = Ce^{t} \tag{2}$$

where C is an arbitrary constant. The equilibria are located at $x = 0$ (stable) and $x = 1$ (unstable). In the limit as $t \to \infty$, the right hand side approaches $\pm\infty$ (depending on C) which implies that $x(t) \to 0$ from the right when the limit is

$-\infty$ and from the left when the limit is $+\infty$. Of course this tells us no more than we already know about solution curves with initial condition $x(0) < 1$ (drawing the phase plot makes this easy to see). However, we made no assumption about the initial condition in equation (2) when taking the limit $t \to \infty$. Consequently, this shows that *all* the solution curves with initial condition $x(0) > 1$ blowup in finite time, since if $\lim_{t\to\infty} x(t)$ existed, then such a solution curve would have to converge to 0 which requires passing through the equilibrium $x = 1$, which is impossible.
We will see that $\deg(P(x)) = 0, 1$ is also necessary for completeness.

Lemma 1 *Suppose that $g(x) > f(x) > 0$ for all $x \in \mathbb{R}$ and that $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are two differntial equations on the real line so that $x(0) = y(0)$. Then for all $t \geq 0$, we have $y(t) \geq x(t)$.*

Proof
Let $h(t) = y(t) - x(t)$. Now $h(0) = 0$ and $h'(0) > 0$ implies that, at least for some possibly short time, $y(t)$ is larger than $x(t)$. If, contrary to the claim of the lemma, $x(t)$ is ever larger than $y(t)$, then there is some time T at which $h(T) = 0$ and $h'(T) \leq 0$. In other words $x(T) = y(T) = z \in \mathbb{R}$ and $h'(T) = g(y(T)) - f(x(T)) = g(z) - f(z) \leq 0$, a contradiction. Therefore no such time exists, and $y(t) \geq x(t)$ for all $t \geq 0$.
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Proposition 2 *The vector field $X = \pm x^{n+1}\frac{\partial}{\partial x}$, $n \geq -1$ is complete if and only if n is -1 or 0.*

Proof
We already proved that $n = -1, 0$ is sufficient. To prove it is necessary suppose that $n \geq 1$ and suppose that

$$\dot{x} = x^{n+1}.$$

Since this holds for all t,

$$\int_0^t \frac{\dot{x}(s)}{x^{n+1}(s)}\, ds = t,$$

or

$$-\frac{1}{n}x^{-n} + \frac{1}{n}x_0^{-n} = t$$

where $x_0 := x(0)$.
After some rearrangement we get

$$x^n = \frac{x_0^n}{1 - n x_0^n t}. \tag{3}$$

The right hand side blows up at $t = \frac{1}{n x_0^n}$, so $x(t)$ cannot be defined for all t. Note that the larger x_0 is, the quicker the blowup. In the case that $X = -x^{n+1}\frac{\partial}{\partial x}$, we obtain the expression

$$x^n = \frac{x_0^n}{1 + nx_0^n t}. \tag{4}$$

which clearly has a blowup at $t = -\frac{1}{nx_0^n}$.
■

This is almost enough to show that $\deg(P(x)) > 1$ has finite time blowups. To see why, suppose that $P(x)$ has degree $n > 1$ and with positive leading coefficient. Then for some $c \in (0, 1)$ and sufficiently large positive integer N, $x > N$ implies that $P(x) > cx^n$. Therefore if the initial condition is taken to be large enough, e.g. $x_0 > N$, it follows that for $t \geq 0$,

$$\dot{x}(t) > cx(t)^n \tag{5}$$

By proposition 2, $\dot{y}(t) = cy(t)^n$ is not complete, so apply lemma 1 to deduce that $x(t)$ is not complete. If the leading coefficient is negative, then for N sufficiently large, $x < -N$ implies that

$$P(x) < -cx^n$$

That is,

$$\cdot x(t) < -cx^n(t)$$

and again apply the proposition and lemma.