Assignment 5.

Recall "Koopmanism":if Φ a measure preserving map of a probability space (X, μ) then the induced unitary operator U on $L^2 = L^2(X, \mu)$ given by $Uf(p) = f(\Phi(p))$ is a unitary operator. In class we showed, or I at least I hope we showed:

- (i) Φ is ergodic if and only if the only L^2 solutions to $Uf = f$ are the constant functions f. and that
- (ii) Φ is mixing if and only if for all $f, g \in L^2$ we have $\langle U^n f, g \rangle \to \int f \int g$.

Exercises 1 and 2 apply these consideratiosn to the cat map from assignment 3, exercise 1. Recall that any $f \in L^2(\mathbb{T}^2)$ can be expanded in a Fourier series: $f = \Sigma a_{p,q} e_{p,q}$ where $e_{p,q}(x, y) = exp(2\pi i(px + qy)/$ Here (x, y) are standard toral coordinates so both x, y are to be viewed mod 1.

1. Complete the following exercise to show the ergodicity of the cat map.

a) Show that $Ue_{p,q} = e_{A(p,q)}$ where A is the linear operator defining the cat map: $A(p, q) = (2p + q, p + q)$.

b) Show that $Uf = f$ if and only if the Fourier coefficients $a_{p,q}$ of f satisfy $a_{p,q} = a_{A^{-1}(p,q)}$

c) Think of $(p, q) \in \mathbb{Z}^2$, the integer lattice. The operators A and A^{-1} act on the integer lattice. Re-interpret (b) as follows: $Uf = f$ if and only if the Fourier coefficients are constant along the orbits of A acting on $\mathbb{Z}^2$. What do these orbits look like? Are they ever bounded?

d) Show that the only bounded orbit for A acting on the integer lattice is the orbit of $(0, 0)$.

e) We have $\int |f|^2 = \Sigma |a_{p,q}|^2$ from which it follows that if $f \in L^2$ then the Fourier coefficients must satisfy $a_{p,q} \to 0$ as $(p, q) \to \infty$.

f) Combine these steps to conclude that the only L^2 solution to $Uf = f$ is $f = a_{0,0} =$ const.

2. Complete the following exercise to show mixing of the cat map.

a) Argue by linearity and limits that it is enough to check item (ii) above for f, g pairs of basis functions $e_{p,q}$.

b) The only basis function with average nonzero is $e_{0,0}$.

c). If $(p, q) \neq (0, 0)$ and $(r, s) \in \mathbb{Z}^2$ is given, then for all n large enough $A^n(p, q) \neq (r, s)$.

d) Now verify (ii) for all pairs f, g of basis functions.

3. [Variational vector field] Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, Φ_t its flow, and $c(t) = \Phi_t(x_0)$ the solution to $\dot{c} = X(c)$ with initial condition x_0. Fix a time τ. Then $\Phi_\tau : \mathbb{R}^n \to \mathbb{R}^n$ is a (typically nonlinear) diffeomorphism. Show that $D\Phi_\tau(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is the linear operator defined as follows. Let $A(t) = DX(c(t)) : \mathbb{R}^n \to \mathbb{R}^n$ be the time-dependent linear operator whose matrix entries are the Jacobian matrix of X along $c(t)$. Solve the time-dependent linear system $dv/dt = A(t)v(t)$, $v(t) \in \mathbb{R}^n$, with initial condition $v(0) = v_0$. **Show** that $D\Phi_\tau(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is given by $D\Phi_\tau(x_0)(v_0) = v(\tau)$.

Hint: associated to v_0 we have the curve of initial conditions $x(\epsilon) = x_0 + \epsilon v_0$ and associated solution curves $c_\epsilon(t) = \Phi_t(x(\epsilon))$. Let $\delta c(t) = \frac{d}{d\epsilon}|_{\epsilon=0} c_\epsilon(t)$. Show that $\delta c(t)$ satisfies the same time-dependent linear system as $v(t)$ does, and has the same initial condition.

Special case: If $X(p_0) = 0$ and $A = DX(p_0)$, then show that $D\Phi_\tau(p_0)v_0 = exp(\tau A)v_0$.