Assignment 1.

1. For which polynomials $P(x), x \in \mathbb{R}$ is the flow of $\dot{x} = P(x)$ a complete flow on the real line?

2. Let $P(x)$ be polynomial of degree N on the real line having N distinct zeros x, all satisfying $-1 \le x \le 1$. (For example, P might be a Legendre polynomial.)

a) Sketch a phase portrait of the system $\dot{\theta} = P(\cos(\theta))$ on the standard circle, parameterized by $\theta \in [0, 2\pi]$. (Take N a small integer to make it simple.)

b) Suppose you start off with $\theta(0) = \theta_*$ such that $\cos(\theta_*)$ lies between two consecutive zeros of $P(x)$, say, $x_i < \cos(\theta_*) < x_{i+1}$ Show that, as $t \to \infty$, we have $\theta(t) \to \theta_\infty$ where $\cos(\theta_\infty)$ equals either x_i or x_{i+1}. How can you know which one $\theta(t)$ tends to , the angle corresponding to x_i or to x_{i+1}?

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Assignment 2.

1. We defined the Poincare section of a periodic orbit for a vector field. Let Σ, Σ' be two Poincare sections to the same orbit c, and passing through the same point $p = c(0)$ of c, and let $\Psi : \Sigma, p) \to (\Sigma, p), \Psi' : (\Sigma', p) \to (\Sigma', p)$ be the corresponding Poincare maps. Prove that the two maps are locally conjugate: there is a smooth map $(\Sigma, p) \to (\Sigma', p)$ which is a diffeomorphism at p and is such that $\Phi \circ \Psi = \Psi' \circ \Phi$. (All maps are only defined in a neighborhood of p and take p to itself.)

2. Suppose that the unit circle is a periodic orbit for a smooth vector field on the plane $\mathbb{R}^2$. The line segment joining $(1-\epsilon, 0)$ to $(1+\epsilon, 0)$ is a slice to this orbit, so defines a Poincare section Use the interval $(-\epsilon, \epsilon)$ to coordinatize this slice by sendig x to $(1+x, 0)$. Prove that it is not possible for the map $x \mapsto -x$ to represent the Poincare map associated to our slice.

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Assignment 3. [Some maps]

1. Let $F : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the standard flat torus. The Arnol'd cat map is the map $\mathbb{T}^2 \to \mathbb{T}^2$ defined by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ Thus the cat map is the map $(x, y) mod 1 \mapsto (2x + y, x + y) mod 1$.

a) Find the fixed points of F.

b) Find the points of order 2, ie, those points p such that $F(F(p)) = p$.

c) Describe all periodic points for F.

d) Find the stable and unstable manifolds of the fixed point you found in a). Hint: look at how points on the eigenlines of A evolve upon iterating A forward or backward.

2. Define the continuous piecewise linear "tent maps $f_k : \mathbb{R} \to \mathbb{R}$ by the conditions

- $f(0) = 0$
- $f(1/2 + x) = f(1/2 - x)$
- $f'(x) = k$ for $x < 1/2$.

a) For k small, show that 0 is a global sink: $x_n \to 0$ if $x_{n+1} = f(x_n)$ is an orbit.

b) Show for $k >> 1$ almost all orbits escape to minus infinity.

c) The case $k = 3$. Show that the set of all x whose orbit under f_3 is bounded is equal to the standard Cantor set d) Take the case $k = 2$. Show that f_2 maps the unit interval to itself and that there is a semi-conjugacy between the doubling map $g(x) = 2x$ on the circle, viewed as the has a unique maximum

3. [borrowed from Devaney] a) Show that the unimodal map $f(x) = 1 - 2x^2$ from the interval $[-1, 1]$ to itself is semi-conjugate to the doubling map $D : \theta \mapsto 2\theta$ on the standard circle by making the substitution $p(\theta) = -\cos(\theta)$. That is, we have a commutative diagram involving f, p, D so that $f \circ p = p \circ D$

b) Conclude that f has orbits of any desired period, and that the set of periodic orbits is dense in the interval.