Format. Weekly HW. Lectures. Some in-class work. On Wednesdays students will select from their HW and present solutions. There will be a short midterm and a take home final.

This course is the second course in the geometry-topology graduate sequence at UCSC.

Summary. In essence, manifolds are the most general topological spaces on which calculus makes sense. The fundamental theorem of calculus on manifolds is an integral equation called Stokes' theorem. Besides generalizing the fundamental theorem, Stokes' simultaneously generalizes the big three integral theorems from vector calculus: Green's theorem, the divergence theorem, and the Stokes' theorem you may remember from vector calculus. The integrands of Stokes' theorem are called k-forms, and they are the main subject of this course.

In the course we cover the algebra of the exterior algebra (also known as the Grassmann algebra), integration, differentiation (exterior derivative) and Lie differentiation of forms. We learn how to pull forms back, and Cartan's magic formula. As applications of forms we will formulate and use the de Rham theorem, the Frobenius theorem, and the structure equations of surface theory. de Rham's theorem relates the differential complex based on the exterior derivative to the underlying algebraic topology of the manifold (Betti numbers;, cohomology spaces). The Frobenius theorem gives necessary and sufficient conditions for a family of "infinitesimal surfaces" (called distributions) to be filled in so as to form a family of smooth surfaces. The structure equations of a surface yield a simple way to understand and compute the Gaussian curvature of a surface. Additional topics, time and stamina permitting: Lie groups. Clifford algebras. Riemannian metrics. Non-integrable distributions. Hodge theory.

History. Poincare used differential forms. Cartan extended and refined their use into the powerful tool, known as the exterior differential calculus and used them in a central way to classify the compact simple Lie groups. Grassmann is credited with the invention of the exterior algebra.

Primary texts. Geometry of Differential Forms, by Morita, mainly ch. 2 and 3. Singer-Thorpe (primarily ch. 7). Abraham-Marsden (or just Abraham-Ratiu-Marsden). Differential Forms in Algebraic Topology - Bott-Tu. (In a pinch you can use Lee, but I generally do not.)

Week-by-week schedule,

Week 1. k-forms on $\mathbb{R}^n$. The Grassmann algebra in terms of generators (dx^i). Abstractly: multi-linear forms on a real vector space. On manifolds, in charts and invariantly. Covectors (one-forms at a point) as 1 jets of functions vanishing at p. As m_p/m_p^2. The cotangent bundle. Pictures of one-forms and two-forms. Forms as integrands (1): Integrating one- forms over curves; two-forms over (oriented) surfaces...

Differentiating forms. Pulling back forms. Pairing vector fields and forms.

HW. Forms. Clifford algebras (?). $\omega^n/n!$. $d\theta$.

Week 2. k-forms as integrands. How to integrate them. What to integrate them over. Stokes.

Morita. Bott-Tu.

3. Poincare lemma: closed vs exact. Baby de Rhams: on $\mathbb{R}^n$, $\mathbb{T}^n$, $\mathbb{R}^n\setminus$ (points). Winding number and degree.

Statement of de Rham (?).

Morita. Bott-Tu. Abraham-Marsden-Ratiu.

4. Application 1. Degree of a map $X^n \to Y^n$ via top forms. Via generic inverse images. Use of in solvability.

Poincare Hopf (?) . de Rham,

ref: Milnor. Bott-Tu.

5. Application 2: Frame bundle of a Riemannian surface. Structure formula. Examples. Sphere. Hyperbolic space. Gauss-Bonnet 1.

ref: Singer-Thorpe.

6. Frobenius theorem on integrability. Forms and vector fields version. Examples: flatness. Lie subgroups. Contact.

ref: Morita. Abraham-Marsden.

7. de Rham II; Hodge theory

8. Vector bundles; circle bundles, connections; 1st chern class.

9. Loose ends. Poincare-Hopf. Topology of bundles. Lie groups.

10. discussion of take home.

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