Problem 1. Problem: Prove that the 'cross" (in algebraic geometry, the 'normal crossing') $xy = 0$ is not a topological manifold

Problem 2. Prove that the cone $x^2 + y^2 = z^2, z \geq 0$ is a topological manifold but is not a smooth embedded manifold in Euclidean 3-space.

Problem 3. . Show that $u = xy, v = y$ is not a good change of coordinates near the origin of the xy plane, while $u = (x + .005)(y + .001), v = y$ is a good change of coordinates near the origin.

Problem 4. Prove that $exp : so(3) \to gl(3)$ is a smooth map and that its rank is 3 at the origin. Here exp is the matrix exponential, $so(3)$ is the vector space of 3 by 3 skew-symmetric matrices and $gl(3)$ is the space of all 3 by 3 real matrices.

Problem 5. Prove that the image of exp from the previous problem is $SO(3) \subset gl(3)$

Problem 6. View x as the affine coordinate for $\mathbb{RP}^1$ so as to identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$. Show that the vector field $\frac{\partial}{\partial x}$ on the open set $\mathbb{R} \subset \mathbb{RP}^1$ extends to a smooth vector field on all of $\mathbb{RP}^1$. Does this vector field vanish at ∞?

Problem 7. Consider the vector field $V(x) = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$. Show that it is incomplete, by showing solutions blow up in finite time.

Problem 8. Continuing with the notation of the previous two problems, so that $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$, show that the vector field $V(x)$ IS complete when viewed as a vector field on $\mathbb{RP}^1$.

Problem 9. Compute stereo projection $S^2 \setminus \{pt\} \to \mathbb{R}^2$ and its inverse. [See Lee, problem 1.5]

Problem 10. For any point $x_0 \in S^2$ we have the stereo projection map $\phi_x \in S^2 \setminus \{x_0\} \to x_0^{\perp}$.

For $x = N$ and $x = S$ we have $x^{\perp} = \mathbb{R}^2$, where $N = (0, 0, 1), S = -N$ and $\mathbb{R}^2$ denotes the xy plane. Compute the overlap map $\phi_S \circ \phi_N^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$.

Problem 11. Prove that multiplication $(A, B) \mapsto AB$ is a smooth map $SO(3) \times SO(3) \to SO(3)$.

HINT: first prove the analogous assertion for multiplication on the space $gl(3)$ of all 3 by 3 real matrices.

Problem 12. Prove that inversion $A \mapsto A^{-1}$ is a smooth map $SO(3) \to SO(3)$.

HINT: first prove the analogous assertion for the space of invertible 3 by 3 real matrices.

Problem 13. Let

$$E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Prove that $X(g) = gE_3$ defines a smooth vector field on $SO(3)$.

Problem 14. Prove that the map $\pi : SO(3) \to S^2$ given by $\pi(g) = ge_3$ is a submersion.

Problem 15. Let $SO(2) \subset SO(3)$ be the rotations about the z- axis. Prove that for each $g_0 \in SO(3)$ the map $F_{g_0} : SO(2) \to SO(3)$ given by $F(\lambda) = g_0\lambda$ is an embedding.

Problem 16. Using the notation of the last two problems , prove that $\pi(g) = \pi(g_0)$ if and only if $g \in ImF_{g_0}$. In other words, π has fibers the circles which are the image of the F_{g_0} – the $SO(2)$ cosets.

Problem 17. In using the IFT to prove that $SO(3)$ is a manifold we used the function $F : gl(3) \to sym(3)$ given by $F(A) = AA^t$. Prove, by hand, that Sard holds for F: almost every value of F is regular.

Hint: show that if $c \in sym(3)$ is invertible then it is a regular value for F.

Problem 18. Describe a vector field on the 2-sphere which is nowhere zero along the equator

Problem 19. Describe,analytically, a vector field on the 2-sphere whose only zeros are at N and S.

Problem 20. . Let A be a symmetric n by n real matrix. Show that $f([v]) = \langle Av, v\rangle / \langle v, v\rangle$ is a smooth function on the projective space $\mathbb{RP}^{n-1}$. Relate the critical values of f to the eigenvalues of A. Here $[v] = span(v) \in \mathbb{RP}^{n-1}$ for $v \neq 0$ a vector in $\mathbb{R}^n$.

Bracket Problems

Problem 21. Prove or disprove: there is a smooth vector field v on the plane such that $[x\frac{\partial}{\partial y}, v] = \frac{\partial}{\partial x}$

Problem 22. Find a radial solution $f = f(r)$ to the 1st order linear PDE $(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})f = 3f$ in the plane.

Riemannian geometry problems

Problem 23. Write down a diffeomorphism between $SO(3)$ and the unit tangent bundle to S^2.

Problem 24. Describe a diffeomorphism between $SO(3)$ and $\mathbb{RP}^3$.

Problem 25. Let $ds^2 = \Sigma g_{ij}(x^1, x^2, x^3)dx^i dx^j$ be a Riemannian metric on the 3 dimensional disc. Show that there exist coordinates u, v, w centered at the origin such that $ds^2 = du^2 + dv^2 + dw^2 + \Sigma\beta_{ij}(u, v, w)du^i du^j$ where the β_{ij} are smooth functions which all vanish at the origin, and where, for notational convenience I've set $u^1 = u, u^2 = v, u^3 = w$.

LIE BRACKET and LIE DERIVATIVE PROBLEMS.

Problem 26. A non-vanishing one-form α defines a hyperplane field $D \subset TQ$ by $D_q = ker(\alpha(q))$. Suppose that V is a vector field on Q with flow Φ_t. Prove that $L_V\alpha = f\alpha$ for some function f if and only if the flow of V preserves D, i.e: $\Phi_t^* D = D$

Problem 27. Let $\alpha = dz - ydx$, a one-form on $\mathbb{R}^3$. Find nonvanishing vector fields V on $\mathbb{R}^3$ such that $L_V\alpha = 0$. Find other vector fields such that $L_V\alpha = f\alpha$ $f \neq 0$ a function.

Problem 28. Repeat the previous problem with the one-form $\alpha = dz$.