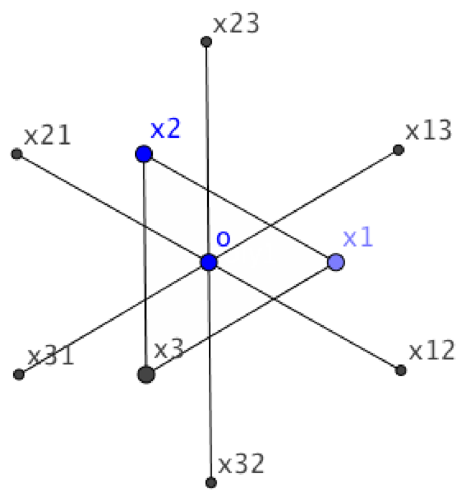

FIGURE 1. Fundamental weights: xi. Roots: $xij = xi - xj$.

Roots and Weights for $SU(3)$

The maximal torus T of $SU(3)$ consists of the diagonal matrices in $SU(3)$:

$$T = \{\begin{pmatrix} expi2\pi x_1 & & \\ & expi2\pi x_1 & \\ & & expi2\pi x_1 \end{pmatrix} : x_i \in \mathbb{R}, x_1 + x_2 + x_3 = 0\}$$

where blank entries means zeros are placed there. Write

$$t = t(x_1, x_2, x_3) = diag(e^{i2\pi x_1}, e^{i2\pi x_2}, e^{i2\pi x_3}) \in T$$

for the above typical element of T.

BIRTHDAY REP. The "birthday representation" of $SU(3)$ is the usual action of $SU(3)$ on $\mathbb{C}^3$ by matrix multiplication, with vectors $z \in \mathbb{C}^3$ viewed as column vectors. Then

$$te_j = exp(2\pi i x_j)e_j, j = 1, 2, 3$$

where e_1, e_2, e_3 is the usual basis for $\mathbb{C}^3$. The x_i are int* being linear functions on $\mathfrak{t}$. Thus the *weights* of the birthday representation are $x_1, x_2, x_3 \in \Lambda^* \subset \mathfrak{t}^*$. The and the corresponding weight spaces are the $\mathbb{C}$-spans of the e_j. The Killing form induces an inner product on $\mathfrak{t}$, and hence $\mathfrak{t}^*$ and with respect to this inner product the x_i form the vertices of an equilateral triangle with center at the origin.

As we will see momentarily, the *edge vectors* of this triangle, being the six difference vectors $x_i - x_j$ together with $0 = x_i - x_i$ form the *roots* for $SU(3)$, which is to say, the weights of its Adjoint representation.

ADJOINT REPRESENTATION. We have

$$Ad_t X = tXt^{-1}.$$

We compute that:

$$Ad_t\begin{pmatrix} 0 & \xi & 0 \\ -\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i2\pi(x_1-x_2)}\xi & 0 \\ e^{-i2\pi(x_1-x_2)}\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Ad_t\begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & 0 \\ -\bar{\xi} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & e^{i2\pi(x_1-x_3)}\xi \\ 0 & 0 & 0 \\ e^{-i2\pi(x_1-x_3)}\bar{\xi} & 0 & 0 \end{pmatrix}$$

And:

$$Ad_t\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\bar{\xi} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i2\pi(x_2-x_3)}\xi \\ 0 & e^{-i2\pi(x_2-x_3)}\bar{\xi} & 0 \end{pmatrix}$$

while

$$Ad_t D = D, D \text{ diagonal}$$

which exhibits the root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha; \alpha = \alpha_{12}, \alpha_{13}, \alpha_{23} \tag{1}$$

of the Lie algebra $\mathfrak{g} = su(3)$ as the sum of the diagonal matrices $\mathfrak{t}$ and the "three root spaces" indicated above. Note that $x_i \in \mathfrak{t}^*$, since the x_i are linear functions on $\mathfrak{t}$. The roots are the three linear functions:

$$\alpha_{12} = x_1 - x_2, \alpha_{13} = x_1 - x_3, \alpha_{23} = x_2 - x_3$$

and their negatives $\alpha_{ji} = -\alpha_{ij}$ which appear in the exponentials for these expressions for Ad_t. Thus for example the root space for $\alpha = \alpha_{12}$ is described by the first Ad_t expression, so that

$$\mathfrak{g}_\alpha = \{\begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & 0 \\ -\bar{\xi} & 0 & 0 \end{pmatrix} : \xi \in \mathbb{C}\}, \text{ for } \alpha = \alpha_{12}$$

Because $Ad_t = Id$ on $\mathfrak{t}$ and $e^{i0} = 1$, we say that $\mathfrak{t}$ corresponds to the zero-root space.

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Computations and theory proceeds more simply if we differentiate and complexify the actions.

Differentiating the adjoint action. Write E_{ij} for the matrix with a 1 in the ij place and 0 everywhere else. Then $\mathfrak{t}$ is the spanned by the $\sqrt{-1}E_{ii},\ i = 1, 2, 3$ (subject to the single linear relation $\Sigma x_k E_{kk} = 0$) while the mk root space above is the *real span* of $E_{km} - E_{mk}$ and $iE_{km} + iE_{mk}, m \neq k$.

Differentiating the Adjoint action gives the Lie bracket or "little ad" action of $\mathfrak{g}$ on $\mathfrak{g}$. Specifically, set $t_\epsilon = exp(\epsilon 2\pi i \Sigma x_k E_{kk})$. Then,

$$\frac{d}{d\epsilon}|_{\epsilon=0} Ad_{t_\epsilon} X = 2\pi i[\Sigma x_k E_{kk}, X]$$

Compute:

$$[\Sigma x_k E_{kk}, E_{ij}] = (x_i - x_j)E_{ij}.$$

But wait. $E_{ij} \notin \mathfrak{g}$. However, $E_{ij} \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} \subset gl(3, \mathbb{C})$. Indeed $E_{ij} = \frac{1}{2}(E_{ij} - E_{ji}) - \sqrt{-1}(\sqrt{-1}E_{ij} + \sqrt{-1}E_{ji})$ is a linear combination of two elements from $\mathfrak{g}$ but with an imaginary coefficient for the second element. We have that $\mathfrak{g}_{\mathbb{C}} = sl(3, \mathbb{C})$, the complex 3 by 3 matrices with trace zero.

Now the action of $sl(3,\mathbb{C})$ on itself by bracket decomposes as

$$\mathfrak{t}_{\mathbb{C}} \oplus \Sigma_{i \neq j} \mathbb{C} E_{ij} \tag{2}$$

with $\mathfrak{t}$ acting ot $\mathfrak{t}_{\mathbb{C}}$ by zero, and on E_{ij} by multiplication by $(x_i - x_j)$. Here $\mathfrak{t}_{\mathbb{C}}$ are complex diagonal matrices with trace 0. Exponentiating and taking appropriate linear combinations, we get the previous splitting ...

Recapitulation. We took the adjoint action of G on $\mathfrak{g}$. We differentiated to get the action of $\mathfrak{g}$ on $\mathfrak{g}$ by Lie bracket. We complexified to get the action of $\mathfrak{g}_{\mathbb{C}} = sl(3,\mathbb{C})$ on $\mathfrak{g}_{\mathbb{C}}$ by Lie bracket. This latter action decomposes into a bunch of one-dimensional complex eigenspaces, the spans of the E_{ij}, upon being restricted to $\mathfrak{t}_{\mathbb{C}}$.

We can turn this procedure around. Start with $\mathfrak{g}_{\mathbb{C}} = sl(3,\mathbb{C})$. Take the adjoint (bracket) action of $\mathfrak{g}_{\mathbb{C}}$ on itself. Restrict this action to $\mathfrak{t}_{\mathbb{C}}$, the space of complex 3 by 3 diagonal matrices of trace zero, to obtain a Lie algebra represenation of $\mathfrak{t}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$. This representation decomposes into a direct sum of a bunch (8 to be exact) of complex one-dimensional "root spaces" , each root space spanned by an E_{ij}. The result is (2). Exponentiate to get the Adjoint representation of $T_{\mathbb{C}}$, the space of complex diagonal matrices with determinant 1, on $g_{\mathbb{C}}$). The restriction of this complexified adjoint action to $T \subset T_{\mathbb{C}}$ maps $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ to itself and the complexified ij root space, together with the ji root space, conspire together with the correct complex scalar factors to yield the root space decomposition (1) for $T \subset SU(3)$ actiing on $\mathfrak{g}$.