Weyl Group.

The standard definition of the Weyl group W is $W = N(T)/T$.

We will show how to construct W out of the root space decomposition of $\mathfrak{g}$. The trick is to see that selecting a root pair $\{\pm\alpha\}$ induces a homomorphism

$$\phi_\alpha : SU(2) \to G$$

such that $d\phi_\alpha(su(2))$ contains the root space $\mathfrak{g}_{\pm\alpha}$. The image G^α of ϕ_α is a copy of either $SU(2)$ or $SO(3)$ canonically attached to the root hyperplane $\{\alpha = 0\}$. G^α is contained in the centralizer $G_\alpha = Z(T_\alpha)$ of the codimension 1 torus T_α, and so leavesT_α invariant when acting by conjugation. Set

$$\sigma_\alpha = \phi_\alpha(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}).$$

Conjugation by σ_α induces the desired reflection about the root hyperplane $\alpha = 0$ (and, in T, about T_α.)

Let X_α, Y_α be an oriented orthonormal basis for $\mathfrak{g}_{\pm\alpha}$.

Proposition 1. *(i)* $[X_\alpha, Y_\alpha] \in \mathfrak{t}$. *(ii)* $[X_\alpha, Y_\alpha] \neq 0$. *(iii)* $[X_\alpha, Y_\alpha] \perp ker(\alpha)$.

Proof of item (i). Let $H \in \mathfrak{t}$. Then

$$[H, X_\alpha] = \alpha(H) Y_\alpha \text{ and } [H, Y_\alpha] = -\alpha(H) X_\alpha.$$

Using this fact, and the Jacobi identity we see that for all $H \in \mathfrak{t}$ we have $[H, [X_\alpha, Y_\alpha]] = 0$. Maximality of T now implies that $[X_\alpha, Y_\alpha] \in \mathfrak{t}$.

Proof of item (ii) and (iii). The inner product on $\mathfrak{g}$ is Ad invariant: $\langle Ad(g)U, Ad(g)V\rangle = \langle U, V\rangle$. Differentiating this identity with respect to g we see that

$$\langle [Z, U], V\rangle + \rangle U, [A, Y] \langle = 0$$

for all $Z, U, V \in \mathfrak{g}$. We compute

$$\langle H, [X_\alpha, Y_\alpha]\rangle = -\langle [X_\alpha, H], Y_\alpha\rangle = +\alpha(H)\langle Y_\alpha, Y_\alpha\rangle$$

from which it follows that $[X_\alpha, Y_\alpha] \perp ker(\alpha)$. Thus $[X_\alpha, Y_\alpha]$ must lie in the line in $\mathfrak{t}$ orthogonal to $ker(\alpha)$. To see that $[X_\alpha, Y_\alpha]$: is not zero, choose $H = e_\alpha$ orthogonal to $ker(\alpha)$ such that $\alpha(e_\alpha) = 1$ to conclude that $[X_\alpha, Y_\alpha] \neq 0$. QED.

The root $SU(2)$. Choose a basis H_α for the line $ker(\alpha)^\perp$ in $\mathfrak{t}$. By appropriate scaling, we can make sure that $X_\alpha, Y_\alpha, H_\alpha$ satisfy the Lie algebra relations of $su(2)$. Hence we have a Lie algebra homorphism $\phi'_\alpha : su(2) \to \mathfrak{g}$ whose image $\mathfrak{g}^\alpha = \mathbb{R}H_\alpha \oplus \mathfrak{g}_{\pm\alpha}$ is the LIe subalgebra generated by $\mathfrak{g}_{\pm\alpha}$. By general theory, (eg. see Hsiang ch. 2, Thm 3) this Lie algebra homomorphism integrates up to a group homomorphism ϕ_α with aforementioned properties.

Proposition 2. *Conjugation by $\sigma_\alpha \in G^\alpha$ induces reflections about $\mathfrak{t}_\alpha$ and T_α.*

Proof. Take the maximal torus T_1 of $SU(2)$ to be the diagonals. Conjugation by $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ maps T to itself, acting by reflection: $\theta \mapsto -\theta$ on T_1. (Coordinatize T_1 by $diag(e^{i\theta}, e^{-i\theta})$.) I claim that conjugation by $\sigma_\alpha = \phi_\alpha(w)$ acts on the maximal torus T by reflection across about $T_\alpha = ker(\alpha)$. Indeed, $\sigma_\alpha \in G^\alpha \subset G_\alpha$ and $G_\alpha = Z(T_\alpha)$, the centralizer of the codimension 1 torus $T_\alpha = ker(\alpha)$. Being an element of G^α, σ_α acts under conjugation as the identity on T_α. But σ_α acts by $H_\alpha \to -H_\alpha$ and H_α generates the line in $\mathfrak{t}$ (resp. T) orthogonal to the root hyperplane $\{\alpha = 0\}$ (resp. orthogonal to T_α). QED.

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Examples. $SU(3)$. Write the maximal torus as the diagonal matrices $diag(e^{i2\pi\theta_1}, e^{i2\pi\theta_2}, e^{i2\pi\theta_3})$, the θ_j subject to the constraint that $\theta_1+\theta_2+\theta_3 = 0$. Selecting any two of the three θ_j yield angular coordinates on the torus. We also view the θ_j as linear functions on $su(3)$, which is to say, elements of the dual $\mathfrak{t}^* = su(3)^*$. As such, $\theta_1, \theta_2, \theta_3$. generated the weight lattice in $\mathfrak{t}^*$, isometric to a hexagonal lattice in the plane. (The θ_j are the weights of the birthday representation of $SU(3)$ on $\mathbb{C}^3$.) The dual lattice Λ^* in $\mathfrak{t}$ generated by $2\pi i diag(1,-1,0), 2\pi i diag(1,0,-1), 2\pi i diag(0,1,-1)$.

There are 3 pairs of roots $\pm(\theta_1-\theta_2), \pm(\theta_2-\theta_3), \pm(\theta_3-\theta_1)$. This '3' is the same as the '3' of $dim(SU(3)) = 2+2*3$. The root spaces are

$$\mathfrak{g}_{12} = \{\begin{pmatrix} 0 & z & 0 \\ -\bar{z} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : z \in \mathbb{C}\}; \mathfrak{g}_{23} = \{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{C}\}; \mathfrak{g}_{31} = \{\begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ -\bar{z} & 0 & 0 \end{pmatrix} : z \in \mathbb{C}\}$$

The corresponding 3 root $SU(2)$'s in $SU(3)$ are obtained by selecting out the corresponding plane ij out of $\mathbb{C}^3$. For example, the 1st root $SU(2)$ is for the 12 plane and consists of block matrices

$$\begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $|a|^2 + |b|^2 = 1$ – in other words, its elements comprise the $SU(2)$ associated to the decomposition $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}^1$ with the $SU(2)$ acting on the copy of $\mathbb{C}^2$.

The Weyl group W is generated by the 3 permutation matrices σ_{ij}

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

lying in their respective root $SU(2)$'s. They act by reflections about the root hyperplanes, which is by transposition of the θ_i; so for example σ_{12} switches θ_1 with θ_2, keeping θ_3 fixed. They generate W which is the permutation group S_3 on three letters, acting on $\mathfrak{t}$ ($\mathfrak{t}^*$, T) by permuting the three coordinates (weights, angular coordinates) $\theta_1, \theta_2, \theta_3$.

$SU(n)$. All of this business for $SU(3)$ generalizes in a straightforward way to $SU(n)$. Its Weyl group W is S_n, the symmetric group on n letters. The action of W on $\mathfrak{t}$ is obtained by realizing $\mathfrak{t}$ as $\{\theta_1 + \ldots + \theta_n = 0\}$ in $\mathbb{R}^n$ and having S_n act by permuting the indices of the coordinates (weights) θ_j.

Proposition 3. *The center of G is discrete if and only if the roots span $\mathfrak{t}^*$.*

Suppose that the roots do not span $\mathfrak{t}^*$. Then there is an $H \in \mathfrak{t}$, $H \neq 0$ such that for all roots α, we have $\alpha(H) = 0$. It follows that the one-parameter subgroup generated by H acts trivially with respect to the adjoint representation. Integrating, up to AD, we see this subgroup commutes with all elements of G, and consequently the center of G is at least one-dimensional.

Conversely, suppose that the center Z is not discrete, and hence has dimension at least one. Consider its identity component Z_0. This is an Abelian Lie group of dimension at least one which commutes with all elements of T and so is contained in T. Let $H \in Lie(Z_0) \subset \mathfrak{t}$, $H \neq 0$. And let $\alpha \in R$ be any root. I claim that

$\alpha(H) = 0$. For otherwise, $Ad(exp(tH))$ will be non-trivial, (not the identity) and as a consequence $AD(exp(tH))$ will be non-trivial. (and in every maximal torus), contradicting the fact that $exp(tH)$ lies in the center of G. QED

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For any $\sigma \in N(T)$, conjugation by σ is an automorphism of T, Since $Ad(\sigma t \sigma^{-1}) = Ad(\sigma)Ad(t)Ad(\sigma^{-1})$, the map $t \mapsto Ad(\sigma)Ad(t)Ad(\sigma^{-1})$ defines another representation of T on $\mathfrak{g}$ equivalent to the restriction to T of the adjoint representation. It follows that the set of roots for this new representation are the same as for the original. Thus, whenever α is a root so is $Ad(\sigma^{-1*})(\alpha)$, where $Ad(\sigma^{-1*})$ denotes the co-adjoint action. Also this co-adjoint action by σ maps the weight lattice $\Lambda \subset \mathfrak{t}^*$ to itself, for similar reasons, and the dual lattice $\Lambda \subset \mathfrak{t}$ to itself.

We now follow the Grothendieck-Demazure definition of roots and the Weyl group, as explained in Vogan.